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Abstract

A common feature of dynamic interactions is that the environment in which they occur changes, possibly stochastically, over time. We study a fluctuating contracting environment with symmetric information and develop a notion of a *separable activity* that corresponds to a large class of prevalent contractual components. We provide a tight condition under which these components manifest a form of seniority: any change that occurs in these components over time favors the agent. We illustrate how our results can be applied in various economic settings.

Keywords: Dynamic contracting, Stochastic opportunities.

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Dynamic interactions between a principal and an agent often take place in a changing environment. Seasonality and random shocks affect periodic demands; workers accumulate skills; business opportunities and threats arrive and disappear at random; and technological innovations make existing practices obsolete. In this paper, we identify a general feature of optimal contracts that arises directly from such fluctuations in the contracting environment. Our results offer new insights into the general phenomenon of seniority and can be applied in various contexts. The results also afford a unified and more general perspective on several seemingly unrelated (classic as well as more recent) papers that study dynamic contracting in a broad range of economic settings.

To illustrate the forces at play, we first consider a stylized *model of random opportunities*. In this model, different types of "tasks" arrive stochastically over time in a manner that is i.i.d. across periods and the principal offers periodic wages to incentivize the agent to exert effort on available tasks. The agent's marginal utility from wage and the marginal productivity of effort are both assumed to be decreasing. To isolate the effect we want to study, we abstract away from frictions that arise from informational asymmetries and assume that the agent's effort and the arrival of tasks are perfectly observed. We show that, although the arrival of tasks is governed by a stationary distribution, the unique optimal contract generates a "promotion-based" dynamics where the periodic wage increases and the required effort (on every type of task) decreases over time.

The above model offers new insights into wage ladders and seniority by drawing a clear connection between exogenous intertemporal variability in business conditions and the dynamics of effort and compensation. Naturally, the fine details of the optimal contract rely on a number of simplifying assumptions. For this reason, even though the contract exhibits several interesting features, we mainly emphasize the monotonic way in which the terms of the contract change over time: when the required effort or wage is updated, *the shift is always in the direction that favors the agent.* This type of monotonicity turns out to be a general feature of optimal contracts in a wide range of contracting settings.

Understanding this property beyond the stylized model of random opportunities is useful since, in practice, long-term interactions usually take place in highly complex environments. Even in the simple context of random opportunities, the distribution by which the opportunities arrive may sometimes depend on the players' past actions, or simply not satisfy the i.i.d. assumption across periods. Deriving optimal contracts for such environments directly may be eminently difficult.

We develop a conceptual framework that circumvents this difficulty and allows us to establish a general version of the above monotonicity property for general (two-player) contracting environments. The key restrictions we impose are that information is symmetric and that only the principal has commitment power. Examples of economic settings where the contracting problem falls within this class include labor contracts with dynamic uncertainty or search frictions (Harris and Holmström 1982; Holmström 1983; Postal-Vinay and Robin 2002a,b), dynamic risk sharing (Marcet and Marimon 1992; Krueger and Uhlig 2006), foreign investment and entrepreneur financing (Thomas and Worrall 1994; Albuquerque and Hopenhayn 2004), and project selection (Forand and Zápal 2019).

Our objective is to identify a property of optimal contracts that does not depend on the specific features of a particular setting. Therefore, in our analysis we will deliberately leave many elements of the contracting environment unspecified. Using this detail-free approach, one can derive partial yet economically meaningful implications even in complex settings in which there is little hope of obtaining a complete characterization of optimal contracts. Due to the flexibility and generality of our modeling approach, our results draw connections between several existing models and can be easily applied to new economic settings.

The main notion we develop is that of *activity*. Broadly speaking, an activity represents a recurring component of the interaction for which the players have monotone and opposite preferences. Examples include periodic wage, financing to an entrepreneur, worker effort, level of authority of a bureaucrat or a unit in an organization, production volume, quality of supplied products, degree of risk-sharing, and more.

At the contractual level, all elements of the interaction are interrelated through incentives. Yet, certain independence conditions may be satisfied at the environmental level. For instance, in the model of random opportunities, the distribution of task arrivals is assumed to be i.i.d. across periods — an assumption that separates future task availability from past events. In our general model, we define a weaker notion of separability between a given activity and all other components of the environment where the interaction takes place. Our main requirement is that changes in the *level* of the activity in question must not affect the environment in the future (e.g., changing the agent's effort must not impact the distribution of future task availability). It is worth pointing out that, unlike in the stylized model of random opportunities, our notion of separability allows the availability of separable activities to depend (in certain ways) on past events. Therefore, situations where the availability of a separable activity is endogenous and/or path-dependent fall under the framework of our general model.

An additional separability requirement that we impose is that the players' payoffs from the activity be additively separable from the payoffs obtained from all other parts of the interaction. An activity-related payoff frontier can thus be constructed without fully specifying the contracting environment where the interaction occurs. Separable activities with convex payoff frontiers are the main object of interest in this paper. We refer to such activities as *convex separable activities*. The agent's periodic wage and the (task-specific) effort in the model of random opportunities are examples of convex separable activities.

In the main result of the paper, we provide a sufficient condition on convex separable activities that guarantees that, regardless of the details of the contracting problem, as time goes by, the levels of such activities change only in the direction that favors the agent, in any optimal contract (Theorem 1). Furthermore, our condition is tight in the sense that, under mild technical requirements, for every convex separable activity that does not meet the condition, there exist contracting problems in which it is optimal to update the activity in the opposite direction (Theorem 2).

In many cases, our condition is easy to verify. For example, it will become clear below that any activity that is unilaterally controlled by one of the players satisfies our condition. All of the activities in the model of random opportunities are solely controlled either by the agent (effort) or by the principal (wage). This gives rise to two general observations: regardless of the details of the contracting problem, the agent's effort on tasks can only decrease over time, while his wage can only increase over time. This yields a general and detail-free downward wage rigidity result, as well as an upward effort rigidity result.

In general, activities can be jointly controlled by both players. For example, consider a production process where first the principal provides the agent with the resources required to produce a given level of output and then the agent either exerts the effort needed to finalize production or deviates by reallocating the capital to alternative uses (this is similar to the situation analyzed in Thomas and Worrall 1994). A key feature of this example is that the agent's value from deviating, which plays a central role in our condition, depends on the intended level of production. Whether or not the condition is satisfied depends on the details of the production function and the agent's value from reallocating capital; we illustrate and discuss examples of both cases.

The rest of the paper is organized as follows. Section 1 introduces the general contracting environment. Section 2 analyzes the optimal contract in a stylized model of random opportunities. In Section 3 we define and discuss the notion of activity and in Section 4 we present and discuss our main result. In Section 5 we address the robustness of our result, and in Section 6 we present examples of applications. Section 7 surveys the related literature and Section 8 concludes. All proofs are relegated to the appendix.

1 Contracting Environment

We consider dynamic contracting settings that can be represented as follows. In each period $t \in \{1, 2, ..., T\}$, where $T \leq \infty$, the players observe a randomly drawn (strategic-form) game, take actions in the game, observe its outcome, and receive payoffs. The game in period t is drawn from a commonly known distribution $f(h_t)$, where the (public) history h_t contains a full description of the realized (periodic) games and the players' actions in periods $1, 2, ..., t-1$. We refer to the stochastic process $f(\cdot)$ as the contracting environment.

As the calendar time, previous realizations of periodic games, and the players' past moves may affect the periodic games the players will play in the future, this specification is fairly general. This class of environments can accommodate a wide variety of settings, including, but not limited to, settings where the agent's cost of effort depends on past events, there is seasonality in demand, there is uncertainty about the principal's ability to provide compensation in the future, there are long-term (or storable) investment opportunities, or there are R&D-type opportunities that may change future production methods and costs.

A *contract* specifies, for every finite history h_t, a suggested strategy profile for any game in the support of $f(h_t)$. A contract is *incentive compatible* if, for every finite history and game that is played after that history, the agent's suggested continuation strategy is a best response to the principal's continuation strategy specified by the contract. The players maximize (discounted) expected utility and use the same positive discount factor δ (if $T = \infty$, we require that $\delta < 1$).

In order to study a particular setting, one needs to specify the stochastic process $f(\cdot)$. If the specification is sufficiently narrow, or "stylized," it may be possible to fully characterize the optimal contract. In Section 2, we offer an example of an analysis along these lines.

The role of this analysis is twofold. First, it provides a transparent and easy illustration of the effect of exogenous fluctuations in task arrival on the dynamics of effort and wage. In doing so, the model is arguably interesting in its own right since it offers new insights into seniority in labor contracts. Second, it lays the groundwork for the more abstract exposition of the main concepts and results of the paper in subsequent sections.

2 A Model of Random Opportunities

Consider an interaction where the set of possible (strategic-form) games is $\Gamma = \{G^1, ..., G^I\}$, and the probability of G^i being played after any history is $q_i > 0$, such that $\sum_{i=1}^{I} q_i = 1$. The set of the principal's and the agent's possible actions in every game in Γ is $[0, \infty)$, with the generic elements denoted by u and e, respectively. The payoffs from the strategy profile $\langle u, e \rangle$ in G^i are $u - e$ for the agent and $\pi_i(e) - c(u)$ for the principal, where $c(\cdot)$ is an increasing, strictly convex, and differentiable function for which $c(0) = 0$, and $\pi_i(\cdot)$, for $i \leq I$, is an increasing, strictly concave, and differentiable function for which $\pi_i(0) = 0$. In addition, we assume that for all $i < I$ and $e \in [0, \infty)$, $\pi_i'(e) < \pi_{i+1}'(e)$, and that for all $i \leq I$, $\pi_i'(0) > c'(0)$ and $\lim_{e_i \to \infty} \pi_i'(e_i) < \lim_{u \to \infty} c'(u)$.

A possible interpretation (to which we adhere throughout the section) is that the principal offers a periodic wage to incentivize the agent to exert costly effort on randomly arriving short-lived tasks (or "opportunities"). The game G^i is played in periods in which a task of type i is available. In every period, u and e represent, respectively, the agent's utility from compensation and (the cost of) effort.[1] The marginal cost of compensating the agent is increasing. This may reflect, for example, the agent's decreasing marginal utility from a nominal wage. For reasons that will become clear later on, we measure compensation directly in terms of the agent's "utils" and assume that the principal's cost of compensation $c(u)$ is convex. The marginal productivity of effort on all tasks is taken to be decreasing. Moreover, we assume that tasks can be ordered with respect to the marginal productivity of effort. Since $\pi_i(0) = 0$ for all i, this ordering assumption means that tasks are also ordered with respect to the total productivity of effort. Accordingly, we say that task i is *better* than task j if $i > j$.

[1] The assumption that $u \geq 0$ reflects a type of limited liability condition for the agent.

2.1 The Phase Mechanism

We construct a particular contract, referred to as the "phase mechanism" (henceforth PM), that we will later prove to be uniquely optimal.[2] In brief, PM consists of multiple hierarchical phases. Within each phase, the agent enjoys a fixed periodic compensation and, whenever a task of a given type arrives, he is instructed to exert the same (type-specific) level of effort. The contract transitions to a new phase upon the arrival of a task that is better than all previously available tasks. Thus, the contract moves monotonically (perhaps with jumps) through the phases until the final absorbing phase is reached. The key qualitative property of PM is that when the contract changes phases, the agent's periodic compensation increases and the task-specific effort requirements decrease. Consequently, as time goes by, both the periodic compensation and the effort exerted on every type of task shift monotonically in the direction that favors the agent.

Auxiliary Problems

The crux of the argument in the characterization of the optimal contract is to show that whenever a best-to-date task becomes available, there should be no debt carried over from the past (i.e., the agent's continuation utility should be zero). Given this insight (which we establish below), the whole interaction can be split into parts that can be analyzed separately. We now define I auxiliary problems, one for each type of task, which constitute the building blocks of the optimal contract.

For $i \in \{1, 2, \ldots, I\}$, let $P^{(i)}$ (referred to as "auxiliary problem i") denote the principal's optimization problem in an auxiliary setting where:

(1) In the first period, G^i is played with certainty.

(2) The interaction ends upon the arrival of a task that is better than i. The period in which that task arrives is referred to as the "last period" of the interaction.

(3) The principal is restricted to selecting stationary levels $(e_j)_{j \le i}$ and u, where e_j is the required effort whenever G^j is played, and u is the periodic compensation granted from the *second* to the last period of the interaction, inclusive.

For a contract to be incentive compatible in this auxiliary setting, the agent's cost of effort on the currently available task must not exceed the expected discounted difference

[2] Up to a redundant multiplicity that exists off the path of play.

between the future compensation and the cost of future effort. Let $\lambda_i = 1 - \sum_{k>i} q_k$ denote the probability that the task available in a given period is no better than i. Since the arrival of tasks is i.i.d. and the principal is restricted to contracts as specified in point (3), the incentive compatibility constraint when game G^j is realized in period t of $P^{(i)}$ is

$$e_j \leq \sum_{s=t+1}^{\infty} \lambda_i^{(s-(t+1))} \delta^{(s-t)} \left(u - \sum_{k \leq i} q_k e_k \right),$$

where $\lambda_i^{(s-(t+1))}$ is the probability that the interaction reaches period s, conditional on G^j, $j \leq i$, being played in t. The above incentive compatibility constraint for game G^j in $P^{(i)}$ simplifies to

$$e_j \leq \frac{\delta}{1 - \delta\lambda_i} (u - \sum_{k \leq i} q_k e_k). \qquad (IC_j^{(i)})$$

Formally, $P^{(i)}$ is given by

$$\max_{u, (e_j)_{j \leq i}} \pi_i(e_i) + \frac{\delta}{1 - \delta\lambda_i} (\sum_{j \leq i} q_j \pi_j(e_j) - c(u)),$$

where $IC_j^{(i)}$ holds for all $j \leq i$.

Since $P^{(i)}$ is a convex optimization problem, it has a unique solution, which we denote by $\langle u^{(i)}, (e_j^{(i)})_{j \leq i} \rangle$. We now derive several important properties of the solutions to the auxiliary problems; we will later use these properties to establish the optimality of PM.

Lemma 1. *The only binding constraint in the solution to $P^{(i)}$ is $IC_i^{(i)}$.*

The intuition for this lemma is as follows. Clearly, in an optimum, at least one constraint must be binding. Suppose that there is $j < i$ for which $IC_j^{(i)}$ is binding. As all the incentive compatibility constraints of $P^{(i)}$ are the same on the RHS, we get $e_j \geq e_i$. This implies that the marginal return on tasks of type j is strictly lower than the marginal return on tasks of type i. As task i is both better than j and available in the initial period, it is profitable to marginally increase e_i and decrease e_j in a manner that does not alter the agent's expected amount of effort.

By Lemma 1, the solution to $P^{(i)}$ can be obtained by maximizing the following Lagrangian function:

$$\max_{u,(e_j)_{j\leq i}} \pi_i(e_i) + \frac{\delta}{1-\delta\lambda_i}\left(\sum_{j\leq i} q_j\pi_j(e_j) - c(u)\right) - \mu_i\left(e_i - \frac{\delta}{1-\delta\lambda_i}(u - \sum_{j\leq i} q_j e_j)\right)$$

Lemma 2. *For all $j \in \{1,...,i\}$, $\pi_j'(e_j^{(i)}) \leq c'(u^{(i)})$ with equality if $e_j^{(i)} > 0$.*

This lemma, which follows directly from the FOCs of the Lagrangian, stipulates that the marginal cost of periodic compensation is equal to the marginal productivity of effort from every implemented task. This condition may seem more obvious than it actually is. The condition, would not hold, for example, if the auxiliary problem did not assume the availability of its best admissible task in the initial period, or if it offered compensation from the first rather than from the second period onwards.

The next lemma, which underpins the main result of this section, ranks the agent's periodic compensation in the different auxiliary problems.

Lemma 3. *The sequence $(u^{(1)}, u^{(2)}, \dots, u^{(I)})$ is strictly increasing.*

To see why this is so, suppose that $u^{(i+1)} \leq u^{(i)}$ and note that the combination of Lemma 2, the concavity of $\pi_j(\cdot)$, and the convexity of $c(\cdot)$ would then imply that $e_j^{(i+1)} \geq e_j^{(i)}$, for all $j \leq i$. Now, consider the continuation of the interaction in auxiliary problem $P^{(i+1)}$, which begins with the arrival of a task of type i. From the arrival of that task to the first arrival of a task of type $k \geq i+1$, the agent exerts weakly more effort on all tasks and receives a weakly lower periodic compensation, compared to the agent in the solution to $P^{(i)}$. By Lemma 1, none of the constraints $IC_j^{(i+1)}$, $j \leq i$, is binding in the solution to $P^{(i+1)}$. Therefore, a periodic compensation strictly lower than $u^{(i)}$ would suffice to incentivize (weakly) more effort than $\{e_j^{(i)}\}_{j=1}^{i}$ until the first arrival of a task of type $k \geq i+1$. This contradicts the optimality of the solution to $P^{(i)}$.

The combination of Lemmas 2 and 3, together with the concavity/convexity assumptions, delivers some simple but important comparison results *within* a given auxiliary problem and *across* different problems. It follows immediately that higher effort is exerted on better tasks within each auxiliary problem, and that lower effort is exerted on tasks of a particular type in the higher-indexed problem (one that starts with a better task) than on tasks of that same type in a lower-indexed problem. These results are the basis for the main qualitative properties of the unique optimal contract, which we define next.

Corollary 1. *Let $j \leq i$.*

1. For $j > 1$, $e_j^{(i)} \geq e_{j-1}^{(i)}$ with strict inequality whenever $e_j^{(i)} > 0$,

2. For $i < I$, $e_j^{(i)} \geq e_j^{(i+1)}$, with a strict inequality whenever $e_j^{(i)} > 0$.

Definition of the Phase Mechanism. — Denote by $\mathscr{I}(h_t)$ the index of the best task that has been available at least once in h_t (for the null history set $\mathscr{I}(\emptyset) = 0$). For every h_t and G^i, let $r(h_t, i) = e_i^{(max\{i, \mathscr{I}(h_t)\})}$ denote the required effort if G^i is played after h_t. After $h_t \neq \emptyset$ that is consistent with $r(\cdot, \cdot)$, the compensation under PM is $u = u^{(\mathscr{I}(h_t))}$; after all other histories, the compensation is zero.[3]

Even though task arrival is i.i.d. across periods, the terms of PM exhibit a seniority-based dynamics.[4] The relationship between the principal and the agent can be described using the metaphor of a ratchet that allows advancement only in the direction that favors the agent. That is, for any realization of task arrival, the periodic wage and effort exerted on every type of task are given by monotonic step functions. When the periodic wage or effort requirements are updated, they jump to a new level where they stay until the next stochastic event causes another jump.

Proposition 1. *PM is the unique optimal contract.*

To develop some intuition for the result consider the case of two possible types of tasks: the *low-productivity* task, $i = 1$, and the *high-productivity* task, $i = 2$. Suppose for a moment that the high-productivity task arrives in period 1. In this case, the solution to $P^{(2)}$ specifies an optimal incentive-compatible contract that is stationary over time.[5] Under this contract, the agent's expected payoff is zero and, as mentioned in Corollary 1, the effort exerted on a high-productivity task is *higher* than the effort exerted on a low-productivity task.

By construction, the periodic wage is identical in all periods. Therefore, while the agent's continuation utility is zero in periods in which a high-productivity task is available, his continuation utility is *strictly* positive whenever a low-productivity task is available. The

[3] A history h_t is consistent with $r(\cdot, \cdot)$ if, along that history, the agent's effort equals the effort specified by $r(\cdot, \cdot)$ in every period.

[4] Since we show that PM is optimal, this augments existing explanations for seniority, such as asymmetric information (Lazear 1981; Carmichael 1983; Milgrom and Roberts 1988), relationship-specific investment (Becker 1962; Parsons 1972), and collective bargaining constraints (Blair and Crawford 1984; Abraham and Farber 1988).

[5] In the proof we show that, on the path of play, the "within-phase" solutions are necessarily stationary (in the sense defined in the definition of the auxiliary problems) and, thus, the solution to $P^{(2)}$ corresponds to the optimal contract provided that the high-productivity task arrives first.

positive continuation utility in these periods serves as an efficient compensation method for the high effort exerted on previous high-productivity tasks.

Now, suppose that the interaction does not begin with the high-productivity task. We refer to the time interval prior to the first arrival of the high-productivity task as *phase 1* of the interaction. The stationary solution to $P^{(2)}$ is still an incentive-compatible contract (for the whole interaction), however, during phase 1 the agent *strictly* prefers not to deviate from the contract. Therefore, in this phase of the interaction, the principal can reduce the agent's periodic compensation and require more effort, without violating the agent's incentive compatibility constraints. In optimum the periodic wage is reduced and the effort requirement is increased until the incentive compatibility constraints in phase 1 are binding, while keeping the marginal productivity of effort equal to the marginal cost of compensation (the resulting wage and effort correspond to the solution to $P^{(1)}$).

As a result, in phase 1, the marginal productivity of effort and the marginal cost of compensation are lower than they are after the first arrival of the high-productivity task (*phase 2*). This leads to the key observation that any additional modification "between" phases is either unprofitable or infeasible. For example, it is not in the interest of the principal to incentivize additional effort in phase 1 via increasing compensation in phase 2. On the other hand, even though the principal would benefit from increasing compensation in phase 1 in return for a higher effort on future high-productivity tasks in phase 2, doing so is not feasible as the agent lacks commitment power.

It is worth noting that if the concavity and convexity assumptions are replaced with their weak versions, PM remains an optimal contract, albeit not necessarily the unique optimal contract. In the extreme case of a linear cost of compensation, in addition to PM, a trivial optimal contract exists where, upon observing the desired effort, the principal fully compensates the agent in the following period. While this contract seems natural in the case of a linear cost of compensation, it cannot be approximated as a limit of optimal contracts where the cost of compensation is strictly convex.

3 Separable Activities

We now return to our general environment. In this section, we develop and discuss extensively the notion of separable activities — unidimensional components of an interaction for which the players have opposed preferences and that satisfy certain separability conditions. Before we present our reduced-form definition, it is useful to motivate our modeling approach by an informal discussion and illustration.

A possible (non-reduced-form) way to model a particular activity A is to specify a simultaneous-move game G_A whose outcome determines the players' activity-related payoffs. The game G_A can be thought of as a ready-to-use "add-on" that contains all the relevant information about A that, in principle, could be added to any contracting environment.

The activity A is *separable* with respect to a given (dynamic) interaction if there are histories after which the players may face a periodic game that can be decomposed into G_A and some other game that corresponds to the rest of that period's interaction[6] such that (1) the periodic payoff of each player is the sum of his payoffs in the two games, and (2) the outcome of G_A does not affect the distribution of periodic games to be played in the future.

We illustrate this decomposition using the model of random opportunities from Section 2. Consider the game G^i (which is played when a task of type i is available). This game can be artificially decomposed into two "degenerate" games, G^i_e and G^i_u, which represent different activities: in the former, the agent chooses an effort level and the principal is inactive, while in the latter, the principal chooses a compensation level and the agent is inactive.

An outcome of G^i_e specifies an effort level e and the corresponding payoffs to the agent and principal, $-e$ and $\pi_i(e)$, from this particular component of G^i, namely, the *activity* of the agent's effort on a task of type i. This activity is part of the periodic interaction only in periods in which a task of type i is available. Similarly, the outcome of G^i_u determines a level of compensation u and the players' payoffs, u and $-c(u)$, generated by the activity of compensating the agent. This activity is identical for all games G^i because, by assumption, all aspects of compensation provision do not depend on the identity of the available task. We can therefore drop the index i from G^i_u and denote the activity of compensating the agent by G_u.

In addition to the players' payoffs from each possible level of periodic wage and effort, the games G_u and G^i_e specify what payoffs the players can obtain from deviations. In G_u, only the principal's action set contains more than one action. Since he is committed to a long-term strategy, the details of G_u are important only insofar as the principal's ex-ante

[6]This second game need not be identical for all occurrences of G_A.

considerations are concerned.

The situation is different in the game G_e^i, where the action set of the agent (the player who lacks commitment power) contains more than one element. When the periodic game G_e^i is considered in isolation, the action that gives the agent his *maximal* payoff is that of exerting zero effort. In an incentive-compatible contract, the agent must find it optimal to select the level of effort suggested by the contract, and the incentives for him to do so must come from other parts of the interaction. The agent's maximal activity-related payoff as a function of the intended level of the activity will play a key role in our characterization.

The agent's periodic wage and task-specific effort are among the simplest possible examples of separable activities. First, since G_u^i and G_e^i are single-player decision problems, the levels of these activities are unilaterally determined by one of the players. Second, every outcome of each game can be described as a level of the activity. However, many activities do not satisfy these two properties. Consider, for example, a variant of the model of random opportunities where, occasionally, there is an opportunity for joint production from "labor" (or effort) provided by the agent and "capital" provided by the principal. Assume that the production process works as follows. First, the principal provides the agent with the capital needed to meet the intended production level, and then the agent decides whether to supply the appropriate labor or to reallocate the capital to alternative uses for his private benefit.[7]

The game underlying this activity is a nondegenerate two-player game in which the agent's maximal activity-related payoff depends on the principal's action. Furthermore, a deviation by the agent creates a discrepancy between the capital provided by the principal and the capital used for production by the agent. Thus, a deviation induces an outcome that is very different in nature from outcomes that can be simply described as a possible production *level* when production is performed correctly. This demonstrates that a formal definition of an activity as a fully specified game must separate the outcomes of the proposed game into outcomes that correspond to *activity levels* (e.g., the good was produced) and outcomes that do not correspond to activity levels (e.g., the agent stole the capital).[8]

Decomposing the complex game played by the players after each history into a fully specified game that represents the activity and a component that corresponds to the rest of the interaction is cumbersome. The additional need to further define the activity within the isolated game makes such a direct approach even less appealing. However, as our sole

[7]This is exactly the production process described in Albuquerque and Hopenhayn (2004).

[8]In Online Appendix B we provide a non-reduced-form definition of an activity.

objective is to emphasize the dynamic monotonicity of an activity, we can take a "reduced-form" approach.[9] Our definition will specify an ordered set of activity levels (e.g., possible effort levels), the players' payoffs at every intended level, and the agent's maximal activity-related payoff at every intended level (e.g., exert no effort or enjoy the stolen capital). We will then impose certain separability conditions between the activity and the rest of the contracting environment.

3.1 Reduced-Form Approach: Definitions and Discussion

In this section we first define activities (and standardized activities) and demonstrate examples thereof. Then we define when an activity is convex and when it is separable with respect to a given contracting environment.

An *activity* $\langle L, u, \kappa, D\rangle$ consists of an interval $L \subset \mathbb{R}$, two continuously increasing functions $u, \kappa : L \to \mathbb{R}$, where $u(l)$ and $\kappa(l)$ are, respectively, the agent's utility and the principal's cost associated with the activity level $l \in L$, and a function $D : L \to \mathbb{R}$ that specifies the supremum of the agent's activity-related payoff at every $l \in L$.

In applications, it is often natural to use ad-hoc activity-specific units to measure an activity (e.g., quantity of good produced, duration of work, amount of compensation paid, etc.). To abstract away the activity-specific details, it is useful to reduce all of the activities to a common denominator. The advantage of doing so is analogous to that of transforming a general normal distribution into a standard normal distribution: seemingly unrelated activities can be considered and compared using the same units of measurement.

A *standardized activity* $\langle L, \kappa, D\rangle$ is an activity that is measured in terms of the agent's utility from the activity, that is, an activity for which[10] $u(l) = l$. In what follows, all general results will be stated in terms of standardized activities; when there is no risk of confusion, the qualifier "standardized" will be dropped.

For example, in the model of random opportunities, the agent's periodic wage is measured in agent's utils and hence it is an example of a standardized activity: the set of possible levels is $[0, \infty)$, $\kappa(\cdot)$ coincides with $c(\cdot)$, and $D(\cdot)$ is the identity function since there are no

[9] This approach is akin to the modeling strategies of Ray (2002) and Albuquerque and Hopenhayn (2004).

[10] Notice that every activity can be expressed in standardized form.

possible deviations for the agent within the activity. By contrast, the agent's task-specific effort is measured in the cost of effort and hence to standardize this activity one needs to look at the negative of effort. Thus, for task i in the standardized form, the set of possible levels is[11] $(-\infty, 0]$, $\kappa(l) = -\pi_i(-l)$, and as the most profitable deviation for the agent within the activity is to exert no effort, $D(\cdot) \equiv 0$. These cases, where the slope of $D(\cdot)$ is either zero or one turn out to be important bounds in our characterization.

The usefulness of our reduced-form approach is even more apparent in more complex activities that are jointly controlled by both players. For such activities, the agent's deviation payoffs depend not only on his actions, but also on the principal's, and our approach provides a concise way to capture the agent's incentives to deviate through the function $D(\cdot)$. We illustrate this using traditional complement/substitute production technologies.

Consider again the aforementioned production example where, first, the principal provides y units of capital to the agent, who then decides whether to combine the capital with x units of labor (to produce the good as intended), or to reallocate the capital to alternative uses for his private benefit. Assume that both the marginal benefit from reallocating capital and the marginal cost of labor equal one. The latter assumption implies that the standardized activity levels are measured by the negative of the agent's labor: $l(x) = -x$.

First, suppose that the production function is $x + \alpha y$, and that there is a production target of one unit. Then, the amount of capital provided by the principal as a function of labor is $y(x) = \frac{(1-x)}{\alpha}$, and it follows that $D(l) = \frac{1+l}{\alpha}$. In particular, the function $D(l)$ is *increasing*: an increase in l is equivalent to a decrease in x, which leads to an increase in y and a larger potential benefit from deviating. Whether the slope is above or below 1 depends on α. Alternatively, let the production function be $min\{x, \alpha y\}$. In this case, $y(x) = \frac{x}{\alpha}$, and $D(l) = -\frac{l}{\alpha}$ is a decreasing function.

We say that an activity is *convex* if the payoff frontier induced by the activity is convex.[12] For standardized activities, this means that $\kappa(l)$ is a strictly convex function. Note that the activities studied in the model of random opportunities are convex.

[11] Recall that $\kappa(\cdot)$ is the principal's cost, while $\pi_i(\cdot)$ is his profit.

[12] Since the players' have opposite preferences for the activity, every level of the activity is associated with a different point on the frontier.

We say that an activity is *separable* with respect to a contracting environment $f(\cdot)$ if:

(1) For every two histories h_t and $\hat{h}_t$ that differ only in the selected levels of the activity, $f(h_t) = f(\hat{h}_t)$.

(2) In periods in which the activity is available, the action set of each player can be written as a cross product of the actions related to the activity and the actions not related to the activity.

(3) The payoff from the activity-related component is additively separable from all other components of the interaction.

Note that the definition is unit-free, and so it applies equally to standardized and non-standardized activities. Thus, one does not need to express an activity in its standardized form in order to check its separability.

The first two conditions state that altering the level of the activity in a given period has no impact on the environmental terms of the interaction: the first condition states that changing the level of the activity does not affect the stochastic process of future periodic games, and the second condition states that changing the level of the activity does not force a player to change his actions related to other components of the interaction. These conditions, together with the condition that payoffs are additively separable (condition 3), enable us to analyze the optimal use of a separable activity while imposing minimal structure on $f(\cdot)$. Such an approach allows us to derive qualitative results on the activity regardless of the exact details of the contractual environment in which the activity occurs.

In the model of random opportunities the agent's periodic wage and effort on every type of task are separable activities. However, in that model the availability of activities is i.i.d., and hence the degree of separability between the activities and the environment is considerably stronger than that which satisfies our separability conditions. While the possibility of affecting the future environment through the selection of specific levels of the separable activity is ruled out, the separability conditions are compatible with contracting environments in which the availability of the separable activity is determined endogenously by past actions or affected by previous realizations of the periodic games.

4 Main Result

We define the following condition on a *standardized* activity (L, κ, D).

Condition $\mathcal{D}$.

$$\textit{For all } l' < l'' \in L, \quad 0 \leq \frac{D(l'') - D(l')}{l'' - l'} \leq 1.$$

The condition, which we state in terms of the slope of the function $D(\cdot)$, has an alternative intuitive representation: Condition $\mathcal{D}$ is satisfied if and only if the agent's maximal attainable payoff, $D(l)$, is a (weakly) increasing function and his *incentive to deviate*, $[D(l) - l]$, is a (weakly) decreasing function. The discussion in Section 3.1 offered examples of activities that satisfy Condition $\mathcal{D}$ and of activities that do not satisfy it (either the right or left inequality thereof). Note that the activities in the model of random opportunities satisfy Condition $\mathcal{D}$ on the boundaries, namely the slope $D'(l) \equiv 1$ (for wage) and the slope $D'(l) \equiv 0$ (for effort).[13]

In many cases, it is not necessary to express an activity in its standardized form in order to verify Condition $\mathcal{D}$. A convenient way to think of the condition for a nonstandardized activity is as follows. Consider two possible levels l^1 and l^2 of a given activity. Two magnitudes need to be compared: (i) the difference between the agent's payoffs from activity levels l^1 and l^2, and (ii) the difference between the agent's maximal attainable payoffs when the activity is set at levels l^1 and l^2. The condition is satisfied if, *for any two levels*, the values of (i) and (ii) do not have opposing signs, and the *absolute* value of (i) is larger than that of (ii). This alternate expression shows directly that the agent's (nonstandardized) effort on every type of task in the model of random opportunities satisfies Condition $\mathcal{D}$.

Specifications that satisfy Condition $\mathcal{D}$ have been used in the past to derive monotonicity results. The activity of wage/transfer that is the focus of the wage dynamics literature (e.g., Holmström 1983) and of the risk-sharing literature (e.g., Marcet and Marimon 1992) is unilaterally controlled by the principal and thus, as we mentioned before, it satisfies the condition. Other papers that study more complex activities make explicit assumptions in the spirit of Condition $\mathcal{D}$. Albuquerque and Hopenhayn (2004) study entrepreneur financing and impose a bound on the change in the liquidation value of the firm (to the entrepreneur) due to an increase in the firm's capital. In a more abstract setting, where a deterministic

[13] The property that $D'(l) = 1$ ($D'(l) = 0$) is common to all activities that are unilaterally controlled by the principal (agent).

contracting problem is faced repeatedly and the principal can commit only to the current period's actions, Ray (2002) requires that increasing the transfer the agent receives have a greater impact on his utility if he stays on the job than if he quits.

We now turn to our main result. Fix a contracting environment and an incentive-compatible contract therein. We say that an activity A is *nondecreasing* under the contract if, from the ex-ante perspective, there is a zero probability of observing two periods $t_1 < t_2$ in which A is available and the level of A is strictly higher in t_1 than in t_2.

Theorem 1. *Let A be a standardized convex activity that is separable with respect to $f(\cdot)$. If A satisfies Condition $\mathcal{D}$, then A is nondecreasing under any optimal contract.*

The powerful implications of Theorem 1 can be illustrated already in the environment of our simple model of random opportunities. The theorem significantly generalizes the key qualitative results that we obtain under arguably restrictive assumptions by solving directly for the optimal contract. In particular, even if the arrival of tasks were not i.i.d. and the tasks could not be ranked according to the marginal productivity of effort over time, the task-specific effort would still decrease while the wage would still increase. Furthermore, this would hold even if the players were able to affect the arrival of tasks endogenously. This result would be all but impossible to derive directly by fully characterizing the optimal contract in such complex environments.

More generally, Theorem 1 unifies and generalizes a large class of results that show that an employee's wage rises over time when there are fluctuations in the value of his outside option. See, for example, Harris and Holmström (1982), Holmström (1983), Postal-Vinay and Robin (2002a, 2002b), and Burdett and Coles (2003) for exogenous fluctuations, and Shi (2009) for endogenous ones.[14] To derive their results, these papers specify a full-blown model of the labor market that embeds fluctuations in the worker's outside option, and then use the specific structure to obtain the downward rigidity of wage directly. This standard approach is inherently limited as it derives the downward rigidity of wage only for the particular, and oftentimes narrow, specification under consideration. By contrast, our result establishes the downward rigidity of wage for any environment where the separability

[14]Although many of these papers study general equilibrium models, their settings can be easily adapted to our framework. Competition over the worker can be incorporated into the model by assuming that occasionally the agent has a "quit" action that: 1) provides him with an immediate payoff equal to the value of his outside offer and 2) after this action is used both players have a payoff of zero in all future games.

conditions hold. Therefore, fluctuations in the agent's outside option (or business conditions in general) generically lead to increasing wages over time.

Finally, the generality of the framework developed in this paper enables us to draw connections between seemingly unrelated monotonicity results. For example, Marcet and Marimon (1992), Krueger and Uhlig (2006), and Grochulski and Zhang (2011) study dynamic risk sharing, Forand and Zápal (2019) study dynamic project selection, and the papers mentioned in the previous paragraph study wage dynamics. The object of interest in each of these papers is a convex separable activity[15] that satisfies Condition $\mathcal{D}$, and hence Theorem 1 delivers the qualitative monotonicity results derived directly in all of these papers.

Intuition for Theorem 1. — We now use a simple example to illustrate the intuition for Theorem 1 in an informal way. Consider a four-period contracting problem where the agent "works" in periods 1 and 3, and the principal compensates him in periods 2 and 4. The agent can quit the contract at any time, his outside option is zero, and the players do not discount the future.

Let x_1 and x_3 be the (exogenously given) effort levels the agent must provide in periods 1 and 3, respectively. Assume first that compensation is provided via the convex activity of paying a "periodic wage" (as in Section 2). The cheapest way to compensate the agent for his total effort $(x_1 + x_3)$ is to pay him half of that amount in periods 2 and 4. If $x_1 \geq x_3$, this compensation plan satisfies all incentive compatibility constraints and is thus uniquely optimal. However, if $x_1 < x_3$, this form of compensation is not incentive compatible in period 3. To restore incentive compatibility (without increasing the total compensation), the principal must shift some of the compensation from period 2 to period 4. A decreasing compensation plan (where the wage in period 4 is strictly lower than that in period 2) is suboptimal for all x_1 and x_3. An alternative way to frame the argument is as follows. Suppose that a decreasing compensation plan is proposed. Reducing the compensation in period 2 by a small amount and increasing it in period 4 so that the total remains the same decreases the overall cost of compensation (this is a basic smoothing argument). If the original compensation plan was part of an incentive-compatible contract, then, a fortiori, so is the modified compensation plan: back-loading compensation can only *relax* some of

[15] In Forand and Zápal (2019) projects can be thought of as *weakly* convex separable activities (that is, activities where $\kappa(\cdot)$ is only weakly convex) that satisfy Condition $\mathcal{D}$. Therefore, a small adaptation of Theorem 1 to weakly convex cost functions would imply that there exists an optimal contract in which each project is used monotonically over time and all shifts are in the direction that favors the agent. The strict convexity of $\kappa(\cdot)$ is only required in order to establish that this occurs under every optimal contract.

the incentive compatibility constraints of the forward-looking agent. This simple argument relies on the implicit assumption that changing the level of compensation in a given period does not create new deviation opportunities for the agent. Formally, this is reflected in the property $D'(l) = 1$, since this form of compensation is unilaterally controlled by the principal.

Assume now that compensation in our example is provided via a more complex convex activity under which the agent's deviation options vary with the level of compensation. Again, start with a decreasing and incentive-compatible compensation plan, and consider the same smoothing modification as before. What is now unclear is whether the modification leads to an incentive-compatible contract. Condition $\mathcal{D}$ guarantees that this is indeed the case. To see this, we now analyze the agent's considerations in periods 2 and 4.

Period 2: The compensation in this period is lowered. By Condition $\mathcal{D}$, the function $D(\cdot)$ is nondecreasing, and so decreasing this period's compensation does not increase the agent's maximal payoff in case of deviation. The compensation schedule is modified such that the agent's continuation payoff from following the contract does not change. Thus, it remains optimal for the agent to adhere to the terms of the (modified) contract in period 2.

Period 4: The compensation in this period is increased, and so the agent's payoff from following the contract is now higher. The other part of Condition $\mathcal{D}$ states that the slope of $D(\cdot)$ is no greater than one. This implies that the increase in the agent's deviation payoff in period 4 is bounded by the increase in his payoff from following the contract in that period. Hence, the modified contract is incentive compatible in period 4 as well.

Tightness of Condition $\mathcal{D}$

The characterization in Theorem 1 is tight in the sense that, whenever Condition $\mathcal{D}$ is violated, there exist contracting environments in which the level of the activity shifts in the principal's favor over time. Condition $\mathcal{D}$ stipulates that a small change in the level of the activity does not have a large impact on his deviation payoff (since the slope of $D(\cdot)$ is bounded). A natural class of activities that violate Condition $\mathcal{D}$ are those for which $D(\cdot)$ is not continuous. For example, consider the joint production example described above, and assume that the principal can assign capital only in discrete increments. Due to the usual potential nonexistence problems, a general treatment of activities with discontinuous $D(\cdot)$ involves technicalities and notation that do not add substance. Therefore, in the main text we avoid this problem and consider the case where $D(\cdot)$ is continuous. In Online Appendix B we generalize this result and show that if $D(\cdot)$ is discontinuous for some activity, there exist contracting problems in which that activity is decreasing under all approximately optimal contracts.[16]

To state the converse to Theorem 1, we need the following definition. Fix a contracting environment and an incentive-compatible contract therein. We say that an activity A is *decreasing* under the contract if, from an ex-ante perspective, there is a probability of one of observing two periods $t_1 < t_2$ in which A is available and the level of A is strictly higher in t_1 than in t_2.

Theorem 2. *Let A be a standardized convex activity with a continuous $D(\cdot)$ that violates Condition $\mathcal{D}$. There exists a contracting environment with respect to which A is separable and in which A is decreasing under any optimal contract.*

To better understand this result we revisit the joint production activity discussed in Section 3. For this activity the agent's incentive to deviate depends on the amount of capital he is entrusted with. However, different production functions lead to different relations between the capital under the agent's control and the standardized level of the activity, and hence to different slopes of $D(\cdot)$. We consider two different production functions—one for each type of violation of Condition $\mathcal{D}$—and, for each function, construct a counterexample by choosing appropriate compensation opportunities. In both examples, in addition to assuming the periodic incentive compatibility constraints, we assume that the agent can reject the contract at time zero, and that the players do not discount the future.

Pay at the End. — For the case where $D(\cdot)$ is decreasing, suppose that the production function is $\min\{x, \frac{y}{2}\}$, where, as before, x is the agent's effort and y is the capital provided by the principal, and assume that there are production opportunities in periods 1 and 2. To further simplify the example, suppose that the principal has very limited discretion on how to provide compensation to the agent: the principal can only decide whether or not to pay the agent a compensation of 1 at the end of the interaction (period 3). We can express the capital provided by the principal as a function of the agent's effort by $y(x) = 2x$. Assuming that the marginal cost of effort and the marginal utility from reallocating capital is 1, we obtain a standardized form where $L = (-\infty, 0]$ and $D(l) = -2l$.

[16] We use the standard notion of ϵ-optimality. A formal definition is given in the appendix.

As the main focus of the present illustration is on the agent's incentives, we prefer not to add details on how to directly turn the activity into a convex one. Instead, we simply assume that the principal seeks to maximize the agent's *aggregate* effort and, among the profiles of the agent's effort (x_1, x_2) that add up to the same total, the principal prefers the one with the minimal difference between x_1 and x_2. Under this assumption, the agent's effort under the optimal contract can be identified by solving the following linear programming problem:

$max \quad \{x_1 + x_2\}$ such that:

$(IC_0) \quad 1 - x_1 - x_2 \geq 0;$

$(IC_1) \quad 1 - x_1 - x_2 \geq 2 \cdot x_1;$

$(IC_2) \quad 1 - x_2 \geq 2 \cdot x_2,$

where IC_t is written such that the LHS corresponds to the agent's payoff from following the (continuation of) the contract from period t onward, and the RHS corresponds to the agent's payoff from (optimally) deviating in period t. The unique solution to this LP problem is $(x_1 = \frac{2}{9}, x_2 = \frac{1}{3})$. Therefore, the optimal contract has an *increasing* effort schedule.

To understand the intuition for the example, observe that the agent's continuation utility at the end of the second period is 1, while his continuation utility at the end of the first period is $1 - x_2$. Therefore, the *loss* of continuation utility from deviating is greater in the second period than in the first period, which, in turn, implies that the activity-related incentive to deviate $(D(l) - l)$ can be greater in the second period than in the first period. Under the assumed production technology, this observation leads to an increasing effort schedule.

Carrot and Stick. — For the case where the slope of $D(\cdot)$ is greater than one, consider the production function $x + \frac{y}{2}$ and suppose that there is an "output target" of 1 that must be fulfilled in each of periods 1 and 3. As in the previous example, the principal can provide a compensation of 1 at the end of the interaction (period 4), but, in addition, in period 2 he can either offer a compensation of 1/2 (carrot) or impose a fine of 1 (stick). Under the assumed production technology, the principal's capital as a function of the agent's effort is $y(x) = 2(1 - x)$. Maintaining the same assumptions on the marginal cost of effort and reallocated capital, we obtain a standardized form where $L = [-1, 0]$ and $D(l) = 2 + 2l$.

In this environment the *only* incentive-compatible contract has $x_1 = \frac{1}{2}$ and $x_3 = 1$, and hence the agent's effort *increases* over time under the optimal contract. To see this, note that the agent's period-3 continuation payoff from following the contract is $1 - x_3$, and his payoff from (optimally) deviating in period 3 is $D(l(x_3)) = D(-x_3) = 2(1-x_3)$. Hence, a contract is incentive compatible only if $x_3 = 1$. The agent's incentive compatibility constraint in period 1 and his participation constraint are given by $1 + \frac{1}{2} - x_1 - x_3 \geq 2(1-x_1) - 1$ and $\frac{1}{2} + 1 \geq x_1 + x_3$, respectively. These constraints, together with $x_3 = 1$, jointly imply that $x_1 = \frac{1}{2}$.

5 Robustness

Our main results show that when an activity does not satisfy Condition $\mathcal{D}$ there exist contracting environments where, over time, the activity shifts in the direction that favors the principal. A natural question that arises is whether activities that "almost" satisfy Condition $\mathcal{D}$ can admit arbitrary fluctuations, or whether only small changes in the principal's favor can be observed. It turns out that violations of the upper and lower bounds of Condition $\mathcal{D}$ have an asymmetric impact on the possible dynamics of a separable activity. In particular, if for a standardized separable activity the slope of $D(\cdot)$ is negative, but close to zero, only small decreases in the activity can be observed under an optimal contract. By contrast, if the slope of $D(\cdot)$ exceeds one, there exist interactions in which large decreases in the level of the activity are observed.

For the robustness result, we need the following definition. Fix a contracting environment and an incentive-compatible contract therein. We say that an activity A is *ϵ-nondecreasing* under the contract if, from the ex-ante perspective, there is a zero probability of observing two periods $t_1 < t_2$ in which A is available and the level of A is higher in t_1 than in t_2 by more than ϵ.

Proposition 2. *Fix a compact interval $\hat{L}$ and an increasing convex function $\hat{\kappa} : \hat{L} \to \mathbb{R}$. There exists a function $\epsilon : \mathbb{R}_+ \to \mathbb{R}_+$ for which $\lim_{c\to 0} \epsilon(c) = 0$ such that if for all $l' < l'' \in \hat{L}$, $\frac{D(l'')-D(l')}{l''-l'} \in [-c, 1]$ and $(\hat{L}, \hat{\kappa}, D)$ is separable with respect to $f(\cdot)$, then the activity $(\hat{L}, \hat{\kappa}, D)$ is $\epsilon(c)$-nondecreasing under any optimal contract.*

To illustrate the intuition behind this result, fix $\hat{L} = [0, 1]$ and $\hat{\kappa}(l) = l^2/2$, and consider a parametrized family of functions $D_c(l) = 1 + c - cl$, for $c > 0$. This specification is convenient as $\frac{D_c(l'')-D_c(l')}{l''-l'} = -c$ for all $l'' > l' \in \hat{L}$. Furthermore, assume that the players do not discount the future and that the activity is available in periods 1 and 2.

Assume that, under an optimal contract, $l_1 > l_2$ where l_t is the level of the standardized activity $(\hat{L}, \hat{\kappa}, D_c)$ in period t. By a standard smoothing argument, decreasing the level of the activity in period 1 by a small ε and increasing it in period 2 by the same ε is profitable for the principal. Therefore, our assumption that $l_1 > l_2$ is part of an optimal contract implies that the aforementioned modification is not incentive compatible. Since the slope of $D_c(\cdot)$ is below 1, increasing the level of the activity in period 2 cannot create any incentive compatibility problems in that period. Therefore, the above modification must violate the incentive compatibility constraint in period 1.

Reducing the activity by ε in period 1 has two effects. First, it reduces the agent's periodic utility (on the path of play) by ε; second, since $D_c'(\cdot) = -c$, it *increases* the agent's payoff from deviating in period 1 by εc. Thus, increasing the level of the activity in period 2 by $(1 + c)\varepsilon$ restores incentive compatibility. We refer to the modification where l_1 is decreased by ε and l_2 is increased by $(1+c)\varepsilon$ as an *ε-modification*. The principal's marginal profit from an ε-modification is $\hat{\kappa}'(l_1) - \hat{\kappa}'(l_2)(1+c)$. Since $\hat{\kappa}'(l) = l$ and $l_t \leq 1$ the marginal profit from the ε-modification is *positive* whenever $l_1 - l_2 > c$. Therefore, $l_1 > l_2$ can be consistent with optimality only if $l_1 - l_2 \leq c$.

While the above example has a very specific structure, the main part of the argument—by which smoothing a decrease in the (standardized) activity can violate the incentive compatibility constraint only in the *earlier* period—is general. This plays a fundamental role in Proposition 2, since it guarantees that the principal can always restore incentive compatibility by increasing the level of the activity in the future. This, in turn, enables us to put an upper bound on the size of a decrease in the activity under an optimal contract by comparing the marginal gain from smoothing with the marginal cost of increasing the aggregate (discounted) level of the activity.

By contrast, when the slope of $D(\cdot)$ is greater than one, smoothing a decrease in the level of the activity between two periods can violate the incentive compatibility constraint only in the *later* period. Consequently, smoothing a decrease in the (standardized) activity may require the principal to increase the agent's continuation utility from the non-activity part of the interaction; however, this requirement may be impossible in some contracting environments.

Proposition 3. *Let A be a (standardized) convex activity with a continuous $D(\cdot)$, and assume that there exists an interval $\tilde{L} \subset L$ and $c > 0$ such that $\frac{D(l'')-D(l')}{l''-l'} \geq 1 + c$ for all $l' < l'' \in \tilde{L}$. There exists a contracting environment with respect to which A is separable in which, over time, the level of the activity decreases by $|\tilde{L}|$ in all optimal contracts.*

6 Additional Examples of Potential Applications

In this section we briefly offer several additional economic settings into which our results may provide insights.

Quality Provision over Time. — Consider a dynamic interaction between a supplier and a client. The quality of the supplied goods may be an important component of the interaction and an interesting question is whether some general qualitative properties can be identified even without fully specifying the details of the contracting environment.

In some cases, the supplier has commitment power and quality provision constitutes a convex separable activity.[17] Since the activity is unilaterally controlled by the player with commitment power, Condition $\mathcal{D}$ holds and, by Theorem 1, quality increases over time. Furthermore, if the quality distribution is subject to supply shocks, the same monotonicity result would hold conditional on each realization of the supply shock.

Power Allocation in Organizations. — Within large organizations it is well known that incentives are often provided via the reallocation of power rather than via monetary transfers (e.g., Cyert and March 1963; Aghion and Tirole 1997; Li, Matouschek, and Powell 2017). "Excess power," i.e., the power a division manager has beyond what is required for him to perform his job, can sometimes be represented as a separable activity, and, if such power is allocated efficiently, it can be represented as a convex separable activity.

Our result shows that the evolution of a division manager's power is inherently related to his potential benefit from abusing his power. If the potential for abuse of power is low, then increasing the manager's power should have a small impact on his incentive to deviate and Condition $\mathcal{D}$ is likely to hold. Thus, Theorem 1 implies that the manager's power will increase over time. If, on the other hand, the potential for abuse of power is high, the dynamics of power need not be monotone and will depend on the details of the environment.

[17] For example, if the supplier has access to a competitive market where he can sell the goods he does not sell to client, and the market's valuation for quality is higher than the client's.

Foreign Investments. — An additional setting into which our result can provide general insights is that of foreign direct investment or entrepreneur financing à la Thomas and Worrall (1994) and Albuquerque and Hopenhayn (2004). In this setting an entrepreneur must borrow funds from a lender to operate a production facility in an environment with demand shocks. Moreover, the entrepreneur cannot be compelled to repay the loan and if he defaults he may be able to keep some of the working funds he received in that period that are needed to finance the periodic production.

The level of production/size of loan (conditional on the demand shock) constitutes a convex separable activity if the entrepreneur's profit is concave in the level of capital invested and the periodic profit satisfies the separability conditions. If the entrepreneur's payoff from defaulting satisfies Condition $\mathcal{D}$, our result implies that the level of production will increase over time regardless of the exact specification of the interaction.[18]

A similar monotonicity result is derived in Albuquerque and Hopenhayn (2004); however, there are important aspects in which their result differs from ours. They assume that at the start of the interaction the entrepreneur owes a large debt to the lender and hence has a low continuation utility. Since the entrepreneur's deviation payoff is increasing in the size of the periodic loan, the entrepreneur's continuation utility limits the size of the periodic loan he can receive. As time goes by, the initial debt is repaid and the entrepreneur's continuation utility increases, and hence larger, and more efficient, periodic loans become incentive compatible. Due to this additional structure, Albuquerque and Hopenhayn (2004) can impose weaker assumptions than ours. Namely, they assume that the (standardized) activity is only quasiconvex and impose fewer restrictions on the entrepreneur's deviation payoff than the ones imposed by Condition $\mathcal{D}$.

7 Literature Review

The main motivation for this this paper is to understand the impact of environmental fluctuations in general contracting environments. Thus, this paper is related to papers that have analyzed the impact of such fluctuations in a specific environment. Möbius (2001), Hauser and Hopenhayn (2008), and Samuelson and Stacchetti (2017) study games where each player occasionally has an opportunity to grant a favor to his counterpart (at a cost to

[18]Condition $\mathcal{D}$ will hold if, for example, the marginal value of increasing production is greater than the marginal value of stealing the capital.

himself) and find that efficient equilibria have a cyclical element and hence are nonmonotone. Bird and Frug (2019) and Forand and Zápal (2019) take a mechanism design approach to (different) variants of the model of random opportunities. Forand and Zápal (2019) allow for arbitrary arrival processes and assume that information is symmetric; they derive directly a particular manifestation of our main result. By contrast, Bird and Frug (2019) assume arrival is i.i.d., but that task availability is *privately* observed by the agent; they find that even though all components of the interaction are (weakly) convex separable activities, the agent's task-specific effort may increase (compensation may decrease) over time under the unique optimal contract. Intuitively, when the agent privately observes task availability, his continuation utility must decrease if no tasks are implemented for a long time in order to incentivize him to reveal available tasks.

This paper derives a general monotonicity result that arises under optimal contracts in a wide range of economic settings. Certain manifestations of this result have been established directly in many specific environments that have been mentioned throughout the paper. Harris and Holmström (1982), Holmström (1983), Postal-Vinay and Robin (2002a, 2002b), Burdett and Coles (2003), and Shi (2009) analyze labor markets and establish that an employee's wage does not decrease over time. Marcet and Marimon (1992), Krueger and Uhlig (2006), and Grochulski and Zhang (2011) study dynamic risk sharing and show that the transfer received by the insured does not decrease over time. Albuquerque and Hopenhayn (2004) show that, conditional on the demand shock, an entrepreneur's access to capital and his level of production do not decrease over time. Forand and Zápal (2019) prove that in a model of random project arrival, once the agent is allowed to pursue a project he desires, he is allowed to do so indefinitely. Our work generalizes and (with the exception of Albuquerque and Hopenhayn 2004) unifies the monotonicity results derived in these settings.

Ray (2002) studies a repeated (deterministic) contracting problem where the principal can commit only to his actions in the present period, and derives a monotonicity result that is reminiscent of our own. In particular, he shows that when the agent's utility is quasilinear, the principal's limited commitment power leads to the periodic contract converging to the agent's preferred self-enforcing contract. Moreover, he shows that the agent's continuation utility increases monotonically until this contract is offered. To identify this qualitative impact of limited commitment power, Ray imposes an assumption that is similar to our Condition $\mathcal{D}$. This similarity suggests that conditions of this type may have an even deeper connection to the monotonicity of optimal contracts.

8 Concluding Remarks

Asymmetric Information. — The methodology developed in this paper can be extended beyond symmetric information environments and used to derive monotonicity results in contracting environments endowed with certain forms of asymmetric information. Consider, for example, a variant of our model of random opportunities, where the agent may have multiple types that differ in the cost of effort. Specifically, assume that $\theta \cdot e$ is the cost of effort level e for an agent of type $\theta \in \Theta \subset \mathbb{R}_+$, and that the value θ is privately known to the agent at the beginning of the interaction (and is constant over time).

This modification introduces a screening element into the principal's problem. Since the arrival of tasks is stochastic and the agent can quit at any time, finding the optimal intertemporal allocation of information rents is a complex problem. Nevertheless, we can establish that the optimal (task-specific) effort of every agent type is nonincreasing over time.

In this screening problem there are two possible reasons for offering a contract with an increasing effort schedule to a certain type of agent. First, it may maximize the principal's profit from the interaction with that agent type. Second, it may reduce the cost of providing information rents to other agent types. From Theorem 1, we know that an increasing effort schedule cannot be justified by the first reason. Next, we consider the impact of front-loading effort on the cost of providing information rents.

Incentive compatibility in this environment requires that, given a "menu of contracts," every agent-type (weakly) prefer accepting (and never quitting) the contract intended for him to selecting a contract intended for another type and (perhaps) quitting that contract during the interaction. Notice that front-loading effort (weakly) reduces the agent's payoff from quitting a contract at any point in time. Hence, it does not increase the cost of information rents. Since offering an increasing effort schedule cannot be justified by the second reason either, it follows that, in optimum, the principal will only offer contracts with a nonincreasing effort schedule to all agent types.

Unconnected Support for Activity Levels. — Our definition of an activity requires that the set of possible activity levels be a real-valued interval. This assumption, which may seem like a mere simplification, is, in fact, necessary for our main result. Consider an infinite interaction with a discount factor of $\delta = \frac{1}{2}$ and assume that in each period the principal may provide compensation worth 2 utils to the agent. In each of the first two periods, the agent can exert an effort of $e \in \{0, 1, 2\}$ on a task and thus generate a profit to the principal that is concave in e. Therefore, effort is a convex separable activity with a discrete support.

Requiring high effort ($e = 2$) in period 1 and low effort ($e = 1$) in period 2 in return for the maximal compensation is not incentive compatible: the agent's discounted utility in period 1 is $\frac{2}{1-\delta} - 2 - \delta = \frac{3}{2}$, while his utility from exerting no effort is 2. Thus, requiring high effort in both periods is also not a viable option. However, requiring low effort in period 1 and high effort in period 2 is incentive compatible: in both periods the agent's discounted continuation payoff is 2, which is exactly his optimal deviation payoff. Therefore, under the optimal contract, the agent's effort increases over time, even though Condition $\mathcal{D}$ holds.

Multiple Activities. — The main theme of this paper is to analyze the dynamic use of a *single* convex separable activity. However, in many contracting problems there exist multiple such activities (e.g., the model of random opportunities). A closer look at the logic behind the proof of Theorem 1 reveals that our analysis is also informative about the dynamics of multiple convex separable activities that satisfy Condition $\mathcal{D}$. First, it shows that the marginal cost that the principal incurs from an activity today is (weakly) less than the marginal cost he will incur from any activity in the future. Thus, observing the level of a single activity today establishes a lower bound on the chosen level of *any* activity in the future. Second, it can be shown that, within a given period, the marginal cost of two convex separable activities can be different (as occurs in PM at each phase transition) only if the slope of $D(\cdot)$ for the activity with the lower marginal cost is greater than the slope of $D(\cdot)$ for the activity with the higher marginal cost.

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Appendix A - Proofs

Proof of Lemma 1. We start with two simple observations. First, at least one constraint must be binding, as otherwise slightly reducing u increases the value of the problem without violating any constraint. Second, all tasks are assumed potentially profitable ($\pi'_j(0) > c'(0)$); thus, for at least one task $e_j > 0$.

Assume by way of contradiction that $IC_j^{(i)}$ is binding, for some $j < i$. This implies that $e_j > 0$ and that $e_j \geq e_i$. Consider the following modification for $\epsilon > 0$. Decrease e_j by $\epsilon \frac{1-\delta\lambda_i}{\delta q_j}$ and increase e_i by $\epsilon \frac{1-\delta\lambda_i}{1-\delta\lambda_i+\delta\lambda_i}$. It is straightforward to verify that this modification does not violate any of the constraints of $P^{(i)}$.

The first-order effect of this modification on the value of the problem is

$$\begin{aligned}
&\epsilon \left(\pi'_i(e_i) \frac{1-\delta\lambda_i}{1+\delta(q_i-\lambda_i)}(1+\frac{\delta}{1-\delta\lambda_i q_i}) - \pi'_j(e_j) \frac{1-\delta\lambda_i}{\delta q_j} \frac{\delta q_j}{1-\delta\lambda_i} \right) \\
&= \epsilon \left(\pi'_i(e_i) - \pi'_j(e_j) \right).
\end{aligned}$$

As $e_j \geq e_i$ the ranking of the tasks implies that this effect is positive; thus, for a small enough ϵ this modification increases the value of the problem. $\square$

Proof of Lemma 2. As the objective is concave, the FOCs are necessary and sufficient for optimality. The lemma is obtained by rearranging the FOCs. $\square$

Proof of Lemma 3. Suppose that $u^{(i+1)} \leq u^{(i)}$. From the convexity of $c(\cdot)$ it follows that $c'(u^{(i+1)}) \leq c'(u^{(i)})$. Thus, for all $j \leq i$, Lemma 2 and the concavity of $\pi_j(\cdot)$ imply that $e_j^{(i+1)} \geq e_j^{(i)}$. The binding constraint of $P^{(i)}$ can be written as

$$e_i^{(i)}(1-\delta\lambda_i) = \delta(u^{(i)} - \sum_{k \leq i} q_k \cdot e_k^{(i)}).$$

Since $\lambda_{i+1} = \lambda_i + q_{i+1}$ we can similarly rewrite the binding constraint of $P^{(i+1)}$ as

$$e_{i+1}^{(i+1)}(1-\delta\lambda_i) - \delta q_{i+1} \cdot e_{i+1}^{(i+1)} = \delta(u^{(i+1)} - \sum_{k \leq i} q_k \cdot e_k^{(i+1)} - q_{i+1} \cdot e_{i+1}^{(i+1)}).$$

Adding $\delta q_{i+1} \cdot e_{i+1}^{(i+1)}$ to both sides, using the expression for the binding constraint of $P^{(i)}$

and the consequences of our assumption at the beginning of the proof, we get

$$e_{i+1}^{(i+1)}(1-\delta\lambda_i) \leq \delta(u^{(i)} - \sum_{k\leq i} q_k \cdot e_k^{(i)}) = e_i^{(i)}(1-\delta\lambda_i),$$

which implies that $e_{i+1}^{(i+1)} \leq e_i^{(i)}$. Since $e_i^{(i)} \leq e_i^{(i+1)}$, it follows that $e_{i+1}^{(i+1)} \leq e_i^{(i+1)}$. However, as $\pi'_{i+1}(e) > \pi'_i(e)$ the only way this can occur without violating Lemma 2 is if $e_{i+1}^{(i+1)} = e_i^{(i+1)} = 0$. This, in turn, implies that $e_j^{(i+1)} = 0$ for all $j < i$, a solution that is not optimal due to our assumption that $\pi'_i(0) > c'(0)$. □

Proof of Proposition 1. We start by showing that the "stationarity" imposed in point (3) of the definition of the auxiliary problem $P^{(i)}$ is a property of the optimal solution to the relaxed version of $P^{(i)}$ in which the requirement in point (3) is removed. The relaxed problem (i.e., choosing an incentive compatible, not necessarily stationary contract) is a convex maximization problem in which the objective function is separable in all arguments. Thus, if the solution to the original auxiliary problem $P^{(i)}$ is suboptimal in the relaxed problem, there exists an improvement of the following simple form: the required effort on one specific task is modified at one particular history, and the compensation offered in the following period is set at the lowest level under which all incentive compatibility constraints are satisfied. By Lemma 2 the marginal benefit from every task equals the marginal cost of compensation. Since the cost of compensation is convex and the productivity of effort is concave, every such modification will (strictly) reduce the total expected value for the principal. Therefore, the solution to $P^{(i)}$ is the unique solution to the relaxed problem.

Denote by C_0 the class of all incentive-compatible contracts for which, whenever a task that is better than all previously available tasks arrives, the agent's continuation utility is zero. It is immediate that PM is the unique optimal contract in C_0. To see this, notice that the restriction to contracts in C_0 implies that it is sufficient to show that PM attains the highest expected value between any two (subsequent) earliest arrivals of tasks that are superior to all previously available ones. But this follows directly from the argument in the previous paragraph and the construction of PM.

Finally, suppose that PM is suboptimal in the class of *all* incentive-compatible contracts. Since the principal solves a convex optimization problem that is separable in all arguments, from the claim established in previous paragraph, it follows that there exists a profitable modification of the following form:

i) at a given (h, G) in phase $k < I$, PM is marginally altered in the direction that reduces the agent's periodic payoff (i.e., either the required effort is increased or compensation is decreased), and

ii) at a later (h', G') that is part of phase $k' > k$, where h' is a continuation of h, PM is marginally changed to restore incentive compatibility.

Since under PM the marginal cost of compensation and the marginal benefit from effort during phase k are below those of phase $k' > k$, any such modification reduces the principal's expected payoff, a contradiction. The optimal contract is unique due to the concavity of the objective function and the linearity of the constraints. $\square$

Proof of Theorem 1. Let $\omega_t = (h_t, G_t)$ denote the history at time t and the periodic game that has been realized. We refer to ω_t as a state. For ω_t at which the activity is available, let $l(\omega_t)$ denote the level of the activity at ω_t.

Consider an incentive-compatible contract C in which the activity A is not non-decreasing. There exist $\omega_t, t' > t, \Delta > 0$, and $p > 0$, such that the set Ω of all states along the path of play in period t' that are consistent with ω_t and for which $l(\omega_t) - \Delta \geq l(\omega_{t'})$ satisfies[19] $Pr(\Omega|\omega_t) = p$. Fix an $\epsilon > 0$ for which $\epsilon + \frac{\epsilon}{p\delta^{t'-t}} < \Delta$.

Consider the contract $\hat{C}$ that is obtained from C by modifying the level of the activity as follows: $\hat{l}(\omega_t) = l(\omega_t) - \epsilon$, and, at every $\omega_{t'} \in \Omega$, $\hat{l}(\omega_{t'}) = l(\omega_{t'}) + \frac{\epsilon}{p\delta^{t'-t}}$. Notice that the original contract is modified only at states in which the agent has adhered to the strategy suggested in the contract. Since the activity is separable, this modification does not impact the distribution of periodic games, or force players to change their actions in the non-activity part of the interaction. First, we show that $\hat{C}$ is incentive compatible.

For all states ω_s such that $s \geq t'$ and $\omega_s \notin \Omega$, $\hat{C}$ is identical to C and so $\hat{C}$ is incentive compatible at such states. At $\omega_{t'} \in \Omega$, the agent's continuation utility from following the contract is increased by $\frac{\epsilon}{p\delta^{t'-t}}$ while his deviation payoff increases by

$$D\left(l(\omega_{t'}) + \frac{\epsilon}{p\delta^{t'-t}}\right) - D(l(\omega_{t'}) \leq \frac{\epsilon}{p\delta^{t'-t}},$$

where the inequality follows from the RHS of Condition $\mathcal{D}$. Hence, $\hat{C}$ is incentive compatible at $\omega_{t'}$. Since $\hat{C}$ and C are identical at all states between periods t and t', it follows that $\hat{C}$ is incentive compatible at all such states.

[19]We say that $\omega_{t'} = (h_{t'}, G_{t'})$ is consistent with the ω_t, if the history $h_{t'}$ is a continuation of ω_t.

Now we return to the state ω_t which we used in the definition of the modified contract. By the construction of $\hat{C}$, if the agent does not deviate, the expected increase in the activity in period t' balances the decrease in the activity at ω_t. Moreover, by the LHS of Condition $\mathcal{D}$, decreasing the activity level at ω_t weakly reduces the agent's payoff from deviating at ω_t. Thus, the modified contract satisfies the incentive compatibility constraint at ω_t. For all other states at period t, the contracts C and $\hat{C}$ are identical. Hence, $\hat{C}$ is incentive compatible at all states in period t. Finally, for all states ω_s such that $s < t$, the modified contract is incentive compatible as the agent's continuation utility from any action is unchanged.

Since the payoff from the activity is additively separable, conditional on reaching ω_t the contract $\hat{C}$ outperforms the contract C if:

$$\kappa\left(\hat{l}(\omega_t)\right) + p\delta^{t'-t}\int \kappa\left(\hat{l}(\omega_{t'})\right) d\mu < \kappa\left(l(\omega_t)\right) + p\delta^{t'-t}\int \kappa\left(l(\omega_t')\right) d\mu,$$

where μ denotes the distribution of states in Ω induced by the contract C, conditional on ω_t. By the definition of $\hat{C}$ this condition is equivalent to:

$$\kappa\left(l(\omega_t) - \epsilon)\right) + p\delta^{t'-t}\int \kappa\left(l(\omega_t') + \frac{\epsilon}{p\delta^{t'-t}}\right) d\mu < \kappa\left(l(\omega_t)\right) + p\delta^{t'-t}\int \kappa\left(l(\omega_t')\right) d\mu,$$

Since $\kappa(\cdot)$ is convex, it has a right-hand side derivative. With a slight abuse of notation, we denote this derivative by $\kappa'(\cdot)$:

$$\begin{aligned}
&\kappa\left(l(\omega_t) - \epsilon\right) + p\delta^{t'-t}\int \kappa\left(l(\omega_{t'}) + \frac{\epsilon}{p\delta^{t'-t}}\right) d\mu < \\
&\kappa\left(l(\omega_t)\right) - \epsilon\kappa'\left(l(\omega_t) - \epsilon\right) + p\delta^{t'-t}\left(\int \kappa(l(\omega_{t'}))d\mu + \frac{\epsilon}{p\delta^{t'-t}}\kappa'(l(\omega_t) - \Delta + \frac{\epsilon}{p\delta^{t'-t}})\right) = \\
&\kappa\left(l(\omega_t)\right) + p\delta^{t'-t}\int \kappa\left(l(\omega_{t'})\right) d\mu - \epsilon\left(\kappa'(l(\omega_t) - \epsilon) - \kappa'(l(\omega_t) - \Delta + \frac{\epsilon}{p\delta^{t'-t}})\right) < \\
&\kappa\left(l(\omega_t)\right) + p\delta^{t'-t}\int \kappa\left(l(\omega_{t'})\right) d\mu,
\end{aligned}$$

where the first inequality follows from the convexity of $\kappa(\cdot)$, and the second inequality follows from the convexity of $\kappa(\cdot)$ and the choice of ϵ.

If such ω_t is reached with positive probability, $\hat{C}$ is better than C. Otherwise, our assumption that the realized level of the activity decreases with positive probability under C implies that there is a t for which there is a positive measure of states ω_t such that 1) the activity is available at ω_t and 2) conditional on ω_t, there is a positive measure of states $\omega_{t'}$

that are consistent with ω_t in which the activity is available and $l(\omega_t) > l(\omega_{t'})$. For each such ω_t, perform the aforementioned modification and note that these modifications do not interact with one another as they modify distinct states. It follows that $\hat{C}$ outperforms C. $\square$

Proof of Theorem 2. We construct two counterexamples, one for each possible violation of Condition $\mathcal{D}$, using the following interaction with no discounting: In period 0 the agent chooses whether to accept or reject the contract. In period 1 the activity is available. In period 2 the agent can end the interaction, and the principal can give the agent a reward worth $-l'$, or impose a fine worth $-D(l')$ and end the interaction. If neither player ends the interaction, in period 3 the activity is available, and in period 4 the principal can either get a (large) payoff by giving the agent a reward worth $-l''$, or impose a fine worth $-D(l'')$. For each counterexample we define l', l'' appropriately.

Under an optimal contract the principal must incentivize the agent to participate and then incentivize him to select the correct level of l_t while granting him both rewards. Note that it is without loss to assume that after the agent deviates the principal fines him. Thus, the IC constraints are

$$\begin{aligned} & l_1 + l_3 - l' - l'' \geq 0 && IC_0 \\ & l_1 + l_3 - l' - l'' \geq D(l_1) - D(l') && IC_1 \\ & l_3 - l'' \geq D(l_3) - D(l'') && IC_3, \end{aligned}$$

where l_t is the suggested level of the activity in period t. We now show that there exist $l' > l''$ such that under the unique optimal contract $l_1 = l'$ and $l_3 = l''$. Note that for such a contract all IC constraints are binding.

Case 1: The slope of $D(\cdot)$ between two points in L is negative. Due to the continuity of $D(\cdot)$ there exists an interval $L_1 \subset L$ and $c > 0$ such that between any two points in L_1 the slope of $D()$ is less than $-c$.

Choose $l'' < l' \in L_1$ such that for any $\epsilon \in [0, l' - l'']$,

$$\kappa(l') + \kappa(l'') < \kappa(l' - \epsilon) + \kappa(l'' + \epsilon(1 + c)).$$

The existence of such values is established in the following lemma.

Lemma 4. *If the slope of $D(\cdot)$ in an interval X is bounded from above by $-c < 0$, there exist $x_2 < x_1 \in X$ such that for any $\epsilon \in (0, x_1 - x_2)$,*

$$\kappa(x_1) + \kappa(x_2) < \kappa(x_1 - \epsilon) + \kappa(x_2 + \epsilon(1 + c)). \tag{1}$$

Proof. Rearranging inequality (1) and dividing by ϵ gives

$$\frac{\kappa(x_1) - \kappa(x_1 - \epsilon)}{\epsilon} < (1 + c)\frac{\kappa(x_2 + \epsilon(1 + c)) - \kappa(x_2)}{(1 + c)\epsilon}. \tag{2}$$

Taking the limit $\epsilon \to 0$, the LHS of (2) converges to the left-handed side derivative of $\kappa(\cdot)$ at x_1 while the RHS converges to $(1+c)$ times the right-handed side derivative of $\kappa(\cdot)$ at x_2. As $\kappa(\cdot)$ is convex, it is differentiable a.e., and so there exist $x_2 < x_1$ for which left-handed side derivative of $\kappa(\cdot)$ at x_1 is strictly less than $(1+c)$ times the right-handed side derivative of $\kappa(\cdot)$ at x_2. Since $\kappa(\cdot)$ is continuous, this implies that (2) will also hold for sufficiently small ϵ.

Finally, note that due to the convexity of $\kappa(\cdot)$, if (1) holds for (x_1, x_2, ϵ), it will also hold for any (x_1, x_3, ϵ) such that $x_3 \in (x_2, x_1)$. □

Since $\kappa(\cdot)$ is convex and IC_0 is binding, if there exists a better contract it must have $l_1, l_3 \in (l'', l')$. Recall that $D(\cdot)$ is decreasing and when $l_1 = l', l_3 = l''$ all IC constraints are binding. Therefore, for any $l_1, l_3 \in (l'', l')$, IC_3 is nonbinding and IC_1 implies IC_0. Finally, observe that since IC_3 is binding when $l_1 = l', l_3 = l''$ and the slope of $D(\cdot)$ is bounded from above by $-c$, a necessary condition for $l_1 = l' - \epsilon$ to be part of an incentive-compatible contract is $l_3 \geq l'' + \epsilon(1 + c)$. Thus, relative to the initial contract, l_3 must be increased by at least $(1 + c)$ times the decrease in l_1. However, by the choice of l'', l', any $l_3, l_1 \in (l'', l')$ for which this is true must have $\kappa(l_1) + \kappa(l_3) > \kappa(l') + \kappa(l'')$.

Case 2: The slope of $D(\cdot)$ between two points in L is greater than one. In this case, since $D(\cdot)$ is continuous there exists a closed interval $L_2 \subset L$ such that $D(x') - D(x'') > x' - x''$ for any $x'' < x'$ such that $x'', x' \in L_2$. In this case, we implicitly define l', l'' by $[l'', l'] \equiv L_2$.

In this interval $D(\cdot)$ is strictly increasing and so IC_0 implies IC_1. Since $\kappa(\cdot)$ is a convex and increasing function, and under the suggested contract IC_0 holds with equality, a contract can be more profitable than the one suggested above only if $l_1, l_3 \in L_2$. However, $l_3 = l''$ is the only $l_3 \in L_2$ for which IC_3 holds. □

Proof of Proposition 2. For any $c > 0$ define

$$X_c = \{x : x > 0 \text{ and } \inf_{l, x+l \in \hat{L}} \frac{\hat{\kappa}'(l+x)}{\hat{\kappa}'(l)} < 1 + c\},$$

where $\hat{\kappa}'(\cdot)$ is the right-hand side derivative of $\hat{\kappa}(\cdot)$. In words, X_c is the set of x's for which increasing the level of the activity by x may, for some values of l, increase the principal's marginal cost from the activity by a factor of less than $1 + c$. Let $\epsilon(c) = \sup\{X_c\}$. Since $\hat{L}$ is compact this supremum is finite, and since $\hat{\kappa}(\cdot)$ is convex, $\lim_{c \to 0} \epsilon(c) = 0$.

Assume to the contrary that in an optimal contract with strictly positive probability the level of the activity decreases by more than $\Delta > \epsilon(c)$ between two of its occurrences. There exist $\omega_t, t' > t$ and $p > 0$, such that the set Ω of states $\omega_{t'}$ that are consistent with ω_t and for which $l(\omega_t) > l(\omega_{t'}) + \Delta$ satisfies $Pr(\Omega|\omega_t) = p$. For each such state we show that there exists a profitable modification of the contract that does not violate any IC constraint.

Changing the level of the activity at ω_t to $\tilde{l}(\omega_t) = l(\omega_t) - \epsilon$ and at all $w_{t'} \in \Omega$ to $\tilde{l}(\omega_{t'}) = l(\omega_{t'}) + \alpha\epsilon$ is profitable for sufficiently small ϵ if $\alpha < \frac{1}{p\delta^{t'-t}} \frac{\hat{\kappa}'(l(\omega_t))}{\hat{\kappa}'(l(\omega_{t'}))}$. Moreover, if $\alpha > \frac{1}{p\delta^{t'-t}}$ such a change relaxes the IC constraints at $\{1, 2, \ldots, t-1, t+1, \ldots, t'-1\}$, has no impact on IC constraints after t', and, as the slope of $D(\cdot)$ is less than one, does not violate the IC constraints at t'. Thus, $l(\omega_t) > l(\omega_{t'}) + \Delta$ only if any small decrease in $l(\omega_t)$, which is offset by a subsequent increase in $l(\omega_{t'})$ that maintains incentive compatibility at t, is not profitable.

The supremum of the marginal increase in the agent's on-path payoff (at t) from a profitable modification is $\frac{\hat{\kappa}'(l(\omega_{t'})+\Delta)}{\hat{\kappa}'(l(\omega_{t'}))} - 1$. Thus, a sufficiently small modification of the type suggested above is incentive compatible at t if $\frac{\hat{\kappa}'(l(\omega_{t'})+\Delta)}{\hat{\kappa}'(l(\omega_{t'}))} - 1 \geq c$. Therefore, a decrease of size Δ in the level of the activity can be part of an optimal contract only if $\inf_{l \in \hat{L}} \frac{\hat{\kappa}'(l+\Delta)}{\hat{\kappa}'(l)} < 1 + c$. However, as $\Delta > \epsilon(c)$ this inequality does not hold. □

Proof of Proposition 3. The proof of this result uses the same counterexample used in the second part of the proof of Theorem 2. To construct a decrease of size $|\tilde{L}|$ in the standardized level of the activity under the unique optimal contract, use that example with $L_2 = \tilde{L}$. □

FOR ONLINE PUBLICATION: Appendix B - Supplementary Material

Game Form Definition of an Activity

A two-player simultaneous move game G contains an activity if the following conditions hold. There exists a subset of the possible outcomes of the game, Σ, that are associated with the activity and an interval of possible levels of the activity, L, such that: 1) there exists a surjective ("onto") function $\eta : \Sigma \to L$ and, 2) there exist two continuous and increasing functions, $u, \kappa : L \to \mathbb{R}$, such that for every $\sigma \in \Sigma$ the principal's payoff is given by $-\kappa(\eta(\sigma))$ and the agent's payoff is given by $u(\eta(\sigma))$.

For a given activity in the game G we define the agent's maximal activity-specific payoff as follows. Let X and Y denote the strategy space for the principal and agent in G, respectively, and denote by $V(x, y)$ the agent's payoff from the strategy profile $\langle x, y \rangle$. For each $l \in L$ define the set of principal's strategies that can induce the level l as

$$X(l) = \{x \in X : \exists y \in Y \text{ for which } \langle x, y \rangle \in \Sigma \text{ and } \eta(x, y) = l\}.$$

The agent's maximal activity specific payoff is then given by

$$D(l) = \inf_{x \in X(l)} \sup_{y \in Y} V(x, y),$$

that is, the maximal payoff he can obtain when the principal tries to induce the level l in the least tempting way possible.

Discontinuous $D(\cdot)$

Theorem 2 establishes that for any convex separable activity for which $D(\cdot)$ is continuous and either decreasing or sharply increasing, there exists an environment in which the activity is decreasing under the optimal contract. In this appendix we show that if $D(\cdot)$ is discontinuous (and hence either decreasing or sharply increasing) the same qualitative result holds. First, we illustrate the impact of discontinuities in $D(\cdot)$ with an example.

Consider a (non-standardized) activity from the class of joint production activities presented in Section 3. In particular, assume that the principal must assign capital in integer units (creating a discontinuity in $D(\cdot)$), and that the production technology is such that the agent's level of effort, x, belongs to $[0, 3]$ and his value from deviating is

$$D(x) = \begin{cases} 1 & \text{if } x < 1 \\ 4 & \text{if } x > 1 \end{cases},$$

and $D(1) \in \{1, 4\}$.

Consider the following four period interaction with no discounting. In period 1 the activity is available, and in period 2 the agent can end the interaction. In period 3 the activity is available again, and in period 4 the principal can either pay the agent a wage of 4 or impose a fine of 10.

In an optimal contract the principal fines the agent only after the agent deviates in an earlier period. This simple observation has three implications. First, the fine is large enough to deter deviations in period 3. Second, since the agent's continuation utility after period 2 is positive, he has no incentive to end the interaction in period 2. Third, the agent will end the interaction in period 2 if he deviates in period 1. Thus, the only IC constraint is that of period 1, and this constraint is given by

$$4 - x_1 - x_3 \geq D(x_1). \tag{3}$$

First, note that if $D(x_1) = 4$ then (3) implies that $x_3 \leq -1$, a contradiction. Second, since the principal's payoff is increasing in x_3 constraint (3) is binding. Thus, (3) simplifies to $x_3 = 3 - x_1$ such that $D(x_1) = 1$. Finally, since the principal's profit from the activity is concave in x and incentive compatibility requires that $x_3 > x_1$, the principal's objective simplifies to maximizing x_1 subject to $D(x_1) = 1$.

If $D(1) = 1$, the constraint $D(x_1) = 1$ is equivalent to $x_1 \leq 1$. In this case, the optimal contract exists and the effort in that contract *increases* over time: $x_1 = 1$, $x_3 = 2$. Denote the principal's value from this contract by v. Under any contract in which the effort schedule does not increase over time, $x_1 \leq 1$ and $x_3 \leq 1$. Therefore, the value of such contracts is bounded away from v.

Now, if $D(1) = 4$, then $D(x_1) = 1$ is equivalent to $x_1 < 1$, and an optimal contract does not exist. However, since v can be approximated by choosing x_1 sufficiently close to 1, under any approximately optimal contract (a formal definition of which is provided below) the agent's effort increases over time.

Next, we derive a variant of Theorem 2 that allows for discontinuous $D(\cdot)$. If there exists an interval on which $D(\cdot)$ is continuous and Condition $\mathcal{D}$ is violated, then by Theorem 2 there exists an environment in which the activity is decreasing under the optimal contract. Hence, without loss of generality we assume that Condition $\mathcal{D}$ holds in any interval $L' \subset L$ over which $D(\cdot)$ is continuous. To avoid dealing with pathological cases, we focus on activities for which $D(\cdot)$ has a finite number of discontinuity points and all these discontinuity points are jump discontinuities.[20] With no additional loss of generality, we can further assume that $D(\cdot)$ has a single jump discontinuity point at $l^* \in L$.

To derive the formal result we need the following definitions. We say that a contract C is ϵ-optimal if for any incentive-compatible contract C', the difference between the principal's payoff from C' and C is less than ϵ. Moreover, we say that an activity is decreasing in all approximately optimal contracts, if there exists a $\bar{\epsilon} > 0$ such that for any $\epsilon < \bar{\epsilon}$ the activity is decreasing under all ϵ-optimal contracts.

Theorem 3. *Let A be a standardized convex activity for which $D(\cdot)$ has a jump discontinuity at l^* and satisfies Condition $\mathcal{D}$ on any interval $L' \subset L$ such that $l^* \notin L'$. There exists a contracting environment with respect to which A is separable such that A is decreasing under all approximately optimal contracts.*

Proof. Let A be a standardized convex activity for which $D(\cdot)$ has a jump discontinuity at l^* and satisfies Condition $\mathcal{D}$ in any interval $L' \subset L$ such that $l^* \notin L'$. We analyze separately the case where $D(\cdot)$ is increasing at the jump point and the case where it is decreasing.

Case 1: $D(\cdot)$ is decreasing at the jump point. Consider the following interaction with no discounting. In period 0 the agent chooses whether to accept or reject the contract. In period 1 the activity is available, and in period 2 the agent can end the interaction and get a payoff of C. In period 3 the activity is available, and in period 4 the principal can either get a (large) payoff by providing the agent with a payoff of R or impose a (large) fine on the agent.[21]

In an optimal contract the principal will fine the agent following a deviation. Thus, if the fine is large enough the agent will not deviate in period 3, and will end the interaction

[20] We find it hard to believe that there exist economically relevant activities for which $D(\cdot)$ has infinitely many discontinuity points or a removable/singular discontinuity point.

[21] The principal's payoff in period 4 must be large enough to make him want the agent to accept the contract, and the fine must be large enough to deter certain deviation. The exact values that are required for this can be calculated later on.

in period 2 if he deviated in period 1. Thus, the IC constraints are

$$l_1 + l_3 + R \geq 0 \qquad (IC_0)$$
$$l_1 + l_3 + R \geq D(l_1) + C \ (IC_1).$$

Let $z = \lim_{\eta \downarrow 0} \Big(D(l^* - \eta) - D(l^* + \eta) \Big)$, select $\varepsilon < \frac{z}{2}$ such that $(l^* - \varepsilon, l^* + \varepsilon) \subset L$, and set

$$R = \varepsilon - 2l^*$$
$$C = \varepsilon - \lim_{\eta \downarrow 0} D(l^* + \eta) - \frac{z}{2}.$$

For these parameters, the IC constraints are

$$l_1 + l_3 \geq 2l^* - \varepsilon \qquad (IC_0)$$
$$l_3 + \lim_{\eta \downarrow 0} D(l^* + \eta) - 2l^* + \frac{z}{2} \geq D(l_1) - l_1 \ (IC_1).$$

First, we show that in order to set $l_1 < l^*$ it must be that $l_3 \geq l^* + \frac{z}{2}$. For $l_1 < l^*$, the slope of $D(\cdot)$ is less than one and so the RHS of (IC_1) is decreasing in l_1 over this range. Thus, it is bounded from below by $\lim_{l \uparrow l^*} D(l) - l = \lim_{\eta \downarrow 0} D(l^* + \eta) + z - l^*$. Replacing the RHS of (IC_1) with this upper bound and rearranging gives $l_3 - l^* > \frac{z}{2}$.

If $D(\cdot)$ is right-continuous the contract $l_1 = l^*,\ l_3 = l^* - \varepsilon$ is incentive compatible and minimizes the aggregate level of the activity. Moreover, due to the convexity of $\kappa(\cdot)$ this contract is strictly better than any contract with $l_1 > l^*$. From the previous argument any contract with $l_1 < l^*$ has $l_3 > l^* + \varepsilon$ and so due to the convexity of $\kappa(\cdot)$ such contracts are not optimal. Moreover, the difference in value between the optimal contract and a contract with $l_1' < l^*$ is at least $\varepsilon\kappa'(l^* - \varepsilon)$. Thus, for $\tilde{\epsilon} < \varepsilon\kappa'(l^* - \varepsilon)$ under any $\tilde{\epsilon}$-optimal contracts $l_3 < l^*$ and $l_1 \geq l^*$, and so the activity is decreasing in all approximately optimal contracts. If $D(\cdot)$ is left-continuous the supremum of the principal's value is the same as above and the same argument concludes the proof.

Case 2: $D(\cdot)$ is increasing at the jump point. Consider the following interaction with no discounting. In period 0 the agent chooses whether to accept or reject the contract. In period 1 the activity is available, and in period 2 the principal can either impose a (large) fine of F on the agent and end the interaction, or provide the agent with a payoff of R_2. In period 3 the activity is available, and in period 4 the principal decides whether or not to get a (large) payoff by providing the agent with a reward of R_4.

The principal will fine the agent after a deviation in period 1 and will withhold the reward in period 4 after a deviation in period 3. Thus, the incentive compatibility constraints are

$$\begin{aligned} & l_1 + R_2 + l_3 + R_4 \geq 0 && (IC_0) \\ & l_1 + R_2 + l_3 + R_4 \geq D(l_1) - F && (IC_1) \\ & l_3 + R_4 \geq D(l_3) && (IC_3). \end{aligned}$$

If F is large enough, then (IC_1) will hold. Let $\varepsilon > 0$ such that $(l^* - \varepsilon, l^* + \varepsilon) \subset L$, and set

$$\begin{aligned} R_2 &= -\lim_{\eta \downarrow 0} D(l^* - \eta) - l^* - \varepsilon \\ R_4 &= \lim_{\eta \downarrow 0} D(l^* - \eta) - l^*. \end{aligned}$$

For these parameters the IC constraints are

$$\begin{aligned} & l_1 + l_3 \geq 2l^* + \varepsilon && (IC_0) \\ & \lim_{\eta \downarrow 0} D(l^* - \eta) - l^* \geq D(l_3) - l_3 && (IC_3). \end{aligned}$$

First, we show that if $D(\cdot)$ is left-continuous, then for a sufficiently small ε the unique optimal contract is $l_1 = l^* + \varepsilon,\ l_3 = l^*$. From (IC_0) it follows that this contract minimizes the aggregate level of the activity. Since $\kappa(\cdot)$ is increasing and convex if there exists a better contract, it follows that under this alternative contract $|l_3' - l_1'| < \varepsilon$ and $l_3' > l^*$.

Since IC_3 is binding when $l_3 = l^*$ and $D(\cdot)$ has a discrete increase at l^* there exists a υ such that IC_3 is violated for all $l_3 \in (l^*, l^* + \upsilon)$; set $\varepsilon < \frac{\upsilon}{2}$. Any contract with $l_3' > l^*$ must have $l_3' > l^* + \upsilon$ and so in order to have $l_3 > l^*$ and $|l_3' - l_3'| < \varepsilon$ it must be that $l_1' > l^* + \varepsilon$, but then $l_3' > l_3 + \varepsilon$, $l_1' > l_1 + \varepsilon$ and this contract is suboptimal. Note that the difference in value between the optimal contract and any contract with $l_3 > l^*$ is at least $\varepsilon\kappa'(l^*)$. Therefore, for $\tilde{\epsilon} < \varepsilon\kappa'(l^*)$ the set of $\tilde{\epsilon}$-optimal contracts has $l_3 \leq l^*$ and $l_1 \geq l^* + \varepsilon$ and the activity is decreasing in all approximately optimal contracts. If $D(\cdot)$ is right-continuous the supremum of the principal's value is and the same argument concludes the proof. $\square$