Lexicographic Choice under Variable Capacity Constraints*

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Abstract

In several matching markets, in order to achieve diversity, agents' priorities are allowed to vary across available seats of each institution and each institution is let to choose agents in a lexicographic fashion based on a predetermined ordering of the seats, called a *lexicographic choice rule*. Lexicographic choice rules have been particularly useful in achieving diversity at schools while allocating school seats. We provide a characterization of lexicographic choice rules, which reveals their distinguishing properties from other plausible choice rules. We discuss some implications for the Boston school choice system. We also provide a characterization of deferred acceptance mechanisms that operate based on a lexicographic choice structure.

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Contents

1 Introduction

Many real-life resource allocation problems involve allocation of objects which are available in a limited number of identical copies, or a *capacity*. Choice rules, which are systemic ways of determining how to ration available copies when demand exceeds the capacity, are essential in the analysis of such problems. A well-known example is the school choice problem in which each school has a certain number of seats to be allocated among students. Although student preferences are elicited from the students, endowing each school with a choice rule is an essential part of the design process.

Which choice rule to use is not always evident. The school choice literature, starting with the seminal study by Abdulkadiroğlu and Sönmez (2003), has widely focused on problems where each school is already endowed with a priority ordering over students, and the choice rule of a school only needs to be responsive to the given priority ordering, in which case the choice rule to be used is clear: a *responsive choice rule*.[1] However, when there are additional concerns such as achieving a diverse student body or affirmative action, which choice rule to use is non-trivial.

In order to achieve a diverse student body, many school districts have been implementing affirmative action policies, such as in Boston, Chicago, and Jefferson County. The affirmative action policies that are in use in several school districts reveal that a natural way to achieve diversity is to allow students' priorities to vary across a school's seats, and to let the school choose students in a lexicographic fashion based on a predetermined ordering of the seats, which amounts to using a *lexicographic choice rule*. Although some properties of lexicographic choice rules have already been discovered in the literature, which set of properties distinguish lexicographic choice rules from other plausible choice rules has remained unknown.[2] In this study, we follow the axiomatic approach to discover general principles (axioms) that characterize *lexicographic choice rules* under *variable capacity constraints*, and then explore the implications of our results for school choice and other resource allocation problems.

In our baseline model, we consider a single decision maker who has a capacity

[1] In Section 3.1, we discuss responsive choice rules.

[2] Although lexicographic choice rules are used to achieve diversity in school choice, there are also other types of choice rules that are used to achieve diversity or affirmative action. Among others, Echenique and Yenmez (2015a) and Ehlers et al. (2014) study such choice rules.

constraint, such as a school with a limited number of seats. The decision maker encounters (capacity-constrained) choice problems which consist of a choice set (a set of alternatives, such as students who demand a seat at the school) and a capacity. A (capacity-constrained) choice rule, at each possible choice problem, chooses some alternatives from the choice set without exceeding the capacity. Note that across different choice problems, we allow capacity to vary, since in applications capacity may vary over time (for instance, the number of available seats at a school may change from year to year) and it may be desirable to use a choice rule that responds well to changes in capacity.[3]

Our main focus is on lexicographic choice rules. A choice rule is *lexicographic* if there exists a list of priority orderings over potential alternatives such that at each choice problem, the set of chosen alternatives is obtainable by choosing the highest ranked alternative according to the first priority ordering, then choosing the highest ranked alternative among the remaining alternatives according to the second priority ordering, and proceeding similarly until the capacity is full or no alternative is left. Our main goal is to understand the characterizing properties of lexicographic choice rules.

In the axiomatic analysis of choice rules, each axiom carries with it a consistency requirement or specifies a desirable procedural aspect of a choice rule. These axioms illuminate characteristics of choice rules that may be relevant for the problem, yet may not be evident from the procedural formulations of the choice rules. We consider the set of axioms characterizing a choice rule as a justification for using that choice rule, besides highlighting its distinguishing features from other plausible choice rules. As for the applications, such as school choice, axiomatic characterizations of choice rules pave the way for the schools to choose an appropriate rule that fits their policy desiderata expressed in the form of choice axioms.[4]

We consider the following three properties of choice rules that have already been studied in the axiomatic literature.

Acceptance: An alternative is rejected from a choice set at a capacity only if the capacity is full;

[3] There are earlier studies in the literature which also formulate choice rules by allowing capacity to vary. See, among others, Doğan and Klaus (2016), Ehlers and Klaus (2014), and Ehlers and Klaus (2016).

[4] Echenique and Yenmez (2015a) also follow an axiomatic approach and characterize several choice rules for a school that wants to achieve diversity.

Gross substitutes: If an alternative is chosen from a choice set at a capacity, then it is also chosen from any subset of the choice set that contains the alternative, at the same capacity.

Monotonicity: If an alternative is chosen from a choice set at a capacity, then it is also chosen from the same choice set at any higher capacity.

We introduce a property that requires consistency of the following capacity-wise revealed preference relation: an alternative is *revealed preferred* to another alternative at a capacity if there is a choice set from which the former alternative is chosen over the latter, whereas with one less capacity both alternatives are rejected from the choice set. We say that a choice rule satisfies the *capacity-wise weak axiom of revealed preference (CWARP)* if the revealed preference relation is asymmetric at each capacity.

CWARP is a counterpart of the well-known *weak axiom of revealed preference* (WARP) in the standard revealed preference framework (Samuelson, 1938), where there is no capacity parameter, and a choice rule chooses exactly one alternative from each choice set. In the standard framework, an alternative is said to be revealed preferred to another alternative if there is a choice set at which the former alternative is chosen over the latter. *WARP* requires the revealed preference relation to be asymmetric, which in a sense requires consistency of the choice behavior in responding to changes in the choice set. In our framework, the preference is revealed not only through the choice at a choice set, but also through a change in the capacity. Hence, *CWARP* requires consistency of the choice behavior in responding to changes in the choice set together with changes in the capacity.

We show that a choice rule satisfies *acceptance*, *gross substitutes*, *monotonicity*, and *CWARP* if and only if it is *lexicographic*: there is a list of priority orderings over alternatives such that at each problem, the set of chosen alternatives is obtainable by choosing the highest ranked alternative according to the first priority, then choosing the highest ranked alternative among the remaining alternatives according to the second priority, and proceeding similarly until the capacity is full or no alternative is left (Theorem 1).

We also provide an alternative characterization of lexicographic choice rules with another property that we introduce, called *rejection-monotonicity. Rejection-monotonicity* requires that, if the set of rejected alternatives are the same for two choice sets at a capacity, then at any higher capacity, the set of accepted alternatives that were formerly rejected should be the same for the two choice sets. In other words, in case of an increase in the capacity, *rejection-monotonicity* requires that the new alternatives that will be chosen (if any) should not depend on the already accepted alternatives. *CWARP* together with *acceptance* and *monotonicity* implies *rejection-monotonicity*. As a corollary to our Theorem 1, we show that a choice rule satisfies *acceptance*, *gross substitutes*, *monotonicity*, and *rejection-monotonicity* if and only if it is *lexicographic*.

Boston school district is one of the school districts that uses lexicographic choice to achieve a diverse student body and implement affirmative action policies. Boston school district aims to give priority to neighborhood applicants for half of each school's seats. To achieve this goal, the Boston school district has been using a deferred acceptance mechanism based on a choice structure, where each school is endowed with a "capacity-wise lexicographic" choice rule, that is, at each capacity, the choice rule lexicographically operates based on a list containing as many priority orderings as the capacity, yet the lists for different capacity levels do not have to be related in any way.[5] Dur et al. (2013) provides an analysis of how the order of the priority orderings in the choice rule of a school may affect the outcome in the Boston school choice context. In Section 4, we consider a class of capacity-wise lexicographic choice rules discussed in Dur et al. (2013) that are relevant for the design of the Boston school choice system and show that our analysis enables us to single out one rule from four plausible candidates.

Besides providing a first axiomatic foundation for lexicographic choice rules, our study contributes to the literature on allocation mechanisms that are based on lexicographic choice structures. To illustrate this contribution, in Section 5, we consider the variable-capacity object allocation model. In that model, Ehlers and Klaus (2016) characterize deferred acceptance mechanisms where each object has a choice rule that satisfies *acceptance*, *gross substitutes*, and *monotonicity*.[6] We introduce a novel property for allocation mechanisms, called *demand-monotonicity*, which is motivated by and intimately related to the *rejection-monotonicity* of choice rules. As a corollary to the characterization result by Ehlers and Klaus (2016), we provide a characterization of lexicographic deferred acceptance mechanisms (Corollary 4), which are deferred acceptance mechanisms that operate based

[5]See Dur et al. (2013) for a detailed discussion of Boston's school choice mechanism.

[6]Kojima and Manea (2010) consider a setup where the capacity of each school is fixed, and characterize deferred acceptance mechanisms where each school has a choice rule that satisfies *acceptance* and *gross substitutes*.

on a lexicographic choice structure.

The paper is organized as follows. In Section 2, we review the related literature. In Section 3, we introduce and characterize lexicographic choice rules, and also provide a characterization of responsive choice rules. In Section 4, we discuss some implications for the Boston school choice system. In Section 5, we highlight an implication of our choice theoretical analysis for the resource allocation framework: we provide a characterization of deferred acceptance mechanisms that operate based on a lexicographic choice structure. In Section 6, we conclude by discussing the main features of our analysis.

2 Related Literature

Several studies investigate choice rules that satisfy *path independence* (Plott, 1973), which requires that if the choice set is "split up" into smaller sets, and if the choices from the smaller sets are collected and a choice is made from the collection, the final result should be the same as the choice from the original choice set. Since acceptance together with gross substitutes imply path independence,[7] lexicographic choice rules are examples of path independent choice rules. Aizerman and Malishevski (1981) show that for each path independent choice rule, there exists a list of priority orderings such that the choice from each choice set is the union of the highest priority alternatives in the priority orderings.[8] Among others, Plott (1973), Moulin (1985), and Johnson and Dean (2001) study the structure of path independent choice rules. Path independent choice rules guarantee the existence of stable matchings in the matching context. Chambers and Yenmez (2016) study path independence in the matching context and its connection to stable matchings.

Although the structure of path independent choice rules have been extensively studied, the structure of lexicographic choice rules and what properties distinguish them from other path independent choice rules have not been well-understood. Houy and Tadenuma (2009) consider two classes of choice rules which are both based on "lexicographic procedures", yet different than the ones we consider here. Similar to our setup, choice rules that they

[7] This is also noted in Remark 1 of Doğan and Klaus (2016), and it follows from Lemma 1 of Ehlers and Klaus (2016) together with Corollary 2 of Aizerman and Malishevski (1981).

[8] In the words of Aizerman and Malishevski (1981), each *path independent* choice rule is generated by some mechanism of collected extremal choice.

consider operate based on a list of binary relations.[9] Yet, their model does not include capacity constraints and the lexicographic procedures that operationalize the lists are different. The only study that considers lexicographic choice rules that we study from an axiomatic perspective is Chambers and Yenmez (2013).[10] They show that lexicographic choice rules satisfy *acceptance* and *path independence*, and they also show that there are *path independent* choice rules that are not lexicographic, but they do not provide a characterization of lexicographic choice rules.

Lexicographic choice rules are used in the Boston School Choice system to favor neighbourhood students, and Dur et al. (2013) and Dur et al. (2016) analyse how the order in the priority profile may cause additional biase for or against the neighbourhood students.[11] In Section 4, we discuss implications of our study for the Boston School Choice system.

Kominers and Sönmez (2016) study lexicographic deferred acceptance mechanisms in a more general matching with contracts framework (Hatfield and Milgrom, 2005). Their main focus is on stability and incentive properties of such mechanisms, our focus is on understanding the characterizing properties of such mechanisms and their underlying choice structures. Moreover, the structure of the lexicographic choice rules in Kominers and Sönmez (2016) are different than the ones we consider since their study is in a general matching with contracts framework. Their framework allows for particular feasibility constraints, in the sense that some alternatives cannot be chosen together with some other alternatives, and for that reason their lexicographic choice rules are not fully covered by our analysis. For instance, the lexicographic choice rules in their setup may violate "substitutability", which is a generalization of gross substitutes to the matching with contracts setup (Hatfield and Milgrom, 2005). Yet, for the school choice aplication where such feasibility constraints are not binding, the lexicographic choice rules in Kominers and Sönmez (2016) fall into our setup and covered by our analysis.

Properties for resource allocation mechanisms under variable resources have been

[9] Houy and Tadenuma (2009) do not start with any assumptions on the list of binary relations. They separately discuss under which assumptions on the list of binary relations, the resulting choice rules satisfy certain properties.

[10] Chambers and Yenmez (2013) is a working paper version of the published version Chambers and Yenmez (2016). The discussion on lexicographic choice rules is not included in the published version.

[11] Dur et al. (2013) is an earlier version of Dur et al. (2016).

widely studied in the literature. A well-known such property is *resource-monotonicity*, introduced by Chun and Thomson (1988), and it has been studied in the resource allocation literature (see, for example, Doğan and Klaus (2016), Ehlers and Klaus (2014), and Ehlers and Klaus (2016)). A resource monotonicity condition for the deferred acceptance mechanisms based on choice rules was also studied in Echenique and Yenmez (2015b). A choice rule is an *expansion* of another choice rule if, for any choice set, any alternative chosen by the latter is also chosen by the former. Echenique and Yenmez (2015b), in Theorem E.1., show that, in the resource allocation framework (see Section 5), each agent weakly prefers one deferred acceptance mechanism to another deferred acceptance mechanism based on path independent choice structures if the choice structure of the former is an expansion of the latter.

3 Capacity-Constrained Lexicographic Choice

Let A be a nonempty finite set of n alternatives and let $\mathcal{A}$ denote the set of all *nonempty* subsets of A. A (capacity-constrained) choice **problem** is a pair $(S, q) \in \mathcal{A} \times \{1, \ldots, n\}$ of a choice set S and a capacity q. A (capacity-constrained) **choice rule** $C : \mathcal{A} \times \{1, \ldots, n\} \to \mathcal{A}$ associates with each problem $(S, q) \in \mathcal{A} \times \{1, \ldots, n\}$, a set of choices $C(S, q) \subseteq S$ such that $|C(S, q)| \leq q$. Given a choice rule C, we denote the set of rejected alternatives at a problem (S, q) by $R(S, q) = S \setminus C(S, q)$.

A **priority ordering** $\succ$ is a complete, transitive, and anti-symmetric binary relation over A. A **priority profile** $\pi = (\succ_1, \ldots, \succ_n)$ is an ordered list of n priority orderings. Let Π denote the set of all priority profiles.

A choice rule C is **lexicographic for a priority profile** $(\succ_1, \ldots, \succ_n) \in \Pi$ if for each $(S, q) \in \mathcal{A} \times \{1, \ldots, n\}$, $C(S, q)$ is obtained by choosing the highest $\succ_1$-priority alternative in S, then choosing the highest $\succ_2$-priority alternative among the remaining alternatives, and so on until q alternatives are chosen or no alternative is left. A choice rule is **lexicographic** if there exists a priority profile for which the choice rule is lexicographic.

Remark 1. Note that, if a choice rule is lexicographic for a priority profile $\pi = (\succ_1, \ldots, \succ_n)$, then it is lexicographic for any other priority profile that is obtained from π by replacing $\succ_n$ with an arbitrary priority ordering. In that sense, the last priority ordering is redundant.

We consider four properties of choice rules. The following three properties are already known in the literature.

Acceptance: An alternative is rejected from a choice set at a capacity only if the capacity is full. Formally, for each $(S, q) \in \mathcal{A} \times \{1, \ldots, n\}$,

$$|C(S, q)| = \min\{|S|, q\}.$$

Gross substitutes:[12] If an alternative is chosen from a choice set at a capacity, then it is also chosen from any subset of the choice set that contains the alternative, at the same capacity. Formally, for each $(S, q) \in \mathcal{A} \times \{1, \ldots, n\}$ and each pair $a, b \in S$ such that $a \neq b$,

$$\text{if } a \in C(S, q), \text{ then } a \in C(S \setminus \{b\}, q).$$

Monotonicity: If an alternative is chosen from a choice set at a capacity, then it is also chosen from the same choice set at any higher capacity. Formally, for each $(S, q) \in \mathcal{A} \times \{1, \ldots, n-1\}$,

$$C(S, q) \subseteq C(S, q + 1).$$

Consider the following capacity-wise revealed preference relation. An alternative $a \in A$ is revealed to be preferred to an alternative $b \in A$ at a capacity $q > 1$ if there is a problem with capacity $q - 1$ for which a and b are both rejected and a is chosen over b when the capacity is q. That is, a is **revealed to be preferred** to b at q if there exists $S \in \mathcal{A}$ such that $a, b \notin C(S, q - 1)$, $a \in C(S, q)$, and $b \notin C(S, q)$. We introduce the following property which requires, for each capacity, the revealed preference relation to be asymmetric.

Capacity-wise weak axiom of revealed preference (CWARP): For each capacity $q > 1$ and each pair $a, b \in A$, if a is revealed to be preferred to b at q, then b is not revealed to be preferred to a at q.

Remark 2. The following is an alternative definition of *CWARP*, which is formulated in

[12] *Gross substitutes* was first introduced in the choice literature by Chernoff (1954). It has been studied in the choice literature under different names such as *Chernoff's axiom*, *Sen's* α, or *contraction consistency*. In the matching literature, it was first studied and referred to as *gross substitutes* in Kelso and Crawford (1982) (*substitutability* is also a commonly used name in the matching literature). We follow the terminology of Kelso and Crawford (1982).

line with the common formulations of WARP-type revealed preference relations in the literature.

An alternative definition of CWARP: For each capacity $q > 1$, each pair $S, T \in \mathcal{A}$ and each pair $a, b \in S \cap T$ such that $[C(S, q-1) \cup C(T, q-1)] \cap \{a, b\} = \emptyset$,

$$\text{if } a \in C(S, q) \text{ and } b \in C(T, q) \setminus C(S, q), \text{ then } a \in C(T, q).$$

Next, we introduce another property that is implied by *CWARP* together with *acceptance* and *monotonicity*. We invoke the property in the proof of our main result. Moreover, we believe that the property also has a stand-alone normative appeal. The property, similar to *monotonicity*, considers the impact of an increase in the capacity.

Consider a problem and the set of rejected alternatives for that problem. Suppose that the capacity increases. The property requires that which alternatives among the currently rejected alternatives will be chosen (if any) should not depend on the currently accepted alternatives. In other words, if the set of rejected alternatives are the same for two choice sets, then at any higher capacity, the set of initially rejected alternatives that become accepted should be the same for the two choice sets.

Rejection-Monotonicity: For each $S, S' \in \mathcal{A}$ and each $q \in \{1, \ldots, n-1\}$,

$$\text{if } R(S, q) = R(S', q), \text{ then } C(S, q+1) \cap R(S, q) = C(S', q+1) \cap R(S', q).$$

Lemma 1. *Suppose that a choice rule satisfies acceptance and monotonicity. If the choice rule satisfies CWARP, then it also satisfies rejection-monotonicity.*

Proof. Let C be a choice rule. Suppose that C satisfies *acceptance* and *monotonicity*, but violates *rejection-monotonicity*. By violation of *rejection-monotonicity*, there are $S, S' \in \mathcal{A}$ and $q \in \{1, \ldots, n-1\}$ such that $R(S, q) = R(S', q)$, but $C(S, q+1) \cap R(S, q) \neq C(S', q+1) \cap R(S', q)$. By *monotonicity*, $R(S, q+1) \subseteq R(S, q)$ and $R(S', q+1) \subseteq R(S', q)$. By *acceptance*, $|R(S, q+1)| = |R(S', q+1)|$. Then, there exist $a, b \in R(S, q) = R(S', q)$ such that $a \in C(S, q+1)$, $b \notin C(S, q+1)$, $b \in C(S', q+1)$, and $a \notin C(S', q+1)$. But then, a is revealed preferred to b and vice versa, implying that C violates *CWARP*. □

The following example shows that the converse statement of Lemma 1 does not hold, that is, there exists a choice rule that satisfies *acceptance*, *monotonicity*, and *rejection-monotonicity*, but violates *CWARP*.

Example 1. *Let $A = \{a, b, c, d, e\}$. Let $\succ$ and $\succ'$ be defined as $a \succ b \succ c \succ d \succ e$ and $a \succ' c \succ' b \succ' d \succ' e$. Let the choice rule C be defined as follows. For each problem (S, q), if $d \in S$, then $C(S, q)$ chooses the highest $\succ$-priority alternatives from S until q alternatives are chosen or no alternative is left;*[13] *if $d \notin S$, then $C(S, q)$ chooses the highest $\succ'$-priority alternatives from S until q alternatives are chosen or no alternative is left. Note that C clearly satisfies acceptance and monotonicity. To see that C also satisfies rejection-monotonicity, let $S, S' \in \mathcal{A}$ and $q \in \{1, \ldots, n-1\}$ be such that $R(S, q) = R(S', q)$. If $d \in S \cap S'$ or $d \in A \setminus (S \cup S')$, then $C(S, q+1) \cap R(S, q) = C(S', q+1) \cap R(S', q)$. So suppose, without loss of generality, that $d \in S \setminus S'$. Since $R(S, q) = R(S', q)$, we have $d \in C(S)$. But then, either $R(S, q) = \emptyset$ or $R(S, q) = \{e\}$. In either case, we have $C(S, q+1) \cap R(S, q) = C(S', q+1) \cap R(S', q)$. To see that C violates CWARP, note that $C(\{a, b, c, d\}, 1) = \{a\}$ and $C(\{a, b, c, d\}, 2) = \{b\}$, implying that b is revealed preferred to c at $q = 2$. Also, $C(\{a, b, c, e\}, 1) = \{a\}$ and $C(\{a, b, c, e\}, 2) = \{c\}$, implying that c is revealed preferred to b at $q = 2$.*

Theorem 1. *A choice rule is lexicographic if and only if it satisfies acceptance, gross substitutes, monotonicity, and the capacity-wise weak axiom of revealed preference.*[14]

Proof. Let C be lexicographic for $(\succ_1, \ldots, \succ_n) \in \Pi$. Clearly, C satisfies *acceptance* and *monotonicity*, and it is already known from the literature that C satisfies *gross substitutes* (Chambers and Yenmez, 2016). To see that it satisfies *CWARP*, let $S, S' \in \mathcal{A}$, $a, b \in A$, and $q \in \{2, \ldots, n\}$ be such that a is revealed preferred to b at q. Then, there is $S \in \mathcal{A}$ such that $a, b \in R(S, q-1)$, $a \in C(S, q)$, and $b \in R(S, q)$. But then, $a \succ_q b$. If also b is revealed preferred to a at q, then by similar arguments we have $b \succ_q a$, contradicting that $\succ_q$ is antisymmetric. Thus, the revealed preference relation is asymmetric and C satisfies *CWARP*.

Let C be a choice rule satisfying *acceptance*, *gross substitutes*, *monotonicity*, and *CWARP*. We first construct a priority profile $(\succ_1, \ldots, \succ_n) \in \Pi$ and then show that C is

[13] That is, $C(S, q)$ coincides with the choice rule that is "responsive" for $\succ$. We discuss responsive choice rules in Section 3.1.

[14] Independence of the characterizing properties is shown in Appendix A.

lexicographic for that priority profile. For each $i, j \in \{1, \ldots, n\}$, let a_{ij} denote the j'th ranked alternative in $\succ_i$ (for instance, a_{i1} is the highest $\succ_i$-priority alternative).

To construct $\succ_1$, first set $\{a_{11}\} = C(A, 1)$. For each $j \in \{2, \ldots, n\}$, set $\{a_{1j}\} = C(A \backslash \{a_{11}, \ldots, a_{1(j-1)}\}, 1)$. To construct $\succ_2$, consider $C(A, 2)$. By *acceptance*, $|C(A, 2)| = 2$. Since $a_{11} \in C(A, 1)$, by *monotonicity*, $a_{11} \in C(A, 2)$. Set $\{a_{21}\} = C(A, 2) \setminus \{a_{11}\}$. For each $j \in \{2, \ldots, n-1\}$, set $\{a_{2j}\} = C(A \setminus \{a_{21}, a_{22}, \ldots, a_{2(j-1)}\}, 2) \setminus \{a_{11}\}$. Set $a_{2n} = a_{11}$.

The rest of the priority profile is constructed recursively as follows. For each $i \in \{3, \ldots, n\}$, first set $\{a_{i1}\} = C(A, i) \setminus \{a_{11}, a_{21}, \ldots, a_{(i-1)1}\}$ (Note that by *monotonicity*, $\{a_{11}, a_{21}, \ldots, a_{(i-1)1}\} \subseteq C(A, i)$ and by *acceptance*, $|C(A, i)| = i$). For each $j \in \{2, \ldots, n - i + 1\}$, set $\{a_{2j}\} = C(A \setminus \{a_{i1}, a_{i2}, \ldots, a_{i(j-1)}\}, i) \setminus \{a_{11}, a_{21}, \ldots, a_{(i-1)1}\}$. Note that there are $i - 1$ rankings yet to be set in $\succ_i$, which are $\{a_{i(n-i+2)}, \ldots, a_{in}\}$. For each $j \in \{n - i + 2, \ldots, n\}$, set $a_{ij} = a_{(j+i-n-1)1}$ (which assigns the alternatives $a_{11}, \ldots, a_{(i-1)1}$ to the rankings $a_{i(n-i+2)}, \ldots, a_{in}$, respectively).

Now, let $(S, q) \in \mathcal{A} \times \{1, \ldots, n\}$. Let b_1 denote the highest $\succ_1$-priority alternative in S, b_2 denote the highest $\succ_2$-priority alternative among the remaining alternatives, and so on up to $b_{\min\{|S|,q\}}$. We show that $C(S, q) = \{b_1, \ldots, b_{\min\{|S|,q\}}\}$. If $\min\{|S|, q\} = |S|$, then by *acceptance*, $C(S, q) = \{b_1, \ldots, b_{|S|}\}$. Suppose that $|S| > q$.

The rest of the proof is by induction: we first show that $b_1 \in C(S, q)$; then, for an arbitrary $i \in \{2, \ldots, q\}$, assuming that $b_1, \ldots, b_{i-1} \in C(S, q)$, we show that $b_i \in C(S, q)$. Let $b_1 = a_{1j}$ for some $j \in \{1, \ldots, n\}$. By the construction of $\succ_1$, $b_1 \in C(A \setminus \{a_{11}, \ldots, a_{1(j-1)}\}, 1)$. Then, by *gross substitutes* and *monotonicity*, $b_1 \in C(S, q)$.

Let $i \in \{2, \ldots, n\}$. Assuming that $b_1, \ldots, b_{i-1} \in C(S, q)$, we show that $b_i \in C(S, q)$. Let S' be the choice set obtained from S by replacing b_1 with a_{11} (note that nothing changes if $b_1 = a_{11}$), replacing b_2 with a_{21}, …, and replacing b_{i-1} with $a_{(i-1)1}$. That is, $S' = (S \setminus \{b_1, \ldots, b_{i-1}\}) \cup \{a_{11}, \ldots, a_{(i-1)1}\}$. Let $q' = i - 1$. Note that $\{b_1, \ldots, b_{i-1}\} = C(S, q')$, because otherwise, by *acceptance*, there is $a \in S$ such that $a \in C(S, q')$ and $a \notin C(S, q)$, which is a violation of *monotonicity*. Also, by the construction of the priority profile and by *gross substitutes*, $\{a_{11}, \ldots, a_{(i-1)1}\} = C(S', q')$. Note that $R(S, q') = R(S', q')$. By Lemma 1, C satisfies *rejection-monotonicity*. Then, by *monotonicity* and *rejection-monotonicity*, we have $R(S, q) = R(S', q)$. Since $b_i \in C(S', q)$ by the construction of the priority profile and by *gross substitutes*, we also have $b_i \in C(S, q)$. □

Corollary 1. *A choice rule is lexicographic if and only if it satisfies acceptance, gross substitutes, monotonicity, and rejection-monotonicity.*

Proof. A lexicographic choice rule satisfies *acceptance*, *gross substitutes*, and *monotonicity* by Theorem 1, and also satisfies *rejection-monotonicity* by Lemma 1. To see the other direction, note that in the proof of Theorem 1, we invoked *CWARP* only to claim that *rejection-monotonicity* is satisfied, and therefore the same proof for the if part is valid when we replace *CWARP* with *rejection-monotonicity*. □

A choice rule C can be lexicographic for two different priority profiles. Even more, the priority profile for which a choice rule is lexicographic is never unique. However, if C is lexicographic for two different priority profiles $(\succ_1, \ldots, \succ_n)$ and $(\succ'_1, \ldots, \succ'_n)$, then for each pair of alternatives $a, b \in A$, if $a \succ_q b$ and $b \succ'_q a$ for some $q \in \{1, \ldots, n\}$, then either a or b must be chosen from any choice set (particularly from A) at any lower capacity. That is, a or b is chosen irrespective of its relative ranking at the q-priority ordering.

To state this observation formally, for each priority ordering $\succ$ on A and for each choice set $S \in \mathcal{A}$, let $\succ_i |_S$ stand for the restriction of $\succ_i$ to S. Let $A_1 = A$, and for each $q \in \{2, \ldots, n\}$, let $A_q = A \setminus C(A, q-1)$.

Proposition 1. *If a choice rule C is lexicographic for a priority profile $(\succ_1, \ldots, \succ_n)$, then C is lexicographic for another priority profile $(\succ'_1, \ldots, \succ'_n)$ if and only if $\succ_1 = \succ'_1$ and for each $q \in \{1, \ldots, n\}$, $\succ_q |_{A_q} = \succ'_q |_{A_q}$.*

Proof. In the proof of Theorem 1, the priority profile $(\succ_1, \ldots, \succ_n)$ is constructed such that for each for each $q \in \{1, \ldots, n\}$ and each choice set $S \in \mathcal{A}$, $max(S \setminus C(S, q-1), \succ_q) = C(S, q) \setminus C(A, q-1)$. Now, for each $q \in \{1, \ldots, n\}$, let $\succ^*_q = \succ_q |_{A_q}$, and $\mathcal{A}_q$ stand for the collection of all nonempty subsets of A_q. Next, define the choice function $c_q : \mathcal{A}_q \to A_q$ such that for each choice set $S \in \mathcal{A}_q$, $c_q(S) = \max\{S \setminus C(A, q-1), \succ^*_q\}$. Since C satisfies *gross substitutes*, c_q also satisfies *gross substitutes*. It follows that there is a unique priority ordering $\succ^*_q$ such that $c_q(S) = \max\{S \setminus C(A, q-1), \succ^*_q\}$. □

3.1 Responsive Choice

A well-known example of a lexicographic choice rule is a "responsive" choice rule,[15] which is lexicographic for a priority profile where all the priority orderings are the same. Formally, a choice rule C is **responsive for a priority ordering** $\succ$ if for each $(S, q) \in \mathcal{A} \times \{1, \ldots, n\}$, $C(S, q)$ is obtained by choosing the highest $\succ$-priority alternatives in S until q alternatives are chosen or no alternative is left. Note that C is responsive for $\succ$ if and only if it is lexicographic for the priority profile $(\succ, \ldots, \succ)$.

Chambers and Yenmez (2013) characterize "responsive" choice rules, but in the context of "classical" choice problems which do not explicitly refer to a variable capacity parameter. Formally, a classical choice rule is a function $C : \mathcal{A} \to \mathcal{A}$ such that for each $S \in \mathcal{A}$, $C(S) \subseteq S$. A classical choice rule is *responsive* if there exists a priority ordering $\succ$ and a capacity $q \in \{1, \ldots, n\}$ such that for each $S \in \mathcal{A}$, $C(S)$ is obtained by choosing the highest $\succ$-priority alternatives until the capacity q is reached or no alternative is left. Chambers and Yenmez (2013), in their Theorem 6, show that, a classical choice rule satisfies *acceptance*[16] and the *weakened weak axiom of revealed preference (WWARP)* if and only if it is responsive.[17] *WWARP* was introduced by Ehlers and Sprumont (2008) and requires that for each pair $a, b \in A$ and $S, S' \in \mathcal{A}$ such that $a, b \in S \cap S'$,

$$\text{if } a \in C(S) \text{ and } b \in C(S') \setminus C(S), \text{ then } a \in C(S').$$

To see what Chambers and Yenmez (2013), Theorem 6, implies in our variable capacity setup, consider the following extension of *WWARP* to our setup.

Weakened weak axiom of revealed preference (WWARP): For each $S, S' \in \mathcal{A}$, $q \in \{1, \ldots, n\}$, and each pair $a, b \in S \cap S'$,

$$\text{if } a \in C(S, q) \text{ and } b \in C(S', q) \setminus C(S, q), \text{ then } a \in C(S', q).$$

[15] Responsive choice rules have been studied particularly in the two-sided matching context (Roth and Sotomayor, 1990).

[16] A classical choice rule satisfies *acceptance* if there exists a capacity such that at each choice problem, an alternative is rejected only if the capacity is reached.

[17] Chambers and Yenmez (2013), in their Theorem 7, also provide a characterization of choice rules that are responsive for a known capacity (namely q-responsive choice rules).

The following Proposition 2 directly follows from Chambers and Yenmez (2013), Theorem 6.

Proposition 2. *A choice rule satisfies acceptance and the weakened weak axiom of revealed preference if and only if for each* $q \in \{1, \ldots, n\}$*, there is a priority ordering* $\succ^q$ *such that for each* $S \in \mathcal{A}$*,* $C(S, q)$ *is obtained by choosing the highest* $\succ^q$*-priority alternatives until the capacity* q *is reached or no alternative is left.*

Proposition 2 states that *acceptance* and the *weakened weak axiom of revealed preference* characterizes "capacity-wise responsive" choice rules, which are responsive for each capacity, but the associated priority orderings for different capacities may be different. Yet, a characterization of responsive choice rules in our setup does not directly follow from Chambers and Yenmez (2013).

We show that, the following extension of WWARP, together with *acceptance*, characterizes responsive choice rules in our variable-capacity setup. The property, called the *capacity-wise weakened weak axiom of revealed preference (CWWARP)*, requires that if an alternative a is chosen and b is not chosen at a problem where they are both available, then at any problem where they are both available, a is chosen whenever b is chosen.

Capacity-wise weakened weak axiom of revealed preference (CWWARP): For each $S, S' \in \mathcal{A}$, $q, q' \in \{1, \ldots, n\}$, and each pair $a, b \in S$ such that $a, b \in S \cap S'$,

$$\text{if } a \in C(S, q) \text{ and } b \in C(S', q') \setminus C(S, q), \text{ then } a \in C(S', q').$$

Theorem 2. *A choice rule is responsive if and only if it satisfies acceptance and the capacity-wise weakened weak axiom of revealed preference.*

Proof. It is clear that a responsive choice rule satisfies *acceptance* and *CWWARP*. Let C be a choice rule satisfying *acceptance* and *CWWARP*. Clearly, *CWWARP* implies *WWARP*, and therefore by Proposition 2, for each $q \in \{1, \ldots, n\}$, there is a priority ordering $\succ^q$ such that for each $S \in \mathcal{A}$, $C(S, q)$ is obtained by choosing the highest $\succ^q$-priority alternatives until the capacity q is reached or no alternative is left.

Let $(S, q) \in \mathcal{A} \times \{1, \ldots, n\}$. If $|S| \leq q$, then by *acceptance*, $C(S, q) = S$. Suppose that $|S| > q$. First note that $C(S, q - 1) \subseteq C(S, q)$, since otherwise, by *acceptance*, there is a pair $a, b \in S$ such that $a \in C(S, q-1) \setminus C(S, q)$ and $b \in C(S, q) \setminus C(S, q-1)$, which contradicts *CWWARP*. Now, consider any pair $a, b \in R(S, q-1)$ such that $a \in C(S, q)$ and $b \notin C(S, q)$. By *CWWARP*, for any $S' \in \mathcal{A}$, b is not chosen over a at (S', q), implying that a has $\succ^q$-priority over b. But then, for each $S \in \mathcal{A}$, $C(S, q)$ is obtained by choosing the highest $\succ^{q-1}$-priority alternatives until the capacity q is reached or no alternative is left. Since we started with an arbitrary $q \in \{1, \ldots, n\}$, C is a choice rule that is responsive to $\succ^1$. $\square$

Given *acceptance*, to better highlight the gap between *WWARP* and *CWWARP*, we introduce the following property. The property requires that the choice from a choice set at a given capacity should not change if the choice is made in two steps: first, choosing at a lower capacity, and then choosing from among the remaining alternatives at the remaining capacity.

Composition up: For each $S \in \mathcal{A}$ and $q, q' \in \{1, \ldots, n\}$ such that $q' > q$,

$$C(S, q') = C(S, q) \cup C(S \setminus C(S, q), q' - q).$$

Proposition 3. *Let C be a choice rule satisfying acceptance. The choice rule C satisfies the capacity-wise weakened weak axiom of revealed preference if and only if it satisfies the weakened weak axiom of revealed preference and composition up.*

Proof. Suppose that C satisfies *CWWARP*. Then, it clearly satisfies *WWARP*. Suppose that $S \in \mathcal{A}$ and $q, q' \in \{1, \ldots, n\}$ are such that $q' > q$ and

$$C(S, q') \neq C(S, q) \cup C(S \setminus C(S, q), q' - q).$$

Then, by *acceptance*, either there exist $a, b \in A$ such that

$$a \in C(S, q) \setminus C(S', q') \text{ and } b \in C(S', q') \setminus C(S, q)$$

or there exist $a, b \in A$ such that

$$a \in C(S \setminus C(S, q), q' - q) \setminus C(S', q') \text{ and } b \in C(S', q') \setminus C(S \setminus C(S, q), q' - q).$$

In either case, we have a contradiction to *CWWARP*. Hence, C satisfies *composition up.*

Suppose that C satisfies *WWARP* and *composition up*. Let $S, S' \in \mathcal{A}$, $q, q' \in \{1, \ldots, n\}$, and $a, b \in A$ be such that $a, b \in S \cap S'$, $a \in C(S, q)$, and $b \in C(S', q') \setminus C(S, q)$. Suppose that $a \notin C(S', q')$. If $q = q'$, we have a contradiction to *WWARP*. Suppose, without loss of generality, that $q' > q$. Since $a \in C(S, q)$, by *composition up*, $a \in C(S, q')$. Since $b \notin C(S, q)$, by *composition up*, $b \notin C(S, q')$. But then, $a \in C(S, q')$, $b \in C(S', q') \setminus C(S, q')$, and $a \notin C(S', q')$, which is a contradiction to *WWARP*. Hence, C satisfies *CWARP*. □

Corollary 2. *A choice rule is responsive if and only if it satisfies acceptance, the weakened weak axiom of revealed preference, and composition up.*

The following example shows that, without *acceptance*, *CWWARP* does not imply *composition up*.

Example 2. *Let $A = \{a, b, c\}$. Let $\succ$ be a priority ordering. Let C be the choice rule such that, for each problem (S, q), $C(S, q)$ is a singleton consisting of the $\succ$-maximal alternative in S if $q \in \{1, 3\}$; and C coincides with the choice rule that is responsive for $\succ$ when $q = 2$. Note that C violates acceptance since it chooses a single alternative from any choice set when the capacity is 3. Moreover, C clearly satisfies CWWARP but violates composition up.*

4 Implications for School Choice in Boston

In the Boston school choice system, for each school there are two different priority orderings: a *walk-zone priority ordering*, which gives priority to the school's neighborhood students over all the other students, and an *open priority ordering* which does not give priority to any student for being a neighborhood student. The Boston school district aims to assign half of the seats of each school based on the walk-zone priority ordering and the other half based on the open priority ordering.

To better understand what the Boston school district wants to achieve and how it can be achieved, let us consider the following class of choice rules that is larger than the class of lexicographic choice rules. We say that a choice rule is **capacity-wise lexicographic** if, at each capacity, the rule operates based on a list containing as many priority orderings as the capacity. Unlike a lexicographic choice rule, the lists for different capacity levels are not necessarily related.

Now, the Boston school district's objective can be achieved with a capacity-wise lexicographic choice rule such that, at each capacity, the associated list consists of only the walk-zone priority ordering and the open priority ordering, and the absolute difference between the numbers of walk-zone and open priority orderings in the list is at most one. We formalize this property as follows.

Let $\succ^w$ and $\succ^o$ be walk-zone and open priority orderings. We say that a capacity-wise lexicographic choice rule satisfies the **Boston requirement for** $(\succ^w,\succ^o)$ if for each capacity q, the associated list of priority orderings $(\succ_1,\ldots,\succ_q)$ is such that

i. for each $l \in \{1,\ldots,q\}$, $\succ_l \in \{\succ^w,\succ^o\}$,
ii. difference between the number of $\succ^w$-priorities and $\succ^o$-priorities is at most one, i.e. $\left|\sum_{i=1}^q 1_{\succ^w}(\succ_i) - \sum_{i=1}^q 1_{\succ^o}(\succ_i)\right| \leq 1$.[18]

Now, it turns out that the following class of capacity-wise lexicographic choice rules are the only rules satisfying our set of properties together with the Boston requirement for $(\succ^w,\succ^o)$.

Proposition 4. *A capacity-wise lexicographic choice rule satisfies acceptance, gross substitutes, monotonicity, the capacity-wise weak axiom of revealed preference, and the Boston requirement for* $(\succ^w,\succ^o)$ *if and only if it is lexicographic for a priority profile* $(\succ_1,\ldots,\succ_n)$ *such that*

i. for each $l \in \{1,\ldots,n\}$, $\succ_l \in \{\succ^w,\succ^o\}$,
ii. for each l *that is odd,* $\succ_l=\succ^w$ *if and only if* $\succ_{l+1}=\succ^o$.

Proof. By Theorem 1, a choice rule satisfying the properties must be lexicographic. The rest is straightforward. □

Dur et al. (2013) analyses the School Choice problem in Boston and four plausible choice rules stand out from their analysis, one of which is currently in use in Boston (Open-Walk choice rule). Dur et al. (2013) compare the below four choice rules in terms of how much they are biased for or against the neighbourhood students. We will compare the four choice rules with respect to our set of choice rule properties.

[18] $1_x(y)$ is the indicator function which has the value 1 if $x = y$ and 0 otherwise.

1. *Walk-Open Choice Rule:* At each capacity, the first half of the priority orderings in the list are the walk-zone priority ordering and the last half are the open priority ordering.
2. *Open-Walk Choice Rule:* At each capacity, the first half of the priority orderings in the list are the open priority ordering and the last half are the walk-zone priority ordering.
3. *Rotating Choice Rule:* At each capacity, the first priority ordering in the list is the walk-zone priority ordering, the second is the open priority ordering, the third is the walk-zone priority ordering, and so on.
4. *Compromise Choice Rule:* At each capacity, the first quarter of the priority orderings in the list are the walk-zone priority ordering, the following half of the priority orderings in the list are the open priority ordering, and the last quarter are again the walk-zone priority ordering.

To be precise, let us introduce the following procedures to accommodate the cases where the capacity is not divisible by two or four.

- *Walk-Open Choice Rule:* If the capacity q is an odd number, the first $\frac{q+1}{2}$ are the walk-zone priority ordering.
- *Open-Walk Choice Rule:* If the capacity q is an odd number, the first $\frac{q+1}{2}$ are the open priority ordering.
- *Compromise Choice Rule:* If the capacity q is not divisible by four, let $q = q' + k$ for some q' that is divisible by 4 and some $k \in \{1, 2, 3\}$. If $k = 1$, let the first $\frac{q'}{4} + 1$ orderings be the open priority ordering, the following $\frac{q'}{2}$ orderings be the open priority ordering, and the last $\frac{q'}{4}$ orderings be the open priority ordering. If $k = 2$, let the first $\frac{q'}{4} + 1$ orderings be the open priority ordering, the following $\frac{q'}{2} + 1$ orderings be the open priority ordering, and the last $\frac{q'}{4}$ orderings be the open priority ordering. If $k = 3$, let the first $\frac{q'}{4} + 1$ orderings be the open priority ordering, the following $\frac{q'}{2} + 1$ orderings be the open priority ordering, and the last $\frac{q'}{4} + 1$ orderings be the open priority ordering.

Note that all of the above rules satisfy the Boston requirement for $(\succ^w, \succ^o)$. It is clear from Proposition 4 that the Rotating Choice Rule satisfies *acceptance*, *gross substitutes*, *monotonicity*, and the *CWARP*. We show that the other three rules violate *CWARP* while all the four rules satisfy *monotonicity*.

We first show that each one of the four rules satisfies *monotonicity*. In fact, we show that a larger class of choice rules satisfies *monotonicity*.

Let $\pi = (\succ_1, \ldots, \succ_q)$ and $\pi' = (\succ'_1, \ldots, \succ'_{q+1})$ be priority lists of size q and $q+1$, respectively. We say that π' is **obtained by insertion** from π if there exists $k \in \{1, \ldots, q+1\}$ such that $\succ'_l = \succ_l$ for each $l < k$, and $\succ'_l = \succ_{l-1}$ for each $l > k$. Note that when π' is obtained by insertion from π, a new priority ordering is inserted into the list of priority orderings in π, by keeping relative order of the other priority orderings in the list the same. It is possible that the new ordering is inserted in the very beginning or in the very end of the list.

Proposition 5. *Let C be a capacity-wise lexicographic choice rule. The choice rule C is monotonic if for each $q \in \{2, \ldots, n\}$, the priority list for q is obtained by insertion from the priority list for $q-1$.*

Proof. Let $(S, q) \in \mathcal{A} \times \{1, \ldots, n-1\}$. Let $\pi = (\succ_1, \ldots, \succ_q)$ be the list for capacity q. Let $a \in C(S, q)$. Suppose that, in the lexicographic choice procedure, a is chosen at the t'th step, i.e. a is chosen based on $\succ_t$.

Let $\pi' = (\succ'_1, \ldots, \succ'_{q+1})$ be the list for capacity $q+1$. Note that π' is obtained by insertion from π. Let $k \in \{1, \ldots, q+1\}$ be such that $\succ'_l = \succ_l$ for each $l < k$, and $\succ'_l = \succ_{l-1}$ for each $l > k$.

Now, consider the problem $(S, q+1)$. If $t < k$, clearly a is still chosen at the t'th step of the lexicographic choice procedure and thus $a \in C(S, q+1)$. Suppose that $t \geq k$. The rest of the proof is by induction. First, suppose that $t = k$. Note that at Step k of the choice procedure for the problem $(S, q+1)$, the choice is made based on the inserted priority ordering and at Step $k+1$, the choice is made based on $\succ_t$. Then, a is either chosen at Step k, or at Step $k+1$, the set of remaining alternatives is a subset of the set of remaining alternatives at Step t of the choice procedure for (S, q) where a is chosen, in which case a is still chosen. Thus, $a \in C(S, q+1)$.

Now, suppose that $t > k$ and each alternative that is chosen at a step $t' < t$ of the choice procedure at (S, q) is also chosen at $(S, q+1)$. Then, a is either chosen before step $t+1$ of the choice procedure for $(S, q+1)$, or at Step $t+1$, the set of remaining alternatives is a subset of the set of remaining alternatives at Step t of the choice procedure for (S, q) where a is chosen, in which case a is still chosen. Thus, $a \in C(S, q+1)$. □

It is easy to see that each of the four choice rules satisfies the insertion property, which yields the following corollary.

Corollary 3. *Each one of the four rules satisfies acceptance, gross substitutes, and monotonicity.*

Proof. Each rule is capacity-wise lexicographic (lexicographic for a given capacity) and therefore satisfies *acceptance* and *gross substitutes*. *Monotonicity* follows by Proposition 5. □

Proposition 6. *Among the four rules, only the rotating choice rule satisfies the capacity-wise weak axiom of revealed preference and only the rotating choice rule is lexicographic.*

Proof. Consider $(\succ_1, \ldots, \succ_n) \in \Pi$ such that the first priority ordering in the list is $\succ^w$, the second is $\succ^o$, the third is $\succ^w$, and so on. The rotating choice rule is clearly lexicographic for $(\succ_1, \ldots, \succ_n)$. Moreover, by Theorem 1, it satisfies *CWARP*. We will show that each of the other three choice rules violates *CWARP*.

Walk-Open Choice Rule: Let $A = \{a, b, c, d, e\}$. Let $\succ^w$ be defined as $a \succ^w b \succ^w c \succ^w d \succ^w e$ and $\succ^o$ be defined as $e \succ^o b \succ^o d \succ^o c \succ^o a$. Note that $C(\{a, c, d, e, 2\}) = \{a, e\}$ and $C(\{a, c, d, e, 3\}) = \{a, c, e\}$, and therefore c is revealed preferred to d at $q = 3$. Moreover, $C(\{a, b, c, d, 2\}) = \{a, b\}$ and $C(\{a, b, c, d, 3\}) = \{a, b, d\}$, and therefore d is revealed preferred to c at $q = 3$, implying that C violates *CWARP*.

Open-Walk Choice Rule: Can be shown by interchanging the orderings for $\succ^w$ and $\succ^o$ in the previous example.

Compromise Choice Rule: Let $A = \{a, b, c, d, x, y\}$. Let $\succ^w$ be defined as $a \succ^w b \succ^w c \succ^w d \succ^w x \succ^w y$ and $\succ_o$ be defined as $b \succ^o c \succ^o y \succ^o x \succ^o d$. Note that $C(\{a, b, c, x, y, 3\}) = \{a, b, c\}$ and $C(\{a, b, c, x, y, 4\}) = \{a, b, c, x\}$, and therefore x is revealed preferred to y at $q = 4$. Moreover, $C(\{a, b, d, x, y, 3\}) = \{a, b, d\}$ and $C(\{a, b, d, x, y, 4\}) = \{a, b, d, y\}$, and therefore y is revealed preferred to x at $q = 4$, implying that C violates *CWARP*.

□

Remark 3. Note that the particular procedures we introduced to accommodate the cases where the capacity is not divisible by two or four are not crucial for the proof of Proposition 6. For the other procedures (for example, for the walk-open choice rule, the extra priority when the capacity is odd can alternatively be set to be the open priority ordering), the examples in the proof can simply be modified to show that $CWARP$ is still violated.

Our analysis shows that if $CWARP$ is deemed desirable, then the rotating choice rule should be selected since it is the only choice rule among the four plausible alternatives that satisfies $CWARP$.

5 Implications for Resource Allocation

Let N denote a finite set of agents, $|N| = n \geq 2$. Let $\mathcal{A}$ be the collection of all nonempty subsets of N. Let O denote a finite set of objects. Each agent $i \in N$ has a complete, transitive, and anti-symmetric preference relation R_i over $O \cup \{\emptyset\}$, where $\emptyset$ is the null object representing the option of receiving no object (or receiving an outside option). Given $x, y \in O \cup \{\emptyset\}$, $x \; R_i \; y$ means that either $x = y$ or $x \neq y$ and agent i prefers x to y. If agent i prefers x to y, we write $x \; P_i \; y$. Let $\mathcal{R}$ denote the set of all preference relations over $O \cup \{\emptyset\}$, and $\mathcal{R}^N$ the set of all preference profiles $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$.

An allocation problem with capacity constraints, or simply a **problem**, consists of a preference profile $R \in \mathcal{R}^N$ and a capacity profile $q = (q_x)_{x \in O \cup \{\emptyset\}}$ such that for each object $x \in O$, $q_x \in \{0, 1, \ldots, n\}$ and $q_\emptyset = n$ so that the null object has enough capacity to accommodate all agents. Let $\mathcal{P}$ denote the set of all problems. Given a problem $(R, q) \in \mathcal{P}$, an object x is **available** at the problem if $q_x > 0$.

Given a capacity profile $q = (q_x)_{x \in O \cup \{\emptyset\}}$, an allocation assigns to each agent exactly one object in $O \cup \{\emptyset\}$ taking capacity constraints into account. Formally, an **allocation** at q is a list $a = (a_i)_{i \in N}$ such that for each $i \in N$, $a_i \in O \cup \{\emptyset\}$ and no object $x \in O \cup \{\emptyset\}$ is assigned to more than q_x agents. Let $M(q)$ denote the set of all allocations at q.

Given an allocation $a = (a_i)_{i \in N}$, a preference profile R, and an object $x \in O \cup \{\emptyset\}$, let $D_x(a, R) = \{i \in N : x \; P_i \; a_i\}$ denote the **demand** for x at (a, R), which is the set of agents who prefer x to their assigned object.

5.1 Lexicographic Deferred Acceptance Mechanisms

A **mechanism** is a function $\varphi : \mathcal{P} \to \bigcup_q M(q)$ such that for each allocation problem $(R, q) \in \mathcal{P}$, $\varphi(R, q) \in M(q)$. For mechanisms, we introduce a new property, called *demand-monotonicity*. To introduce demand-monotonicity, consider a problem in which there is only one available object. Next, suppose that the capacity of the object is increased. Now, some of the agents who initially did not receive the object may receive it, that is, some agents may receive the object due to the capacity increase. Demand monotonicity requires that the set of agents who receive the object due to the capacity increase does not depend on the set of agents who initially receive the object. In other words, for two problems with a common capacity, if the demands for the only available object are the same, then whenever the capacity of the object increases, the sets of agents who receive the object due to the capacity increase should be the same for the two problems.

Formally, for each $x \in O$, let 1_x be the capacity profile which has 1 unit of x and nothing else. A mechanism φ satisfies **demand-monotonicity** if for each pair of problems (R, q) and (R', q) and each object $x \in O$, if for each $y \in O \setminus \{x\}$, $q_y = 0$ and $D_x(\varphi(R, q), R) = D_x(\varphi(R', q), R')$, then $D_x(\varphi(R, q + 1_x), R) = D_x(\varphi(R', q + 1_x), R')$.

A **lexicographic choice structure** $\mathcal{C} = (C_x)_{x \in O}$ associates each object $x \in O$ with a lexicographic choice rule $C_x : \mathcal{A} \times \{1, \ldots, n\} \to \mathcal{A}$. Next, we present **the lexicographic deferred acceptance algorithm based on** $\mathcal{C}$. For each problem $(R, q) \in \mathcal{P}$, the algorithm runs as follows:

Step 1: Each agent applies to his favorite object in O. Each object $x \in O$ such that $q_x > 0$ temporarily accepts the applicants in $\mathcal{C}_x(S_x, q_x)$ where S_x is the set of agents who applied to x, and rejects all the other applicants. Each object $x \in C$ such that $q_x = 0$ rejects all applicants.

Step $r \geq 2$: Each applicant who was rejected at step $r - 1$ applies to his next favorite object in O. For each object $x \in C$, let $S_{x,r}$ be the set consisting of the agents who applied to x at step r and the agents who were temporarily accepted by x at Step $r - 1$. Each object $x \in O$ such that $q_x > 0$ accepts the applicants in $\mathcal{C}_x(S_{x,r}, q_x)$ and rejects all the other applicants. Each object $x \in O$ such that $q_x^r = 0$ rejects all applicants.

The algorithm terminates when each agent is accepted by an object. The allocation where each agent is assigned the object that he was accepted by at the end of the algorithm is called the C-lexicographic Deferred Acceptance allocation at (R, q), denoted by $DA^{\mathcal{C}}(R, q)$.

Lexicographic deferred acceptance mechanisms: A mechanism φ is a *lexicographic deferred acceptance mechanism* if there exists a lexicographic choice structure $\mathcal{C}$ such that for each $(R, q) \in \mathcal{P}$, $\varphi(R, q) = DA^{\mathcal{C}}(R, q)$.

Proposition 7. *Each lexicographic deferred acceptance mechanism satisfies demand-monotonicity.*

Proof. Let $\mathcal{C} = (C_x)_{x \in O}$ be a lexicographic choice structure. Let $(R, q), (R', q) \in \mathcal{P}$ and $x \in O$ be such that for each $y \in O \setminus \{x\}$, $q_y = 0$ and $D_x(DA^{\mathcal{C}}(R, q), R) = D_x(DA^{\mathcal{C}}(R', q), R') = T$. Let C_x be lexicographic for the priority profile $(\succ_1, \ldots, \succ_n) \in \Pi$. Let $S(R)$ and $S(R')$ be the sets of agents who prefer x to $\emptyset$ at R and at R', respectively. It is easy to see that $DA^{\mathcal{C}}(R, q) = C_x(S(R))$, $DA^{\mathcal{C}}(R', q) = C_x(S(R'))$, and $T = S(R) \setminus C_x(S(R)) = S(R') \setminus C_x(S(R'))$. Let $i \in T$ be the agent who is highest ranked according to $\succ_{q_x+1}$ in T. Clearly, $DA^{\mathcal{C}}(R, q') = DA^{\mathcal{C}}(R, q) \cup \{i\}$ and $DA^{\mathcal{C}}(R', q') = DA^{\mathcal{C}}(R', q) \cup \{i\}$. Hence, $D_x(DA^{\mathcal{C}}(R, q'), R) = D_x(DA^{\mathcal{C}}(R', q'), R') = T \setminus \{i\}$. □

Ehlers and Klaus (2016), in their Theorem 3, characterize deferred acceptance mechanisms based on a choice structure satisfying *acceptance*, *gross substitutes*, and *monotonicity*, with the following properties of mechanisms: unavailable-type-invariance (if the positions of the unavailable types are shuffled at a profile, then the allocation should not change); weak non-wastefulness (no agent receives the null object while he prefers an object that is not exhausted to the null object), resource-monotonicity (increasing the capacities of some objects does not hurt any agent), truncation-invariance (if an agent truncates his preference relation in such a way that his allotment remains acceptable under the truncated preference relation, then the allocation should not change), and strategy-proofness (no agent can benefit by misreporting his preferences).[19]

Theorem 3. *(Ehlers and Klaus, 2016) A mechanism is a deferred acceptance mechanism based on a choice structure satisfying acceptance, gross substitutes, and monotonicity if and only if it satisfies unavailable-type-invariance, weak non-wastefulness, resource-monotonicity, truncation-invariance, and strategy-proofness.*

[19] See Appendix B for the formal definitions of the properties.

Corollary 4. *A mechanism is a lexicographic deferred acceptance mechanism if and only if it satisfies unavailable-type-invariance, weak non-wastefulness, resource-monotonicity, truncation-invariance, strategy-proofness, and demand-monotonicity.*

Proof. The following notation will be helpful. For each $x \in O$, let R^x be a preference relation such that x is top-ranked and $\emptyset$ is second-ranked. For each $S \in \mathcal{A}$ that is nonempty, let R^x_S be a preference profile such that for each $i \in S$, $R_i = R^x$, and for each $j \notin S$, R_j top-ranks $\emptyset$. For each $x \in O$ and $l \in \{0, \dots, n\}$, let l_x denote the capacity profile where x has capacity l and every other object has capacity zero.

Let φ be a mechanism satisfying the properties in the statement of the theorem. Let $\mathcal{C} = (C_x)_{x \in O}$ be defined as follows. For each $x \in O$, $S \in \mathcal{A}$, and $l \in \{0, \dots, n\}$, $C_x(S, l) = \{i \in S : \varphi_i(R^x_S, l_x) = x\}$.

In their proof of Theorem 3, (Ehlers and Klaus, 2016) show that if φ satisfies *unavailable-type-invariance*, *weak non-wastefulness*, *resource-monotonicity*, *truncation-invariance*, and *strategy-proofness*, then for each $x \in O$, C_x satisfies *acceptance*, *gross substitutes*, and *monotonicity*. Moreover, φ is a deferred acceptance mechanism based on $\mathcal{C}$. It is easy to see that, since φ satisfies *demand-monotonicity*, for each $x \in O$, C_x satisfies *rejection-monotonicity*. Thus, $\mathcal{C}$ is a lexicographic choice structure and φ is a lexicographic deferred acceptance mechanism.

Let φ be a lexicographic deferred acceptance mechanism. *Demand-monotonicity* follows from Proposition 7. The other properties follow from Theorem 3 of Ehlers and Klaus (2016). □

Remark 4. We give an example of a mechanism which satisfies all the properties in the statement of Corollary 4 except for *demand-monotonicity*, and therefore which is not a lexicographic deferred acceptance mechanism. The mechanism in the example is a deferred acceptance mechanism based on a choice structure such that the choice rule of each object is a walk-open choice rule. The example uses some arguments from the proof of Proposition 6, where it was shown that the walk-open choice rule violates *CWARP*.

Example 3. *Let $N = \{a, b, c, d, e\}$ and let O be a finite set of objects. Let $\succ^w$ be defined as $a \succ^w b \succ^w c \succ^w d \succ^w e$ and $\succ^o$ be defined as $e \succ^o b \succ^o d \succ^o c \succ^o a$. Let $(C_x)_{x \in O}$ be the choice structure such that for each object $x \in O$, C_x is the walk-open choice rule based* *on* $(\succ^w, \succ^o)$. *Let* φ *be the deferred acceptance mechanism based on the choice structure* $(C_x)_{x \in O}$.

Since for each $x \in O$, C_x *satisfies acceptance, gross substitutes, and monotonicity, by Theorem 3,* φ *satisfies unavailable-type-invariance, weak non-wastefulness, resource-monotonicity, truncation-invariance, and strategy-proofness.*

Let $x \in O$. *Let* q *be such that* $q_x = 2$ *and for each* $y \in O \setminus \{x\}$, $q_y = 0$. *Let* R *be such that* x *is preferred to* $\emptyset$ *for all the agents except for* b. *Note that* $D_x(\varphi(R,q),R) = \{c,d\}$ *since* $C_x(\{a,c,d,e,2\}) = \{a,e\}$. *Let* R' *be such that* x *is preferred to* $\emptyset$ *for all the agents except for* e. *Note that* $D_x(\varphi(R',q),R') = \{c,d\}$ *since* $C_x(\{a,b,c,d,2\}) = \{a,b\}$. *Thus,* $D_x(\varphi(R,q),R) = D_x(\varphi(R',q),R')$.

Now, note that $D_x(\varphi(R,q+1_x),R) = \{c\}$ *since* $C_x(\{a,c,d,e,3\}) = \{a,c,e\}$ *and* $D_x(\varphi(R',q+1_x),R') = \{b\}$ *since* $C_x(\{a,b,c,d,3\}) = \{a,b,d\}$. *Hence,* φ *violates demand-monotonicity.*

Remark 5. A property that is stronger than *demand-monotonicity* is the following. A mechanism φ satisfies **strong demand-monotonicity** if for each pair of problems (R,q) and (R',q) and each object $x \in O$, if $D_x(\varphi(R,q),R) = D_x(\varphi(R',q),R')$, then $D_x(\varphi(R,q+1_x),R) = D_x(\varphi(R',q+1_x),R')$. Clearly, *strong demand-monotonicity* implies *demand-monotonicity*. The following example shows a lexicographic deferred acceptance mechanism (in fact, a *classical* deferred acceptance mechanism based on a priority profile, which is lexicographic for a priority profile where all the priority orderings are the same) that violates *strong demand-monotonicity*.

Example 4. *Let* $A = \{1,2,3\}$. *Let* $O = \{a,b,c\}$. *Let* $\succ_a$ *be defined as* $1 \succ_a 2 \succ_a 3$, $\succ_b$ *be defined as* $2 \succ_a 3 \succ_a 1$, *and* $\succ_c$ *be defined as* $1 \succ_a 2 \succ_a 3$. *Let* $\mathcal{C} = (C_x)_{x \in O}$ *be a lexicographic choice structure such that for each* $x \in O$, C_x *is lexicographic for the priority profile* $(\succ_x, \succ_x, \succ_x)$. *Note that* DA^C *is a classical deferred acceptance mechanism based on a priority profile. Let the preference profiles* R *and* R' *be as depicted below.*

R_1	R_2	R_3
a	a	b
b	b	a
c	c	c
$\emptyset$	$\emptyset$	$\emptyset$

R'_1	R'_2	R'_3
a	a	a
b	b	c
c	c	b
$\emptyset$	$\emptyset$	$\emptyset$

Let $q = (q_a, q_b, q_c) = (1, 1, 1)$ *and* $q' = (q'_a, q'_b, q'_c) = (2, 1, 1)$*. Note that* $D_a(DA^{\mathcal{C}}(R, q), R) = D_a(DA^{\mathcal{C}}(R', q), R') = \{2, 3\}$*. However,* $D_a(DA^{\mathcal{C}}(R, q'), R) = \{\emptyset\}$ *and* $D_a(DA^{\mathcal{C}}(R', q'), R') = \{3\}$*. Thus,* $DA^{\mathcal{C}}$ *is a lexicographic deferred acceptance mechanism but violates strong demand-monotonicity.*

6 Conclusion

Our formulation of a choice rule and the properties that we consider take into account that the capacity may vary. When designing choice rules especially for resource allocation purposes, such as in school choice, a designer may be interested in how the choice rule responds to changes in capacity. In that framework, our Theorem 1 shows that *acceptance*, *gross substitutes*, *monotonicity*, and *CWARP* are altogether satisfied only by lexicographic choice rules, which identifies the properties that distinguish lexicographic choice rules from other plausible choice rules.

A lexicographic choice rule in our setup operates based on a unique list containing as many priority orderings as the maximum possible capacity. Alternatively, one could consider capacity-wise lexicographic choice rules that we have defined in Section 4, which, at each capacity, operate based on a list containing as many priority orderings as the capacity, yet the lists for different capacity levels do not have to be related in any way. A characterization of capacity-wise lexicographic choice rules, or in other words characterizing lexicographic choice rules for a fixed capacity, is also an important step in the analysis of lexicographic choice rules, which we do not answer in our paper.

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Appendix

A Independence of Properties in Theorem 1

Violating only acceptance: Let $A = \{a, b, c\}$. Let $\succ$ be a priority ordering. Let C be the choice rule such that, for each problem (S, q), $C(S, q)$ is a singleton consisting of the

$\succ$-maximal alternative in S. Note that C violates *acceptance* and clearly satisfies *gross substitutes*. Since the choice does not vary with capacity, C also satisfies *monotonicity* and *CWARP*.

Violating only gross substitutes: Let $A = \{a, b, c\}$. Let $\succ$ and $\succ'$ be defined as $a \succ b \succ c$ and $b \succ' a \succ' c$. Let the choice rule C be defined as follows. For each problem (S, q), $C(S, q)$ consists of the $\succ$-maximal alternative in S if $q = 1$ and $c \in S$; otherwise, $C(S, q)$ coincides with the choice rule that is responsive for $\succ_2$. Note that C satisfies *acceptance*.

Since $a \in C(\{a, b, c\}, 1) = \{a\}$ and $a \notin C(\{a, b\}, 1) = \{b\}$, C violates *gross substitutes*. To see that C satisfies *monotonicity*, first note that there is a unique choice set S such that $C(S, 2) \neq \emptyset$, which is $S = \{a, b, c\}$. Therefore, the only possibility to violate monotonicity is to have $x \in A$ such that $x \in C(\{a, b, c\}, 1)$ and $x \notin C(\{a, b, c\}, 2)$. Since $C(\{a, b, c\}, 1) = \{a\}$ and $C(\{a, b, c\}, 2) = \{a, b\}$, C satisfies *monotonicity*. To see that C satisfies *CWARP*, note the revealed preference relation at $q = 2$ consists of a unique pair: b is revealed preferred to c.

Violating only monotonicity: Let $A = \{a, b, c\}$. Let $\succ$ be defined as $a \succ b \succ c$. Let the choice rule C be defined as follows. For each problem (S, q), $C(S, q)$ consists of the $\succ$-maximal alternative in S if $q = 1$; $C(S, 2) = S$ if $|S| = 2$; and $C(\{a, b, c\}, 2) = \{b, c\}$. Note that C satisfies *acceptance*.

Since $a \in C(\{a, b, c\}, 1)$ and $a \notin C(\{a, b, c\}, 2)$, C violates monotonicity. For $q = 1$, C satisfies *gross substitutes*, since C maximizes $\succ$; for $q \in \{2, 3\}$, C clearly satisfies *gross substitutes*. To see that C satisfies *CWARP*, note that the revealed preference relation is empty at $q = 2$, since $C(\{a, b, c\}, 1) = \{a\}$ and $C(\{a, b, c\}, 2) = \{b, c\}$.

Violating only CWARP: Note that three of the four rules that we have discussed in Section 4 satisfy all the properties but *CWARP*.

B Definitions of the Properties in Section 5

Unavailable-Type-Invariance: Let $(R, q) \in \mathcal{P}$ and $R' \in \mathcal{R}^N$. If for each $i \in N$ and each pair of available objects $x, y \in O$ ($q_x > 0$, $q_y > 0$) we have [$x \ R_i \ y$ if and only if

$x\ R'_i\ y]$, then $\varphi(R,q) = \varphi(R',q)$.

Weak Non-Wastefulness: For each $(R,q) \in \mathcal{P}$, each $x \in O$ such that $q_x > 0$, and each $i \in N$, if $x\ P_i\ \varphi_i(R,q)$ and $\varphi_i(R,q) = \emptyset$, then $|\{i \in N : \varphi_i(R,q) = x\}| = q_x$.

Resource-Monotonicity: For each $R \in \mathcal{R}^N$, and each pair of capacity profiles (q,q'), if for each $x \in O$, $q_x \leq q'_x$, then for each $i \in N$, $\varphi_i(R,q')\ R_i\ \varphi_i(R,q)$.

Truncation-Invariance: Let $(R,q) \in \mathcal{P}$ and $R' \in \mathcal{R}^N$. If for each $i \in N$ and each pair of objects $x,y \in O$ we have $[x\ R_i\ y$ if and only if $x\ R'_i\ y]$ and $\varphi_i(R,q)\ R'_i\ \emptyset$, then $\varphi(R,q) = \varphi(R',q)$.

Strategy-proofness: For each $(R,q) \in \mathcal{P}$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $\varphi_i(R,q)\ R_i\ \varphi_i((R'_i,R_{-i}),q)$.