Auctions with multi-member bidders

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Abstract

I consider an auction in which one of the bidders is a team consisting of several individuals. These individuals need to agree on a bid, and on splitting the payment to the auctioneer if they win the item. Under some conditions, a unique equilibrium is obtained under either a first-price or a second-price format. Under more permissive conditions the equilibrium need not be unique, but the symmetric equilibria of the first-price model are isomorphic to the symmetric equilibria of the second-price model. The free riding problem which stems from collective bidding is studied in detail.

Keywords: Auctions; Multi-member bidders; Public goods.

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1 Introduction

Works in auction theory assume that bidders are individuals. They can be of many different kinds—firms, organizations, or persons—but all are subject to the "one bidder=one agent" rule. In practice, however, bidding is often decided by groups of

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agents. For example, when a multi-million-dollar firm decides on a bid, the decision is made by a group of individuals consisting of management, outside experts, or both. Thus, something is missing from the theory: the theory assumes that bidders are individuals, whereas in many real-life auctions they are not. My goal in the present paper is to address this lacuna and take a first step toward the understanding of auctions with *multi-member bidders*.

When some bidders are multi-member, bid-decisions involve not only the usual strategic considerations, but present additional challenges. These challenges may depend on the item being auctioned off, on the relationships between the agents who place a joint bid, and more. I consider the case where the agents placing a joint bid are ex ante symmetric, and the item up for sale is, from their standpoint, a public good; that is, either all of them win the item together, in which case they all enjoy it, or all lose. Examples of scenarios that fit into this framework range from small-scale instances, such as bidding on a TV set by a couple of roommates, to larger ones of a more commercial character. A recent real-world example of an auction with a multi-member bidder is the Israeli 5G spectrum auction that was carried out in the year 2020. This auction included several bidders, one of which was multi-member, comprising three telecommunication companies. The companies needed to decide on a bidding strategy, and each license they (jointly) won was available for each one of them to use.[1]

I consider an auction environment with two bidders: one who is multi-member, and it comprises n symmetric individuals (players 1 through n) and an additional bidder who is a single individual (player $n+1$, the *regular bidder*). The two bidders compete in an auction for an indivisible item, and the valuations of all $n+1$ individuals are private and independent. From the regular bidder's standpoint, the

[1]Disclosure: I was involved in consulting to one of the operators in the 5G auction. The present paper is not related to this consulting. The models to be developed in the present paper are different from the 5G auction. The content of the present paper, which was developed exclusively by me, should not be attributed to anybody but myself.

auction is an ordinary auction. As for the multi-member bidder, hereafter MUMB,[2] the situation looks as follows: given that players' valuations are $(\theta_1, \cdots, \theta_n)$, if the MUMB wins the auction the payoff of player i is $\theta_i - p_i$, where p_i is the payment he contributes to cover the item's cost.[3] The MUMB's decisions are made via a mechanism to which players 1 through n send reports. The mechanism consists of a *bid aggregation rule*, A, and a *cost sharing rule*, s; player i sends a report (or bid) $b_i \geq 0$, the reports are sent simultaneously, and the bid submitted on behalf of the MUMB is $A(b_1, \cdots, b_n)$; if the auction is won, the cost is split between the players in proportions $(s_1(b_1, \cdots, b_n), \cdots, s_n(b_1, \cdots, b_n))$.[4] I study several versions of this model, which differ in the assumptions made on A, s, the auction format, and on outside competition (i.e., on the regular bidder). I start off with the following version.

The *second-price model* is such that:

- (I) The auction format is second-price (SPA);
- (II) The bid aggregation rule is the average rule: $A(b_1, \cdots, b_n) = \frac{1}{n} \sum_{i=1}^{n} b_i$;
- (III) Cost sharing is proportional to individual bids: $\max\{b_1, \cdots, b_n\} > 0 \Rightarrow s_i(b_1, \cdots, b_n) = \frac{b_i}{\sum_{j=1}^{n} b_j}$.
- (IV) Each MUMB member's valuation is distributed on $[0, 1]$ according to some differentiable distribution F; the regular bidder's valuation is distributed uniformly on $[0, M]$, where M is large.[5]

The model has a unique equilibrium. This equilibrium is symmetric. I characterize the equilibrium and study its properties. In particular, I describe the relations

[2] I use "MUMB" and "team" interchangeably.

[3] The value from not obtaining the item (and not paying anything) is zero.

[4] Throughout, the word "bid" will be used in several senses. It will be used to denote an individual bid of player i—namely, his report to the mechanism—and to denote the actual bid (the MUMB's bid) that results from the reports. It will also be used to denote the regular bidder's bid.

[5] A concrete lower bound on M will be described later. This assumption implies that the team faces a sufficiently fierce competition. I discuss its importance in Subsection 2.1.

between the team's size and the free riding problem that stems from collective bidding.

Next, I consider the *first-price model*, in which the auction format is first-price (FPA), but the other assumptions, (II)-(IV), are maintained. Whereas in the second-price model the regular bidder is assumed to be optimizing—he reports his type truthfully—under the first-price format things are substantially more complicated. For this model, I *assume* that the regular bidder's bid is uniform over $[0, M]$. Thus, outside competition is given exogenously, and is not modeled as the behavior of an optimizing agent. Under this assumption, the first-price model has a unique equilibrium. This equilibrium is symmetric.

With β^{SPA} and β^{FPA} denoting the equilibrium bid functions for the two formats, the following holds:

$$\beta^{FPA} = \frac{1}{2} \cdot \beta^{SPA}. \tag{1}$$

This relation is robust, in the sense that it holds under a variety of bid aggregation and cost sharing rules, not only under the ones described in (II) and (III). I will remark on this point later on; now, I turn to describing the equilibria that are characterized when bid-aggregation and cost-sharing are as in (II)-(III).[6]

Under the second-price format, the equilibrium bid function, β^{SPA}, is continuous and piecewise linear, identically zero up until some cutoff, and then it becomes linearly increasing with slope equal to n. The function has this form regardless of the MUMB members' type distribution—this distribution only affects the value of the cutoff. This cutoff plays a key role in the analysis.

For $n = 1$ the cutoff is zero, hence the bid function is the identity function; that is, the ordinary SPA's weak dominance equilibrium is a special case of the current model's equilibrium. For $n \geq 2$ the cutoff is positive, so there are types to both of its sides.

[6] To keep this Introduction at a reasonable length, I skip some of these equilibria's properties; these can be found in the subsequent sections.

Types below the cutoff free ride, hoping that their partners will submit sufficiently large bids, whereas types above the cutoff bid "aggressively"—their marginal bidding propensity is n. The aggressive bidding of the high types overshadows the low types' free riding, in the sense that the MUMB's expected bid is greater than the expected type (i.e., greater than the expected bid of a participant in the ordinary SPA).

The cutoff is increasing in n, which is intuitive: the more partners one has, the greater is the incentive to free ride. Moreover, the cutoff converges to one as $n \to \infty$. That is, in the limit, free riding assumes an extreme form, in which all types refrain from participation. For a given n, the cutoff is sandwiched between two bounds, and the lower bound is increasing in the type distribution's expectation. This is intuitive as well: if a MUMB member believes that each of his partners' types is expected to be large, then he also believes that they are likely to place high bids and therefore there is a high probability that the auction will be won even if he does not contribute. Despite the fact that free-riding is more of an issue as n gets large, both the MUMB's expected bid and its winning probability are increasing in n.

The second-price format facilitates the analysis because it makes it possible to ignore the regular bidder, as truthful type-reporting is a weakly dominant strategy for him. This, of course, is not true under a first-price format. In the first-price model, I maintain the assumption that the regular bidder's bid is uniform, but there is no claim that this uniformity is optimal for the regular bidder. Thus, outside competition is assumed to be given exogenously, and is not part of the equilibrium analysis.[7] Under this assumption, the first-price model has a unique equilibrium. The equilibrium is symmetric, and its bid function, β^{FPA}, satisfies (1). Therefore, it is piecewise linear and identically zero up until a cutoff—the same cutoff from the second-price model. Therefore, all the results concerning the cutoff continue to hold under the first-price format.

[7]Thus, in the first-price model I make no assumptions on the regular bidder's type distribution—this distribution becomes irrelevant given the postulated (and not derived) behavior.

Next, I turn to a general version of the model, where the bid aggregation and cost sharing rules are general (i.e., not restricted to be the ones described in (II)-(III)) and where outside competition is distributed according to an arbitrary (differentiable) distribution, not necessarily uniform. In this general set-up, and under either of the auction formats, a symmetric equilibrium exists. Qualitatively, it is similar to the equilibria that were characterized under assumptions (II)-(IV): the bid function is identically zero on an interval of low types, and then it is positive-valued and strictly increasing. Under some extra assumptions, the isomorphism between the auction formats described in (1) continues to hold; specifically, it holds if outside competition is uniform over a sufficiently large support, the cost-sharing rule is homogeneous of degree 0, and the bid-aggregation rule is homogeneous of degree 1. However, in the general set-up, equilibrium need no be unique.

I end the paper with an example for this non-uniqueness where, in addition to a symmetric equilibrium, there exists an equilibrium in which only one MUMB member participates in active bidding and everybody else free ride. I call this phenomenon *complete free riding*. An equilibrium with complete free riding can be sustained only if the MUMB members' types are likely to be large, else the designated abstainers will find it profitable to intervene by reporting positive bids. In particular, I show that when the auction format is second-price, complete free riding is impossible if the type distribution is first-order stochastically dominated by the uniform distribution.

The rest of the paper is organized as follows. Subsection 1.1 reviews the literature. In Sections 2 and 3 I consider the second- and first-price models under assumptions (II)-(IV). Section 4 considers the more general version of the models, and Section 5 is dedicated to complete free riding. Section 6 concludes, and proofs that are omitted from the main text appear in the appendix.

1.1 Literature

At a general level, the paper is related to the literature about games that are played by teams. This literature, however, is almost exclusively experimental.[8] A recent exception is Kim et al. (2021), where a theoretical model of team play is studied. In that model, all team members enjoy the same payoff, and they only differ in their ex ante estimate of this payoff. The members' estimates are aggregated via an exogenous rule, to produce the team's action. Thus, there are two important differences with respect to the present paper. First, in Kim et al. (2021) preferences are common, whereas here values are private and independent. Second, in Kim et al. (2021) there is no strategic interaction between team members, whereas here the inter-team interaction is what the paper is all about.[9]

From the existing literature, the model which is closest to the ones studied here is that of private funding of public goods via a *subscription game*. In a subscription game, agents simultaneously make contributions, the public good is provided if the sum of contributions exceeds the good's cost, and otherwise the contributions are refunded.[10] An important difference between this game and my setting is that in the subscription game each agent pays his own contribution when the good is supplied, so when the good is supplied the sum of payments equals the sum of contributions, which may exceed the good's cost.

From the subscription game literature, the work which is closest to the present paper is by Barbieri and Malueg (2010), in which the public good's cost is stochas-

[8]See, e.g., Charness and Sutter (2012) and Kugler et al. (2012). Experimental papers on team bidding include Cox and Hayne (2006) and Sutter et al. (2009).

[9]In subsection 2.1 of their paper Kim et al. write: *"Because it is a common value problem for the team, there is an implicit assumption of sincere reporting."* A similar approach is taken in an earlier paper by Duggan (2001), where groups of players aggregate their actions via social choice rules, with no strategic issues involved in the aggregation.

[10]The term "subscription game" is due to Admati and Perry (1991), who studied a complete information version of this game.

tic. In their model, valuations are private and independent, and when player i with valuation v_i contributes x and all others follow the strategies $\{s_j(\cdot)\}_{j\neq i}$, player i's expected utility is $(v_i - x)Pr(x + \sum_{j\neq i} s_j(v_j) \geq c)$, where c is the stochastic production cost. This expression can be re-written as $(v_i - x)Pr(\frac{x+\sum_{j\neq i} s_j(v_j)}{n} \geq \bar{c})$ where $\bar{c} \equiv \frac{c}{n}$. That is, the expression can be re-written such that the winning probability, $Pr(\frac{x+\sum_{j\neq i} s_j(v_j)}{n} \geq \bar{c})$, is analogous to the one derived from the the average bid aggregation rule, and the random variable $\bar{c}$ plays the role of the regular bidder's bid. However, cost sharing is substantially different from the one assumed in my setting.[11]

In the theoretical literature on auctions, the topic of team bidding is understudied. The literature on collusion can be seen as somewhat relevant, to the extent that the MUMB is viewed as a cartel. However, it is not a cartel in the ordinary sense of the word. First, at the conceptual level, the MUMB is an organic unit, not a fictitious construct the purpose of which is to advance the goals of its individual members. At the operational level, collusion in one-shot auctions is typically based on transfers between the colluding agents, which are required because the cartel allocates the won item to *one* of its members, and the others need to be compensated for leaving the auction empty-handed (see, e.g., McAfee and McMillan 1992).[12] Here, by contrast, transfers are not available, and the losers-compensation issue is irrelevant, because when the MUMB wins the auction, all of its members win together. Another context in which team members win together or lose together is that of group contests (see, e.g., Kobayashi and Konishi 2021 and the references therein). There, however, the focus is typically on moral hazard (members exert unobservable effort), and less on adverse selection, which is the main focus in the auction setting.

[11]In Barbieri-Malueg, each person's contribution (when the good is being supplied) is independent of the others' behavior; in my setting—under both SPA and FPA—the individual's payment does depend on the others' behavior.

[12]In a repeated game, continuation play can substitute for money transfers; see, e.g., Athey and Bagwell (2001), Skrzypacz and Hopenhayn (2004), and Rachmilevitch (2013).

2 The second-price model

The model's description was given in the previous section, in (I)-(IV), and for brevity I will not repeat it. The valuation, or type, of each player (a MUMB member) is distributed on $[0,1]$ according to the cumulative distribution function F, whose density is $f = F'$.[13] Types are private and independent.[14] The expected type is $\mathbb{E}(\theta) \equiv \int_0^1 tf(t)dt$. A (pure) strategy for player i is a bid function $\beta_i\colon [0,1] \to \mathbb{R}_+$. An *equilibrium* means a profile of bid functions, $(\beta_1, \cdots, \beta_n)$, that forms a Bayes-Nash equilibrium.

Theorem 1. *Suppose that $M \geq 2n^2$. Then the second-price model has a unique equilibrium. The equilibrium, $(\beta_1, \cdots, \beta_n)$, is symmetric: $\beta_1 = \cdots = \beta_n = \beta^{SPA}$. The bid function β^{SPA} is given by:*

$$\beta^{SPA}(\theta) = \begin{cases} 0 & \textit{if } \theta \leq a \\ n\theta - na & \textit{otherwise,} \end{cases} \tag{2}$$

where a is the unique solution to:

$$a = \frac{n-1}{n+1} \cdot \left(\int_a^1 tf(t)dt + aF(a)\right). \tag{3}$$

The bid function has this continuous piecewise linear form independent of the MUMB's members type distribution.[15] Note that for $n = 1$ the corresponding cutoff is $a = 0$, hence the above formulas generalize the dominant strategy equilibrium of the ordinary SPA. Similarly to the ordinary SPA, the equilibrium is such that no player pays more than his valuation. To see this, note that the maximum that the MUMB may end up paying if it wins with the bid $\frac{\sum_j \beta_j^{SPA}(\theta_j)}{n}$ is $\frac{\sum_j \beta_j^{SPA}(\theta_j)}{n}$, hence player i's payment is bounded by $\frac{\beta_i^{SPA}(\theta_i)}{\sum_j \beta_j^{SPA}(\theta_j)} \cdot \frac{\sum_j \beta_j^{SPA}(\theta_j)}{n} = \frac{\beta_i^{SPA}(\theta_i)}{n} = \max\{0, \theta_i - a\}$.

[13]In Section 5 I will assume that F is twice differentiable.

[14]They are also independent of the regular bidder's type (which is uniform on $[0, M]$). The regular bidder reports his type truthfully.

[15]The distribution does influence, however, the value of the cutoff a, as seen in (3).

The equilibrium shares similar features with the one obtained by Barbieri and Malueg (2010) for their subscription game. Under the assumption of a uniformly distributed cost (which is analogous to a regular bidder with a uniform valuation), their equilibrium contribution-function is continuous, and, like β^{SPA}, its graph consists of two linear segments: the first is flat at the level zero and the second is increasing. However, whereas the slope of β^{SPA}'s increasing part is n, that of the Barbieri-Malueg function is independent of the number of players. Another point of similarity is that the expected bid/expected contribution appear in the cutoff up to which the function is flat. Finally, when $n = 2$ the cutoffs of the two models coincide: both equal $3-2\sqrt{2}$ in this case.

In equilibrium, types below the cutoff a free ride—they refrain from bidding, hoping that their partners' types be sufficiently large, in which case the auction will be won (with a high probability), but they will not be asked to contribute. Since the cutoff a is associated with free riding in this way, one would expect it to depend positively on n. The following result, which summarizes the relations between n and the cutoff, shows that this is indeed the case; in its statement, a_n denotes the cutoff corresponding to a MUMB of cardinality n.

Proposition 1. *The cutoff a_n satisfies the following:*

1. *a_n is strictly increasing in n.*

2. *$lim_{n\to\infty} a_n = 1$.*

3. *$(\frac{n-1}{n+1})\mathbb{E}(\theta) \leq a_n \leq \frac{n-1}{n}$ for all $n \geq 1$.*

The intuition behind the lower bound in part *3* is that if a MUMB member believes that his partners' types are expected to be large, then he also believes that they are likely to place high bids and therefore there is a high probability that the auction will be won even if he does not contribute.[16] For the same reason, one would

[16] Additionally, the lower bound is increasing in n, and the intuition for that is obvious.

expect the cutoff to increase under first-order stochastic dominance. To establish this monotonicity, the following result, which is of independent interest, will be useful.

Proposition 2. *Let $n = 2$. In the second-price model, the equilibrium-expected-utility of a MUMB member with type θ is:*

$$\pi^*(\theta) = \begin{cases} \frac{2a\theta}{M} & \text{if } \theta \leq a \\ \frac{1}{M}\left[\frac{1}{2}(\theta^2 - a^2) + \theta a + a^2\right] & \text{otherwise.} \end{cases} \tag{4}$$

Similarly to the role it plays in β^{SPA}'s formula, the type distribution affects equilibrium payoffs only through the cutoff a. For $a = 0$ the formula boils down to $\pi^*(\theta) = \frac{\theta^2}{2M}$, which is the expected payoff of a participant in an ordinary 2-bidder second-price auction, whose type is θ and whose opponent's type is distributed uniformly over $[0, M]$. Namely, when one plugs the $n = 1$-cutoff in the $n = 2$-formula, the $n = 1$-payoff obtains.[17]

Based on Proposition 2, the following result can now be proved.

Proposition 3. *Let $n = 2$ and consider two copies of the second-price model—one with the type distribution F and one with G, where F first-order stochastically dominates G. Let a^z be the cutoff corresponding to $z \in \{F, G\}$. Then $a^F \geq a^G$.*

Proof. Assume by contradiction that $a^F < a^G$. Let π^z the expected-utility function under $z \in \{F, G\}$. Consider $\theta < a^F$. By Proposition 2, $\pi^F(\theta) < \pi^G(\theta)$. Therefore, the winning probability under F is smaller than the corresponding probability under G. By the shape of the bid function β^{SPA}, the win-probability under F is:

$$\int_{a^F}^{1} (t - a^F) f(t) dt = \int_{a^F}^{1} t f(t) dt - a^F(1 - a^F).$$

Applying integration by parts to the RHS's first terms yields that the probability is $1 - a^F F(a^F) - \int_{a^F} F(t)dt - a^F(1 - F(a^F)) = 1 - a^F - \int_{a^F} F(t)dt$. Therefore $1 - a^F - \int_{a^F} F(t)dt < 1 - a^G - \int_{a^G} G(t)dt$, hence:

[17] Note, however, that when $n = 2$ the cutoff cannot be zero, by part *3* of Proposition 1.

$$a^G - a^F < \int_{a^F} F(t)dt - \int_{a^G} G(t)dt \leq \int_{a^F} G(t)dt - \int_{a^G} G(t)dt \leq \int_{a^F}^{a^G} G(t)dt,$$

a contradiction. □

Under a strengthening of first-order stochastic dominance, a similar conclusion obtains for $n \geq 3$. This is the content of the following result, in the statement of which $a(F, n)$ denotes the cutoff when the type distribution is F and the MUMB's cardinality is n.

Proposition 4. *Let* $n \geq 3$ *and suppose that:*

1. *F first-order stochastically dominates G;*
2. $\int_{\frac{n-1}{n}}^{1} F(t)dt < \int_{\frac{n-1}{n}}^{1} G(t)dt$*; and*
3. $a(F, 2) > a(G, 2)$.

Then $a(F, n) > a(G, n)$.

Types above the cutoff bid aggressively—their marginal bidding propensity is n. The reason for this behavior is twofold: first, because the bid aggregation rule splits the individual reports by n; and second, in order to compensate for the possible presence of free riding partners. It turns out that the high-types effect dominates the low types' free-riding, and it becomes more and more pronounced as n increases. This is the content of the following result, in the statement of which β_n^{SPA} denotes the equilibrium bid function corresponding to a MUMB of cardinality n.

Proposition 5. $\mathbb{E}(\beta_n^{SPA})$ *is strictly increasing in* n.[18]

Note that, a priori, it is not obvious why n should have the aforementioned effect. On the one hand, bidding becomes cheaper as n increases (increasing an individual report by Δ impacts the collective bid only by $\frac{\Delta}{n}$), but on the other hand the greater number of partners increases the incentives to abstain.

[18]Since $\mathbb{E}(\beta_1^{SPA}) = \mathbb{E}(\theta)$, the result implies that $\mathbb{E}(\beta_n^{SPA}) > \mathbb{E}(\theta)$ for all $n \geq 2$.

Proof. By (2), $\mathbb{E}(\beta_n^{SPA}(\theta)) = n\int_a^1 tf(t)dt - na(1-F(a))$, and by (3) this expression is equal to $\frac{2na}{n-1}$. By (3):

$$a = \frac{n-1}{n+1}\left(\int_a^1 tf(t)dt + aF(a)\right),$$

hence:

$$\frac{2na}{n-1} = \frac{2n}{n+1}\left(\int_a^1 tf(t)dt + aF(a)\right) = \frac{2n}{n+1}\left(1 - \int_a^1 F(t)dt\right).$$

The result follows from the fact that the term in the parentheses is increasing in a. □

A related property is that the MUMB's winning probability increases in n.

Proposition 6. *The probability that the MUMB wins the auction is* $\frac{2na}{(n-1)M}$.

Proof. With u denoting the random variable which is distributed uniformly over $[0, M]$, the above probability is:

$$Pr\left(u \leq \frac{1}{n}\sum_{i=1}^{n}\beta^{SPA}(\theta_i)\right) = \int_0^M Pr(u \leq b)\cdot Pr\left(\frac{1}{n}\sum_{i=1}^{n}\beta^{SPA}(\theta_i) = b\right)db =$$

$$= \frac{1}{M}\int_0^M bPr\left(\frac{1}{n}\sum_{i=1}^{n}\beta^{SPA}(\theta_i) = b\right)db =$$

$$= \frac{1}{M}\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}\beta^{SPA}(\theta_i)\right) = \frac{\mathbb{E}(\beta^{SPA}(\theta))}{M} = \frac{2na}{(n-1)M}.$$

□

Estimating the degree of inefficiency caused by collective bidding is a natural issue to address. The above probability can shed light, if only as a rough proxy, on how severe the issue can be. Specifically, consider the case where $n = M = 2$ and F is uniform. Then, under an efficient allocation, the probability that the MUMB wins the item is $\frac{1}{2}$; according to Proposition 6, the probability that it obtains the item is $2a$, which, is this case, is about $\frac{1}{3}$.

2.1 The role of outside competition

The requirement that M be sufficiently large is imposed in order to guarantee that each bid-increase by a MUMB member increases the probability of winning, no matter the partners' types. When that is not the case, the objective faced by a MUMB member changes significantly relative to the large-M case. To illustrate, consider $n = 2$ and suppose that $M > 0$ is small enough, so that if player 1 reports x and θ_2 exceeds some value—call this value $\theta(x)$—the MUMB wins for sure. Then, the objective faced by type θ_1 of player 1, given that player 2 follows some monotonic reporting function $\tilde{\beta}$, is to maximize the following expression over $x \geq 0$:

$$\frac{1}{M}\int_0^{\theta(x)} \frac{x+\tilde{\beta}(t)}{2}\cdot(\theta_1 - \frac{x}{x+\tilde{\beta}(t)}\cdot\frac{x+\tilde{\beta}(t)}{4})f(t)dt + \int_{\theta(x)}^1 [\theta_1 - \frac{xM}{2(x+\tilde{\beta}(t))}]f(t)dt. \quad (5)$$

The first-order condition associated with this objective is a differential equation that depends on F non-trivially, and for which I have no closed-form solution. Similar to the equilibrium bid function β, the function $\tilde{\beta}$ is identically zero on an interval that starts at the origin and ends at some cutoff, but to the right of this cutoff $\tilde{\beta}$ is non-linear. The function β stems from a more tractable optimization because the largeness of M implies that $\theta(x) \geq 1$ for every x a player may report in equilibrium, hence the second term in (5) disappears.

The requirement $M \geq 2n^2$ is a sufficient to guarantee that, no matter how the equilibrium looks like, it necessarily has the property that any bid-increase by a MUMB member increases the winning probability. Thus, $M \geq 2n^2$ is sufficient for equilibrium uniqueness. However, for the equilibrium from Theorem 1 to exit, a smaller bound on M can be assumed. Specifically, for this equilibrium to exist it is sufficient and necessary that $M \geq n(1-a)$, where a is given by (3). This is because a sufficient and necessary condition for the existence of this equilibrium is that $\beta(1) \leq M$, and $\beta(1) = n(1-a)$.

3 The first-price model

Consider now the first-price model. For this model, I assume that the regular bidder submits a uniform bid over $[0, M]$; that is, the regular bidder is not considered to be an optimizing agent, but is merely taken as representing some exogenously given outside competition.

Theorem 2. *Suppose that $M \geq n^2$. Then, the first-price model has a unique equilibrium. The equilibrium, $(\beta_1, \cdots, \beta_n)$, is symmetric: $\beta_1 = \cdots = \beta_n = \beta^{FPA}$. The bid function β^{FPA} is equal to $\frac{1}{2} \cdot \beta^{SPA}$, where β^{SPA} is the equilibrium bid function of the second-price model. That is, β^{FPA} is given by:*

$$\beta^{FPA}(\theta) = \begin{cases} 0 & \text{if } \theta \leq a \\ \frac{n}{2}\theta - \frac{n}{2}a & \text{otherwise,} \end{cases}$$

where a is the unique solution to (3).

Since the cutoff is the same under both formats, the results concerning the cutoff—Propositions 1, 3, and 4—continue to hold under the first-price format. Since $\beta^{FPA} = \frac{1}{2} \cdot \beta$, a counterpart of Proposition 5 continues to hold as well. Payoffs, however, are not the same.[19]

Theorems 1 and 2 do not extend to general formats; in particular, it is not true that given any standard auction format, the MUMB bid function is piecewise linear with a graph that consists of two segments. For example, it is easy to check that under the all-pay format, if outside competition is given by a uniform bid then each MUMB member's best-response is to bid zero.

[19] Conditional on winning when the MUMB members' valuations are $(\theta_1, \cdots, \theta_n)$ the expected payment is the same under either format, but the winning probability is smaller under SPA.

4 Generalizations

The main limitation of the analysis in the two previous sections, is that it requires very specific assumptions in order to be applicable. I now set to relax them.

Let the *general first- (second-) price model* denote the model from Sec. 2/3, except that s and A are not confined to (II)-(III) and outside competition is not necessarily uniform; instead, the following more permissive assumptions are made. Outside competition is a stochastic bid on $[0, M]$, distributed according to H. Each s_i is strictly increasing in i's report, A is weakly increasing in each individual report, and $A(b'_1, \cdots, b'_n) > A(b_1, \cdots, b_n)$ when $(b'_1, \cdots, b'_n) > (b_1, \cdots, b_n)$. Also, s and A are symmetric and differentiable a.e., and for each number $r > 0$ there exists a report such that when a MUMB member sends it, the collective bid is at least r. Finally, taking $M > A(\bar{r}, \cdots, \bar{r})$ for a large enough $\bar{r}$ implies that under any profile of reports that the MUMB members may report in equilibrium, their probability of winning is smaller than one.

The following results show that a symmetric equilibrium exists, and, qualitatively, it is similar the ones characterized in the previous sections.

Theorem 3. *In the general second-price model, a symmetric equilibrium exists. The equilibrium bid function, β, is identically zero on an interval of the form $[0, a)$ and strictly increasing on $(a, 1]$, for some $a > 0$.*

Theorem 4. *In the general first-price model, a symmetric equilibrium exists. The equilibrium bid function, β, is identically zero on an interval of the form $[0, a)$ and strictly increasing on $(a, 1]$, for some $a > 0$.*

None of these results makes a claim for uniqueness, let alone characterization. However, under some additional conditions, a further result obtains: the symmetric equilibria of the two models are isomorphic, in the sense of (1).

Theorem 5. *Suppose that s is homogeneous of degree 0 and A is homogeneous of degree 1. Then, there exists an M^* such that if $M \geq M^*$ and outside competition is a* *uniform bid over* $[0, M]$*, then* β *is a symmetric equilibrium bid function in the general second-price model if and only if* $\frac{1}{2} \cdot \beta$ *is such a function in the general first-price model.*

The class of team-mechanisms to which Theorem 5 applies includes a variety of economically-plausible ones. For example, it covers the cases where $A(b_1, \cdots, b_n) = c\sum_{i=1}^{n} b_i$ for any $c > 0$, and $A(b_1, \cdots, b_n) = \lambda\min(b_1, \cdots, b_n) + (1-\lambda)\max(b_1, \cdots, b_n)$ for any $\lambda \in [0, 1]$. As for cost-sharing, the rule $s_i(b_1, \cdots, b_n) = \frac{b_i^\rho}{\sum_{j=1}^{n} b_j^\rho}$ is 0-homogeneous for any ρ.

5 Complete free riding

The following result shows that under the conditions of Theorem 5, the equilibrium need not be unique.[20]

Proposition 7. *Suppose that:*

1. *The auction format is second-price;*
2. $3f(x) \leq 2F(x) + 2xf(x) + xf'(x)$*;*
3. $\mathbb{E}(\theta) \geq \frac{3}{4}$*;*
4. *Cost sharing is proportional, namely as in (III).*
5. *The bid aggregation rule* A *is the maximum rule:* $A(b_1, \cdots, b_n) = max\{b_1, \cdots, b_n\}$.

Then there exists, in addition to a symmetric equilibrium, an equilibrium in which one MUMB member reports his type truthfully and every other member reports zero independent of his type.

[20] The result is for the second-price model, but it can be adapted to the first-price format as well.

Requirements *2* and *3* mean that the type distribution is sufficiently convex. For example, these requirements are satisfied by $F(t) = t^\alpha$ for $\alpha \geq 4$. Under sufficient convexity, the designated bidder's type, and therefore his bid, is likely to be large, which sustains the others' incentives to abstain. Additionally, if a free rider contemplates a deviation, then he faces the following concern: if the deviated-to report is lower than the designated bidder's report, then it will not change the allocation, but make the deviator pay a positive amount instead of zero.

Call an equilibrium as above—one in which everybody, with the exception of a single individual, abstains—an equilibrium with *complete free riding*. As just noted, complete free riding depends on the MUMB members' beliefs: a free rider has to believe that the designated bidder is likely to have a large enough type in order incentive him (the free rider) to keep quite and send a zero report. Therefore, if types are, stochastically, low, complete free riding cannot occur. This is formalized in the following result, which applies to a general class of bid aggregation and cost sharing rules, and holds for multiple forms of outside competition and for any n. The auction-format, however, is required to be second-price.[21]

Proposition 8. *Assume the general second-price model. Suppose that the regular bidder's type is distributed according to the differentiable distribution H, defined on $[0, M]$ for some $M \geq 1$. Then, if F is first-order stochastically dominated by the uniform distribution, then there does not exist an equilibrium with complete free riding.*

[21]The proof's main argument shows that if a complete-free-riding-profile is considered as a putative equilibrium, then a player who is supposed to abstain but has a large enough type finds the following deviation profitable: it is profitable for him to send an arbitrarily large report r, given which the regular bidder will lose for sure. Under the second-price format, the price payed upon a deviation is the regular bidder's expected type, but in the first-price case the price can be arbitrarily high, which renders the deviation non-profitable.

6 Conclusion

I end the paper with two comments. First, I have focused on a single team (MUMB). Studying interactions among multiple MUMBs is a line for future investigation. In a model with multiple MUMBs but no regular bidders, the MUMB bid functions would not look like the ones derived in the present paper; in particular, segments of types who bid zero would be impossible in equilibrium. To see this, consider a model with two MUMBs, each with at least two members, and no regular bidders. It is impossible that within each MUMB all members follow a bid function which is zero on a segment of low types, because then every low enough type would have an incentive to deviate to a positive bid: such a deviation would cause a jump in the winning probability, but only a negligible increase in the expected payment.[22]

Second, there is the mechanism design question: given a type distribution F and a number of MUMB members n, what mechanism maximizes the expected sum of the members' payoffs? This is a challenging question, different from typical mechanism design questions. In a typical mechanism design set-up, the designer needs to select an allocation rule and a payment rule, subject to incentive and participation constraints. The allocation and payments are separate from one another. Here, by contrast, they are not. To see this, consider the rule that gives the item to the MUMB with certainty. Clearly, it cannot be implemented with zero payments, because of outside competition. The presence of the regular bidder creates a feasibility constraint that links payoffs to allocation, in a way that has no counterpart in the classical mechanism design set-up.

[22]In the two-MUMB case, there does not exist an equilibrium in which in one MUMB the bid function has the piecewise linear form studied here while the members of the other MUMB follow a continuous, strictly increasing function. This result appears in an earlier draft of the paper.

Appendix

Proof of Proposition 1:

1. Equation (3) can be written as $G \equiv \int_a^1 tf(t)dt - a - (\frac{2a}{n-1}) + aF(a) = 0$. Let us view n as a continuous, rather than discrete, variable. By the implicit function theorem, he sign of the derivative $\frac{\partial a}{\partial n}$ is the same as that of $-[\frac{\partial G}{\partial n}]/[\frac{\partial G}{\partial a}]$. Since $\frac{\partial G}{\partial n} = -2a(\frac{1}{n-1})' > 0$ and $\frac{\partial G}{\partial a} = -1 - \frac{2}{n-1} + F(a) < 0$, the result follows.

2. Let $a^* \equiv \lim_{n\to\infty} a_n$. Equation (3) implies $a^*(1 - F(a^*)) = \int_{a^*}^1 tf(t)dt$, or $a^* = \mathbb{E}(\theta : \theta \geq a^*)$. Therefore $a^* = 1$.

3. This is clear for $n = 1$, so suppose $n \geq 2$.

Lower bound: It follows from (3) that:

$$a \cdot (\frac{n+1}{n-1}) = \int_a^1 tf(t)dt + aF(a) =$$
$$= \int_a^1 tf(t)dt + a\int_0^a f(t)dt > \int_a^1 tf(t)dt + \int_0^a tf(t)dt = E.$$

Upper bound: Since (3) has a unique solution[23], and since at $a = 0$ the RHS of (3) exceeds the LHS, it is enough to show that at $a = \frac{n-1}{n}$ the reserve inequality holds. This inequality is equivalent to $\frac{1}{n} > \frac{1}{n+1} \cdot (\int_a^1 tf(t)dt + aF(a))$, which holds because $\int_a^1 tf(t)dt + aF(a) < 1$. $\square$

Proof of Proposition 3: Note that equation (3) can be written as $a = \frac{n-1}{n+1} \cdot (1 - \int_a^1 F(t)dt)$, hence:

$$a(F,n) - a(G,n) = \frac{n-1}{n+1} \cdot (\int_{a(G,n)}^1 G(t)dt - \int_{a(F,n)}^1 F(t)dt). \tag{6}$$

At $n = 2$ the LHS is positive, by assumption. Therefore, if $a(F,n) \geq a(G,n)$ then

[23]The uniqueness proof appears in Lemma 3's proof, in the appendix.

there exists an $n^* \in [2, n]$ such that $a(F, n^*) = a(G, n^*)$. Plugging n^* into (6) makes the LHS zero but the RHS positive—a contradiction. □

Lemma 1. *Let $(\beta_1, \cdots, \beta_n)$ be an equilibrium. Then each β_i is weakly increasing.*

Proof. Let p (resp. p') and t (resp. t') be the wining probability and expected payment of types θ_i (resp. θ_i') when they send their equilibrium bids, where $\theta_i > \theta_i'$. If $b_i(\theta_i') > b_i(\theta_i)$, then $p' > p$ and $t' > t$. Incentive compatibility implies:

$$p\theta_i - t \geq p'\theta_i - t',$$

and

$$p'\theta_i' - t' \geq p\theta_i' - t.$$

Rearranging these inequalities yields $t' - t \geq \theta_i(p' - p)$ and $t - t' \geq \theta_i'(p - p')$. Summing the rearranged inequalities yields $0 \geq (p' - p) \cdot (\theta_i - \theta_i') > 0$—a contradiction. □

Lemma 2. *Suppose that $M \geq 2n^2$. If $(\beta_1, \cdots, \beta_n)$ is an equilibrium, then $\frac{\sum_{i=1}^n \beta_i(\theta_i)}{n} \leq M$ for all $(\theta_1, \cdots, \theta_n)$.*

In words, Lemma 2 says that any bid-increase by a MUMB member increases the probability that the MUMB will win the auction.

Proof. By Lemma 1, it suffices to prove that $\beta_i(1) \leq M$. Assume by contradiction that there exists an i such that $\beta_i(1) > M$. Therefore, when type $\theta_i = 1$ submits his equilibrium report, the resulting bid of the MUMB is at least $\frac{M}{n}$, which means that: (i) if the MUMB wins, the expected price (conditional on winning) is at least $\frac{M}{2n}$, and (ii) the share of the price for which i is responsible is at least $\frac{1}{n}$. Thus, the price that i pays conditional on winning is at least $\frac{M}{2n^2} \geq 1$, which is impossible. □

Lemma 3. *Suppose that $M \geq 2n^2$. If $(\beta_1, \cdots, \beta_n)$ is an equilibrium, then $\beta_1 = \cdots = \beta_n = \beta$, where β satisfies $(2) - (3)$.*

Proof. Let $(\beta_1, \cdots, \beta_n)$ be an equilibrium. Let I be the set of players who follow a non-null bidding strategy. That is, $I \equiv \{i : \beta_i(\theta_i) > 0 \text{ for some } \theta_i\}$. Obviously, $I \neq \emptyset$. By Lemma 2, for each $i \in I$ the function β_i is positive-valued on $(a_i, 1]$, where $a_i \equiv \inf\{\theta_i : \beta_i(\theta_i) > 0\}$.

I argue that $|I| > 1$. To see this, assume by contradiction that I is a singleton, and, w.l.o.g, that $1 \notin I$. When type θ_1 reports r, his payoff, conditional on $(\theta_2, \cdots, \theta_n) = (t_2, \cdots, t_n)$, is[24]:

$$\frac{1}{M} \cdot \frac{r + \sum_{i \in I, i \neq 1} \beta_i(t_i)}{n} \cdot (\theta_1 - \frac{r}{r + \sum_{i \in I, i \neq 1} \beta_i(t_i)} \cdot \frac{r + \sum_{i \in I, i \neq 1} \beta_i(t_i)}{2n}) =$$
$$= \frac{1}{M} \cdot [(\frac{r + \sum_{i \in I, i \neq 1} \beta_i(t_i)}{n})\theta_1 - \frac{r^2}{2n^2} - \frac{r \sum_{i \neq 1} \beta_i(t_i)}{2n^2}].$$

Clearly, the objective is independent of M, so to ease the notation I assume, in what follows, that $M = 1$. Therefore, when type θ_1 reports r, his expected payoff is:

$$(\frac{r + \sum_{i \in I, i \neq 1} \mathbb{E}(\beta_i)}{n})\theta_1 - \frac{r^2}{2n^2} - \frac{r \sum_{i \in I, i \neq 1} \mathbb{E}(\beta_i)}{2n^2}. \tag{7}$$

The derivative of this expression w.r.t r is:

$$\frac{\theta_1}{n} - \frac{r}{n^2} - \frac{\sum_{i \in I, i \neq 1} \mathbb{E}(\beta_i)}{2n^2}. \tag{8}$$

If $I = \{j^*\}$ for some $j^* \neq 1$, then this j^* plays an ordinary second-price auction against the regular bidder, and therefore sends the report $n\theta_{j^*}$; therefore, $\sum_{i \in I, i \neq 1} \mathbb{E}(\beta_i) = \mathbb{E}(\beta_{j^*}) = nE$. Therefore, at $\theta_1 = 1$ and $r = 0$ the above derivative is equal to $\frac{1}{n} - \frac{E}{2n} > 0$, which implies that 1's bidding function is non-optimal for all large enough types of player 1. Therefore, $|I| > 0$.

Suppose that $1 \in I$. For $\theta_1 > a_1$, the first-order condition is:

$$\frac{\theta_1}{n} - \frac{r}{n^2} - \frac{\sum_{i \in I, i \neq 1} \mathbb{E}(\beta_i)}{2n^2} = 0.$$

[24]This payoff formula applies regardless of the cardinality of I.

The condition is satisfied at $r = \beta_1(\theta_1)$, hence:

$$\beta_1(\theta_1) = n\theta_1 - \frac{1}{2} \cdot \sum_{i \in I, i \neq 1} \mathbb{E}(\beta_i). \tag{9}$$

The analogous formula holds for any other $i \in I$. Therefore, the following holds for each $i \in I$:

$$\beta_i(\theta_i) = \begin{cases} 0 & \text{if } \theta_i < a_i \\ n\theta_1 - \frac{1}{2} \cdot \sum_{j \in I, j \neq i} \mathbb{E}(\beta_j) & \text{if } \theta_1 > a_1 \end{cases}$$

Consider type a_i. This type is indifferent between bidding zero and bidding $\beta_i(a_i)$. This type's expected payoff from bidding zero is a_i times the probability of winning: $a_i \cdot (\frac{\sum_{j \in I, j \neq i} \mathbb{E}(\beta_j)}{n})$. The expected utility from bidding $\beta_i(a_i)$ is $\frac{\beta_i(a_i) + \sum_{j \in I, j \neq i} \mathbb{E}(\beta_j)}{n} \cdot (a_i - \frac{\beta_i(a_i)}{2n})$. The indifference condition is:

$$a_i \cdot (\frac{\sum_{j \in I, j \neq i} \mathbb{E}(\beta_j)}{n}) = \frac{\beta_i(a_i) + \sum_{j \in I, j \neq i} \mathbb{E}(\beta_j)}{n} \cdot (a_i - \frac{\beta_i(a_i)}{2n}),$$

or:

$$\beta_i(a_i) \cdot (a_i - \frac{\beta_i(a_i)}{2n}) = (\frac{\sum_{j \in I, j \neq i} \mathbb{E}(\beta_j)}{2n}) \cdot \beta_i(a_i).$$

I argue that $\beta_i(a_i) = 0$. To see this, note that if $\beta_i(a_i) > 0$ then the above equation implies $a_i - \frac{\beta_i(a_i)}{2n} = \frac{\sum_{j \in I, j \neq i} \mathbb{E}(\beta_j)}{2n}$, which implies $\beta_i(a_i) = 2\beta_i(a_i)$, and therefore $\beta_i(a_i) = 0$.

It therefore follows that the following holds for all $i \in I$:

$$\beta_i(\theta_i) = \begin{cases} 0 & \text{if } \theta_i < a_i \\ n\theta_i - na_i & \text{if } \theta_i \geq a_i, \end{cases}$$

where $na_i = \frac{1}{2} \cdot \sum_{j \in I, j \neq i} \mathbb{E}(\beta_j)$.

Note that $\mathbb{E}(\beta_i) = \int_{a_i}^{1} (nt - na_i) f(t) dt = n \int_{a_i}^{1} t f(t) dt - na_i(1 - F(a_i))$; therefore:

$$\sum_{j\in I, j\neq 1} \mathbb{E}(\beta_j) = n \sum_{j\in I, j\neq 1} \{\int_{a_i}^{1} tf(t)dt - a_i(1-F(a_i))\} = 2na_1,$$

or:

$$\sum_{j\in I, j\neq 1} \{\int_{a_i}^{1} tf(t)dt - a_i(1-F(a_i))\} = 2a_1.$$

Since $|I| > 1$, assume, w.l.o.g, that $2 \in I$. Then:

$$\sum_{j\in I, j\neq 2} \{\int_{a_i}^{1} tf(t)dt - a_i(1-F(a_i))\} = 2a_2.$$

I argue that $a_1 = a_2$. To see this, assume by contradiction, w.l.o.g, that $a_1 > a_2$. Therefore, the above equations imply:

$$\int_{a_2}^{a_1} tf(t)dt - a_2(1-F(a_2)) + a_1(1-F(a_1)) = 2(a_1 - a_2).$$

At $a_1 = a_2$ both sides are equal to zero; the derivative of the LHS w.r.t. a_1 is $1 - F(a_1) \leq 1$ and that of the RHS is 2, hence $a_1 = a_2$ is the unique solution, in contradiction to $a_1 > a_2$. It follows that there exists an a such that $a_i = a$ for all $i \in I$, and therefore all bid functions coincide; the equilibrium bid function, β, is given by:

$$\beta(\theta) = \begin{cases} 0 & \text{if } \theta < a \\ n\theta - na & \text{if } \theta \geq a, \end{cases}$$

where $a = \frac{n-1}{2n} \cdot \mathbb{E}(\beta)$, or $\mathbb{E}(\beta) = \frac{2na}{n-1}$. This condition can be written as:

$$\mathbb{E}(\beta) = \int_{a}^{1} (nt - na)f(t)dt = n\int_{a}^{1} tf(t)dt - na(1-F(a)) = \frac{2na}{n-1},$$

or:

$$\int_{a}^{1} tf(t)dt = a \cdot (\frac{n+1}{n-1}) - aF(a).$$

Note that $a = 0$ the LHS exceeds the RHS, and at $a = 1$ the converse holds; therefore, a solution, a, exists. To see that it is unique, assume by contradiction that there exists a $b \neq a$ such that:

$$\int_b^1 tf(t)dt = b \cdot (\frac{n+1}{n-1}) - bF(b).$$

Suppose, w.l.o.g, that $b > a$. The above equations imply $\int_a^b tf(t)dt = (a-b) \cdot (\frac{n+1}{n-1}) - aF(a) + bF(b)$. At $b = a$ both sides are equal to zero, the derivative of the LHS w.r.t b is $bf(b)$ and that of the RHS is $-(\frac{n+1}{n-1}) + F(b) + bf(b) < bf(b)$, hence the solution is unique.

Finally, it remains to show that $|I| = n$. To see this, assume by contradiction that $|I| = k < n$ and let i be such that $i \notin I$. For $\theta_i = 1$ and $r = 0$, the (counterpart of the) derivative (8) is $\frac{1}{n} - \frac{k\mathbb{E}(\beta)}{2n^2} > \frac{1}{n} - \frac{\mathbb{E}(\beta)}{2n}$. Since $\mathbb{E}(\beta) = \frac{2na}{n-1}$, the RHS of the last inequality is equal to $\frac{1}{n} - \frac{a}{n-1}$. Thus, to complete the proof it is enough to show that $a < \frac{n-1}{n}$. This is indeed the case, because of Proposition 1. □

Proof of Theorem 1: By Lemma 3, if $(\beta_1, \cdots, \beta_n)$ is an equilibrium, then $\beta_i = \beta$ for all i and (2) – (3) hold. Conversely, consider β such that (2) – (3) hold. The arguments from Lemma 1's proof establish that this is an equilibrium; for types $\theta > x$ the FOC is satisfied at $r = \beta(\theta)$ and for types $\theta < x$ not participating is optimal, because the derivative of their objective function w.r.t. r is negative. □

Proof of Proposition 2: Consider first $\theta \leq a$. The probability that he wins the auction is $\frac{1}{M}\int_a^1 (t-a)dt = \frac{1}{M}[\int_a^1 tf(t) - a(1-F(a))] = \frac{1}{M}[3a - aF(a) - a + aF(a)] = \frac{2a}{M}$. Consider now $\theta > a$. His expected utility is:

$$\frac{F(a)}{2M}(\theta^2 - a^2) + \int_a^1 [(\frac{2\theta + 2t - 4a}{2M})(\theta - \frac{2\theta + 2t - 4a}{4} \cdot \frac{2\theta - 2a}{2\theta + 2t - 4a})]f(t)dt.$$

Simplifying this expression yields $\frac{1}{M}[\frac{1}{2}(\theta^2 - a^2) + \theta a + a^2]$. □

Proof of Theorem 2: Verifying that the aforementioned profile is an equilibrium is easy, so I only prove uniqueness. Suppose that $(\beta_1^{FPA}, \cdots, \beta_n^{FPA})$ is an equilibrium. Let I be the set of players who follow a non-null bidding strategy. For each $i \in I$ the function β_i^{FPA} is positive-valued on $(a_i, 1]$, where $a_i \equiv \inf\{\theta_i : \beta_i^{FPA}(\theta_i) > 0\}$.

I argue that $|I| > 1$. To see this, assume by contradiction that I is a singleton, and, w.l.o.g, that $1 \notin I$. When type θ_1 reports r, his payoff, conditional on $(\theta_2, \cdots, \theta_n) = (t_2, \cdots, t_n)$, is[25]:

$$\frac{1}{M} \cdot \frac{r + \sum_{i\in I, i\neq 1} \beta_i^{FPA}(t_i)}{n} \cdot (\theta_1 - \frac{r}{r + \sum_{i\in I, i\neq 1} \beta_i^{FPA}(t_i)} \cdot \frac{r + \sum_{i\in I, i\neq 1} \beta_i^{FPA}(t_i)}{n}) =$$
$$= \frac{1}{M} \cdot [(\frac{r + \sum_{i\in I, i\neq 1} \beta_i^{FPA}(t_i)}{n})\theta_1 - \frac{r^2}{n^2} - \frac{r\sum_{i\neq 1} \beta_i^{FPA}(t_i)}{n^2}].$$

Clearly, the objective is independent of M, so to ease the notation I assume, in what follows, that $M = 1$. Therefore, when type θ_1 reports r, his expected payoff is:

$$(\frac{r + \sum_{i\in I, i\neq 1} \mathbb{E}(\beta_i^{FPA})}{n})\theta_1 - \frac{r^2}{n^2} - \frac{r\sum_{i\in I, i\neq 1} \mathbb{E}(\beta_i^{FPA})}{n^2}. \tag{10}$$

The derivative of this expression w.r.t r is:

$$\frac{\theta_1}{n} - \frac{2r}{n^2} - \frac{\sum_{i\in I, i\neq 1} \mathbb{E}(\beta_i^{FPA})}{n^2}. \tag{11}$$

If $I = \{j^*\}$ for some $j^* \neq 1$, then this j^* plays an ordinary first-price auction against the uniformly-distributed regular bidder, and therefore sends the report $\frac{n}{2}\theta_{j^*}$; therefore, $\sum_{i\in I, i\neq 1} \mathbb{E}(\beta_i^{FPA}) = \mathbb{E}(\beta_{j^*}^{FPA}) = \frac{n}{2}E$. Therefore, at $\theta_1 = 1$ and $r = 0$ the above derivative is equal to $\frac{1}{n} - \frac{E}{2n} > 0$, which implies that 1's bidding function is non-optimal for all large enough types of player 1. Therefore, $|I| > 0$.

Suppose that $1 \in I$. For $\theta_1 > a_1$, the first-order condition is:

[25] The assumption $M \geq n^2$ implies an analog of Lemma 2, hence the payoff formula applies. Note that the formula applies regardless of the cardinality of I.

$$\frac{\theta_1}{n} - \frac{2r}{n^2} - \frac{\sum_{i\in I, i\neq 1}\mathbb{E}(\beta_i^{FPA})}{n^2} = 0.$$

The condition is satisfied at $r = \beta_1^{FPA}(\theta_1)$, hence:

$$\beta_1^{FPA}(\theta_1) = \frac{n}{2}\theta_1 - \frac{1}{2} \cdot \sum_{i\in I, i\neq 1}\mathbb{E}(\beta_i^{FPA}). \tag{12}$$

The analogous formula holds for any other $i \in I$. Therefore, the following holds for each $i \in I$:

$$\beta_i^{FPA}(\theta_i) = \begin{cases} 0 & \text{if } \theta_i < a_i \\ \frac{n}{2}\theta_1 - \frac{1}{2} \cdot \sum_{j\in I, j\neq i}\mathbb{E}(\beta_i^{FPA}) & \text{if } \theta_1 > a_1 \end{cases}$$

Consider type a_i. This type is indifferent between bidding zero and bidding $\beta_i^{FPA}(a_i)$. This type's expected payoff from bidding zero is a_i times the probability of winning: $a_i \cdot (\frac{\sum_{j\in I, j\neq i}\mathbb{E}(\beta_i^{FPA})}{n})$. The expected utility from bidding $\beta_i^{FPA}(a_i)$ is $\frac{\beta_i^{FPA}(a_i)+\sum_{j\in I, j\neq i}\mathbb{E}(\beta_j)}{n} \cdot (a_i - \frac{\beta_i^{FPA}(a_i)}{n})$. The indifference condition is:

$$a_i \cdot (\frac{\sum_{j\in I, j\neq i}\mathbb{E}(\beta_i^{FPA})}{n}) = \frac{\beta_i^{FPA}(a_i) + \sum_{j\in I, j\neq i}\mathbb{E}(\beta_i^{FPA})}{n} \cdot (a_i - \frac{\beta_i^{FPA}(a_i)}{n}),$$

or:

$$\beta_i^{FPA}(a_i) \cdot (a_i - \frac{\beta_i^{FPA}(a_i)}{n}) = (\frac{\sum_{j\in I, j\neq i}\mathbb{E}(\beta_j^{FPA})}{n}) \cdot \beta_i^{FPA}(a_i).$$

I argue that $\beta_i^{FPA}(a_i) = 0$. To see this, note that if $\beta_i^{FPA}(a_i) > 0$ then the above equation implies $a_i - \frac{\beta_i^{FPA}(a_i)}{n} = \frac{\sum_{j\in I, j\neq i}\mathbb{E}(\beta_j^{FPA})}{n}$, which implies $\beta_i^{FPA}(a_i) = 2\beta_i^{FPA}(a_i)$, and therefore $\beta_i^{FPA}(a_i) = 0$.

It therefore follows that the following holds for all $i \in I$:

$$\beta_i^{FPA}(\theta_i) = \begin{cases} 0 & \text{if } \theta_i < a_i \\ \frac{n}{2}\theta_i - \frac{n}{2}a_i & \text{if } \theta_i \geq a_i, \end{cases}$$

where $\frac{n}{2}a_i = \frac{1}{2} \cdot \sum_{j \in I, j \neq i} \mathbb{E}(\beta_j^{FPA})$.

Note that $\mathbb{E}(\beta_i^{FPA}) = \frac{1}{2}\int_{a_i}^{1}(nt - na_i)f(t)dt = \frac{n}{2}\int_{a_i}^{1} tf(t)dt - \frac{n}{2}a_i(1 - F(a_i))$. Therefore:

$$\sum_{j \in I, j \neq 1} \mathbb{E}(\beta_j^{FPA}) = \frac{n}{2} \sum_{j \in I, j \neq 1} \{\int_{a_i}^{1} tf(t)dt - a_i(1 - F(a_i))\} = na_1,$$

or:

$$\sum_{j \in I, j \neq 1} \{\int_{a_i}^{1} tf(t)dt - a_i(1 - F(a_i))\} = 2a_1.$$

Since $|I| > 1$, assume, w.l.o.g, that $2 \in I$. Then:

$$\sum_{j \in I, j \neq 2} \{\int_{a_i}^{1} tf(t)dt - a_i(1 - F(a_i))\} = 2a_2.$$

By the same argument given in the proof of Theorem 1, $a_1 = \cdots = a_n$. Therefore, the equilibrium bid function, β^{FPA}, is given by:

$$\beta(\theta) = \begin{cases} 0 & \text{if } \theta < a \\ \frac{n}{2}\theta - \frac{n}{2}a & \text{if } \theta \geq a, \end{cases}$$

where $a = \frac{n-1}{n} \cdot \mathbb{E}(\beta^{FPA})$, or $\mathbb{E}(\beta^{FPA}) = \frac{na}{n-1}$. This condition can be written as:

$$\mathbb{E}(\beta^{FPA}) = \frac{1}{2}\int_{a}^{1}(nt - na)f(t)dt = \frac{n}{2}\int_{a}^{1} tf(t)dt - \frac{n}{2}a(1 - F(a)) = \frac{na}{n-1},$$

or:

$$\int_{a}^{1} tf(t)dt = a \cdot (\frac{n+1}{n-1}) - aF(a).$$

By Theorem 1, this equation has a unique solution—the cutoff from the second-price model. By the argument from Theorem 1's proof it also follows that $|I| = n$. $\square$

I now turn to proving Theorems 3 and 4. In either one, existence of a symmetric equilibrium follows from Reny (2011), and the fact that the equilibrium bid function is weakly increasing follows from standard arguments (see Lemma 1); it remains to show that, under either auction format, low enough types bid zero.

Proof of Theorem 3: Assume the general second-price model. Let β be the symmetric equilibrium bid function, and suppose that all $j \neq i$ follow this function. When player i with type θ sends the report r, his expected utility is:

$$\int \{H(A(r, \beta_{-i}(t_{-i})))\theta - s_i(r, \beta_{-i}(t_{-i}) \int_0^{A(r,\beta_{-i}(t_{-i}))} th(t)dt\} f_{-i}(t_{-i})dt_{-i}.$$

The derivative of this expression w.r.t r is:

$$\int \{h(A(r, \beta_{-i}(t_{-i}))) \cdot A_1(r, \beta_{-i}(t_{-i})) \cdot \theta - \frac{\partial}{\partial r}[s_i(r, \beta_{-i}(t_{-i}) \int_0^{A(r,\beta_{-i}(t_{-i}))} th(t)dt]\} f_{-i}(t_{-i})dt_{-i}.$$

At $\theta = 0$ this derivative is $-\int\{\frac{\partial}{\partial r}[s_i(r, \beta_{-i}(t_{-i}) \int_0^{A(r,\beta_{-i}(t_{-i}))} th(t)dt]\} f_{-i}(t_{-i})dt_{-i} < 0$. □

Proof of Theorem 4: The arguments are identical to those give in the proof of Theorem 3, except that that are applied to the expected utility $\int\{H(A(r, \beta_{-i}(t_{-i})))\theta - s_i(r, \beta_{-i}(t_{-i})A(r, \beta_{-i}(t_{-i}))\} f_{-i}(t_{-i})dt_{-i}$ □

Proof of Theorem 5: Make the theorem's assumptions and let β be the bid function of a symmetric equilibrium in the second-price model. Assume by contradiction that $\frac{1}{2} \cdot \beta$ is not a symmetric equilibrium function in the first-price model. Then there exists a player i, a type θ and a bids x such that:

$$\int A(x, \frac{1}{2}\beta_{-i}(t_{-i})) \cdot [\theta - s_i(x, \frac{1}{2}\beta_{-i}(t_{-i})) \cdot A(x, \frac{1}{2}\beta_{-i}(t_{-i}))] f_{-i}(t_{-i}) dt_{-i} >$$

$$> \int A(\frac{1}{2}\beta(\theta), \frac{1}{2}\beta_{-i}(t_{-i})) \cdot [\theta - s_i(\frac{1}{2}\beta(\theta), \frac{1}{2}\beta_{-i}(t_{-i})) \cdot A(\frac{1}{2}\beta(\theta), \frac{1}{2}\beta_{-i}(t_{-i}))] f_{-i}(t_{-i}) dt_{-i} =$$

$$= \frac{1}{2} \int A(\beta(\theta), \beta_{-i}(t_{-i})) \cdot [\theta - s_i(\beta(\theta), \beta_{-i}(t_{-i})) \cdot \frac{A(\beta(\theta), \beta_{-i}(t_{-i}))}{2}] f_{-i}(t_{-i}) dt_{-i} \geq$$

$$\geq \frac{1}{2} \int A(2x, \beta_{-i}(t_{-i})) \cdot [\theta - s_i(2x, \beta_{-i}(t_{-i})) \cdot \frac{A(2x, \beta_{-i}(t_{-i}))}{2}] f_{-i}(t_{-i}) dt_{-i} =$$

$$= \int A(x, \frac{1}{2}\beta_{-i}(t_{-i})) \cdot [\theta - s_i(x, \frac{1}{2}\beta_{-i}(t_{-i})) \cdot A(x, \frac{1}{2}\beta_{-i}(t_{-i}))] f_{-i}(t_{-i}) dt_{-i},$$ [26]

which is a contradiction. The analogous considerations establish the implication in the other direction. □

Proof of Proposition 7: W.l.o.g suppose that $n = 2$.[27] Let player 2 be the active member who bids θ_2, as he would have done in an ordinary IPV second-price auction. I will show that, given this behavior of player 2, it is optimal for player 1 to refrain from bidding, no matter his type.

To see this, consider the case where player 1 sends the report x. Conditional on a particular value of θ_2 that satisfies $\theta_2 < x$, player 1's expected utility is $\frac{x}{M}(\theta_1 - \frac{x}{2} \cdot \frac{x}{x+\theta_2})$; given a particular θ_2 that satisfies $\theta_2 > x$, the utility is $\frac{\theta_2}{M}(\theta_1 - \frac{\theta_2}{2} \cdot \frac{x}{x+\theta_2})$. Therefore, player 1's expected utility from reporting x is:

$$\frac{1}{M} \int_0^x x(\theta_1 - \frac{x}{2} \cdot \frac{x}{x+\theta_2}) f(\theta_2) d\theta_2 + \frac{1}{M} \int_x^1 \theta_2(\theta_1 - \frac{\theta_2}{2} \cdot \frac{x}{x+\theta_2}) f(\theta_2) d\theta_2.$$

I will show that the maximal type, $\theta_1 = 1$, finds it optimal to refrain from bidding. That is, the following holds for all $x > 0$.

Consider first $x \leq 1$. Here, the following needs to be proved:

[27] This assumptions is w.l.o.g because of the bid aggregation rule and the cost sharing rule, both of which are insensitive to the number of players; in other words, sustaining the profile as equilibrium for $n = 2$ implies sustainability for any n.

$$\frac{1}{M}\int_0^x x(1-\frac{x}{2}\cdot\frac{x}{x+\theta_2})f(\theta_2)d\theta_2+\frac{1}{M}\int_x^1 \theta_2(1-\frac{\theta_2}{2}\cdot\frac{x}{x+\theta_2})f(\theta_2)d\theta_2\leq\frac{E}{M}.$$

Thus, in this case it is enough to show that:

$$\int_0^x x(1-\frac{x}{2}\cdot\frac{x}{x+\theta_2})f(\theta_2)d\theta_2+\int_x^1 \theta_2 f(\theta_2)d\theta_2\leq\int_0^1 \theta_2 f(\theta_2)d\theta_2,$$

or:

$$\int_0^x x(1-\frac{x}{2}\cdot\frac{x}{x+\theta_2})f(\theta_2)d\theta_2\leq\int_0^x \theta_2 f(\theta_2)d\theta_2.$$

The above is implied by the following (θ_2 is the denominator is replaced by x):

$$\int_0^x x(1-\frac{x}{4})f(\theta_2)d\theta_2\leq\int_0^x \theta_2 f(\theta_2)d\theta_2.$$

This inequality holds as equality at $x=0$, hence it is enough to show that the derivative of its RHS w.r.t x is greater than the one of the LHS:

$$\int_0^x (1-\frac{x}{2})f(\theta_2)d\theta_2+x(1-\frac{x}{4})f(x)\leq xf(x),$$

or:

$$4\int_0^x (1-\frac{x}{2})f(\theta_2)d\theta_2\leq xf(x).$$

Again, this is true when $x=0$, so it is enough to prove the ordering of derivatives: $-2F(x)+4(1-\frac{x}{2})f(x)\leq f(x)+xf'(x)$, or $3f(x)\leq 2F(x)+2xf(x)+xf'(x)$, which holds by assumption.

Now consider a deviation to $x>1$. What needs to be proved is that:

$$\int_0^1 x(1-\frac{x^2}{2(x+\theta_2)})f(\theta_2)d\theta_2\leq E,$$

so it is enough to establish $\int_0^1 x(1-\frac{x^2}{2(x+1)})f(\theta_2)d\theta_2\leq E$. It is easy to check that the integrand, $x-\frac{x^3}{2(x+1)}$, is smaller than $\frac{3}{4}$. □

It needs to be shown that this expression is maximized at $x = 0$, namely that the following holds for all $x \in [0, 1]$:[28]

$$\int_0^x \theta_2^{\alpha-1}[x\theta_1 - \frac{x^3}{2(x+\theta_2)}]d\theta_2 + \int_x^1 \theta_2^{\alpha}[\theta_1 - \frac{x\theta_1}{2(x+\theta_2)}]d\theta_2 \leq \int_0^1 \theta_1\theta_2^{\alpha}d\theta_2.$$

The LHS is bounded from above by:

$$\int_0^x \theta_2^{\alpha-1}[x\theta_1 - \frac{x^3}{2(1+\theta_2)}]d\theta_2 + \int_x^1 \theta_2^{\alpha}[\theta_1 - \frac{x\theta_1}{2(1+\theta_2)}]d\theta_2 =$$

$$= \frac{x^{\alpha+1}}{\alpha}\theta_1 + (\frac{1-x^{\alpha+1}}{\alpha+1})\theta_1 - \int_0^x \theta_2^{\alpha-1}\frac{x^3}{2(1+\theta_2)}d\theta_2 - \int_x^1 \theta_2^{\alpha}\frac{x\theta_1}{2(1+\theta_2)}d\theta_2 \leq$$

$$\leq \frac{x^{\alpha+1}}{\alpha}\theta_1 + (\frac{1-x^{\alpha+1}}{\alpha+1})\theta_1 - \min\{x^3, x\theta_1\}\int_0^1 \frac{\theta_2^{\alpha}}{2(1+\theta_2)}d\theta_2.$$

Therefore, it is enough to prove that:

$$\frac{x^{\alpha+1}}{\alpha}\theta_1 + (\frac{1-x^{\alpha+1}}{\alpha+1})\theta_1 - \min\{x^3, x\theta_1\}\int_0^1 \frac{\theta_2^{\alpha}}{2(1+\theta_2)}d\theta_2 \leq \int_0^1 \theta_1\theta_2^{\alpha},$$

or:

$$\frac{x^{\alpha+1}}{\alpha}\theta_1 + (\frac{1-x^{\alpha+1}}{\alpha+1})\theta_1 - \min\{x^3, x\theta_1\}\int_0^1 \frac{\theta_2^{\alpha}}{2(1+\theta_2)}d\theta_2 \leq \frac{\theta_1}{\alpha+1}.$$

Therefore, it is enough to prove that:

$$\frac{\theta_1 x^{\alpha+1}}{\alpha(\alpha+1)} + \frac{\theta_1}{\alpha+1} - \frac{\min\{x^3, x\theta_1\}}{4}\int_0^1 \theta_2^{\alpha}d\theta_2 \leq \frac{\theta_1}{\alpha+1},$$

or:

$$\frac{4\theta_1 x^{\alpha+1}}{\alpha} \leq \min\{x^3, x\theta_1\}. \tag{13}$$

If $x^3 < x\theta_1$ then (13) becomes $4\theta_1 x^{\alpha-2} \leq \alpha$; otherwise, it becomes $4x^{\alpha} \leq \alpha$. In either case, the desired inequality is satisfied if $\alpha \geq 4$. □

[28]Because of the bid aggregation rule, $x > 1$ is clearly not optimal.

Proof of Proposition 8: Make the proposition's assumptions, and assume by contradiction that there exists an equilibrium with, w.l.o.g, $\beta_i \equiv 0$ for $i = 1, \cdots, n-1$.

Let $\phi(\theta)$ be such that $A(0, \cdots, 0, \phi(\theta)) = \theta$. Clearly, player n's best-response is to bid according to the function $\phi(.)$. Now, consider, w.l.o.g, player 1's. His expected payoff in this equilibrium is $\theta_1 \cdot \int_0^1 H(t)f(t)dt$.

Let $\bar{r}$ denotes a bid that results in sure winning for the MUMB if it is submitted by one of its members, regardless of the behavior of the other members. Therefore, when the maximal type reports $\bar{r}$ his expected utility is:

$$\int_0^1 \{1 - s_1(\bar{r}, t) \int_0^M zh(z)dz\} f(t)dt.$$

Therefore, in order to show that $\theta_1 = 1$ has a profitable deviation, it suffices to show that:

$$\int_0^1 \{1 - s_1(\bar{r}, t) \int_0^M zh(z)dz\} df(t) > \int_0^1 H(t)f(t)dt.$$

Applying integration by parts to the RHS, it follows that the above inequality is equivalent to:

$$1 - \bar{s}_1 \cdot \int_0^M zh(z)dz > 1 - \int_0^1 F(t)h(t)dt,$$

where $\bar{s}_1$ is the expected value of s_1. Since $M \geq 1$ and $\bar{s}_1 \in (0, 1)$ it is enough to show that:

$$1 - \int_0^1 zh(z)dz \geq 1 - \int_0^1 F(t)h(t)dt,$$

or $\int_0^1 F(t)h(t)dt \geq \int_0^1 zh(z)dz$, which holds because F is first order stochastically dominated by the uniform distribution. $\square$

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