On Acceptant and Substitutable Choice Rules (Very Preliminary) *

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May 16, 2017

Abstract

We analyze *acceptant* and *substitutable* choice rules that are prominently used in resource allocation problems. We discipline the structure of *collected maximal representation* of these choice rules due to Aizerman and Malishevski (1981) by restricting the number of priorities that appear in the representation. We constructively show that the number of *prime atoms* of a choice rule determines the smallest size collected maximal representation. We observe that *responsive* choice rules render collected maximal representations of the largest size among all acceptant substitutable choice rules. Finally, we characterize collected maximal choice rules in which the number of priorities equals the capacity. It follows from this characterization that if the difference between the size of the universal set of elements and the capacity is bigger than two, then it is impossible to have such a choice rule.

JEL Classification Numbers: D01, D03, C78, D47, D78.

Keywords: Choice rules, school choice, diversity.

*Battal Doğan gratefully acknowledges financial support from the Swiss National Science Foundation (SNSF) and Kemal Yıldız gratefully acknowledges financial support from the Scientific and Research Council of Turkey (TUBITAK). We thank Ahmet Alkan and Bettina Klaus for valuable comments.

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1 Introduction

Choice rules are essential in the analysis of resource allocation problems in which a set of objects, each of which has a certain capacity, is to be allocated among agents. Although the relevant restrictions on choice rules vary across applications, *acceptant* and *substitutable* choice rules remain as the general prominent class of choice rules for many relevant applications. In this study, we analyze *representation* of acceptant and substitutable choice rules. First, we introduce these two properties of choice rules that have already been studied extensively in the previous literature.

Acceptance: an element is rejected from a choice set only if the capacity is full.

Substitutability: If an element is chosen from a choice set, then it is also chosen from any subset of the choice set that contains the element.

Acceptance of a choice rule is a natural restriction when there is limited number of positions that has to be filled. Substitutable choice rules have been a standard tool in the matching literature following the seminal work of Kelso and Crawford (1982). Beyond its normative appeal, substitutability of choice rules is an "almost necessary" condition for the non-emptiness of the *core* (Hatfield and Kojima, 2008). In particular, if priorities are substitutable, then the core coincides with the set of *stable* allocations. It follows that substitutability of choice rules guarantee the existence of *stable matchings*, which is a central desideratum for applications. Similarly, several classical results of matching literature have been generalized with substitutable choice rules (Roth and Sotomayor (1990), Hatfield and Milgrom (2005)). Despite all their eminence for applications, acceptant and substitutable choice rules lack a *canonical representation*. We try to address this problem and explore its implications.

Acceptance together with substitutability imply another well-known property called *path independence,*[1] which requires that if the choice set is "split up" into smaller sets, and if the choices from the smaller sets are collected and a choice is made from the collection, the final result should be the same as the choice from the original choice set (Plott, 1973).[2] In an early study, Aizerman and Malishevski (1981) show that for

[1]This is also noted in Remark 1 of Doğan and Klaus (2016), and it follows from Lemma 1 of Ehlers and Klaus (2016) together with Corollary 2 of Aizerman and Malishevski (1981).

[2]Among others, Plott (1973), Moulin (1985), and Johnson and Dean (2001) study the structure of

each *path independent* choice rule, there exists a list of *priority orderings*[3] such that the choice from each choice set is the union of the highest priority alternatives in the priority orderings.[4] We call these *collected maximal choice rules*, and discipline the structure of these choice rules by restricting the number of priorities that appear in the representation. That is, for given capacity q, a choice rule C has a *collected maximal representation* of size m (or simply called *collected m-maximal*) if there exists a list of m-many priorities $(\succ_1, \ldots, \succ_m)$ such that for each choice set S that contains more than q elements, $C(S)$ is obtained by collecting the maximizers of the priority orderings in S; if S contains at most q elements, then all elements in S are chosen.

Although it follows from Aizerman and Malishevski (1981) that each acceptant and substitutable choice rule is collected maximal, they remain silent about the minimal size of the collected maximal representation. Moreover, to best of our knowledge, including Aizerman and Malishevski (1981), there is no prior study constructing the priority orderings that render a collected maximal representation of acceptant and substitutable choice rules. That is, so far, acceptant and substitutable choice rules lack having a canonical representation.

We introduce the concept of a "prime atom" of a choice rule, which will be key in finding the minimal number of priorities needed for rendering a collected maximal representation of an acceptant and substitutable choice rule. Given a choice rule C, a choice set T with q elements is a *prime atom* if there exists an element a that is chosen whenever added to T, but no longer chosen whenever any other element is added to $T \cup \{a\}$. In our Theorem 1, we constructively prove that the number of prime atoms determines the smallest size collected maximal representation for each acceptant and substitutable choice rule.

A well-known example of an acceptant and substitutable choice rule is *responsive* choice rule that has been studied particularly in the two-sided matching context (Roth and Sotomayor, 1990).[5] For given capacity q, a choice rule C is *responsive* if there exists

path independent choice rules. Chambers and Yenmez (2016) study *path independence* in the matching context and its connection to stable matchings.

[3]A *priority ordering* $\succ$ is a complete, transitive, and anti-symmetric binary relation over the universal set of elements.

[4]In the words of Aizerman and Malishevski (1981), each *path independent* choice rule is generable by some mechanism of collected extremal choice.

[5]For example, starting with the seminal study by Abdulkadiroğlu and Sönmez (2003), the school choice

a priority ordering $\succ$ such that for each choice set S, $C(S)$ is obtained by choosing the highest $\succ$-priority elements until the capacity q is reached or no element is left.[6] In Proposition 1, we show that the upper bound on the number of prime atoms is achieved by responsive choice rules. That is, responsive choice rules render a collected maximal representation of the largest size among all q-acceptant choice rules that satisfy substitutability.

For any acceptant choice rule, the minimum number of priorities that can render a collected maximal representation is at least equal to the given capacity q. Next, we analyze collected q-maximal choice rules. For the applications, such as school choice, we believe that collected q-maximal choice rules have particular appeal. As pointed out by several studies to achieve a diverse student body, schools implement affirmative action policies. These policies are in the form of designing the choice rules of the schools used in the school choice problem. In particular, as Kominers and Sönmez (2016) put it, schools typically come up with *slot specific priorities* and apply these lexicographically to decide the students to be accepted. In this vein, Dur et al. (2013) and Dur et al. (2016) provide empirical evidence, by using school choice data from Boston school district, indicating that the order in the priority profile may cause additional, possibly unintended, advantage for some group of students. If a school with q-many slots chooses according to a collected q-maximal choice rule, then it could transparently reveal the order to be maximized to fill each slot. Moreover, since a collected q-maximal choice rule would make the same choice independent of the order to be followed in filling the slots, it is immune to, rather debatable, order based affirmative action effects.

In Section 5, we characterize acceptant choice rules that are collected q-maximal. An impossibility result follows from this characterization, in that if the difference between the size of the universal set of elements and the capacity is bigger than two, then there is no collected q-maximal choice rule. On the other hand, we observe that whenever it is possible to have a collected q-maximal choice rule, different representations of the same choice rule are similar to each other in a particular way.

literature has widely focused on problems where the choice rule of a school is responsive to a given priority ordering over students. However, when there are other concerns such as achieving a diverse student body or affirmative action, which choice rule to use is non-trivial.

[6] Chambers and Yenmez (2013), in their Theorem 6, show that, a classical choice rule satisfies *acceptance* and *weakened weak axiom of revealed priority (WWARP)* if and only if it is responsive.

2 Preliminaries

Let A be a nonempty finite set of n elements and let $\mathcal{A}$ denote the set of all *nonempty* subsets of A. A **choice rule** $C : \mathcal{A} \to \mathcal{A}$ associates with each choice set $S \in \mathcal{A}$, a nonempty set of elements $C(S) \subset S$. Let $q \in \{1, \ldots, n\}$ be a given capacity. We analyze choice rules that satisfy the following two properties that are well-known in the literature.

(q-)Acceptance: For given capacity $q \in \mathbb{N}$, an element is rejected from a choice set at a capacity q only if the capacity is full. Formally, for each $S \in \mathcal{A}$,

$$|C(S)| = \min\{|S|, q\}.$$

Substitutability: If an element is chosen from a choice set, then it is also chosen from any subset of the choice set that contains the element. Formally, for each $S \in \mathcal{A}$ and each pair $a, b \in S$ such that $a \neq b$,

$$\text{if } a \in C(S), \text{ then } a \in C(S \backslash \{b\}).$$

Each q-acceptant choice rule C satisfies substitutability if and only if C satisfies *path independence*[7] which requires that if the choice set is "split up" into smaller sets, and if the choices from the smaller sets are collected and a choice is made from the collection, the final result should be the same as the choice from the original choice set (Plott, 1973). Formally, for each $S, S' \in \mathcal{A}$, $C(S \cup S') = C(C(S) \cup C(S'))$.

2.1 Collected maximal choice

Aizerman and Malishevski (1981) argues that a choice rule is *path independent* if and only if there exists a list of priority orderings such that the choice from each choice set is the union of the highest priority alternatives in the priority orderings. In the words of Aizerman and Malishevski (1981), each *path independent* choice rule is *generable by some mechanism of collected extremal choice*. Next, we formally define and add more

[7]This is also noted in Remark 1 of Doğan and Klaus (2016), and it follows from Lemma 1 of Ehlers and Klaus (2016) together with Corollary 2 of Aizerman and Malishevski (1981).

structure on these choice rules that we call *collected maximal choice rules*.

A **priority ordering** $\succ$ is a complete, transitive, and anti-symmetric binary relation over A. A **priority profile** $\pi = (\succ_1, \ldots, \succ_m)$, for some $m \in \mathbb{N}$, is an ordered list of m **distinct** priority orderings. Let Π denote the set of all priority profiles. Given $S \in \mathcal{A}$ and a priority ordering $\succ$, let $\max(S, \succ) = \{a \in S : \forall b \in S \setminus \{a\}, a \succ b\}$.

A choice rule C has a **collected maximal representation** of size $m \in \mathbb{N}$ (or simply called **collected m-maximal**) if there exists $(\succ_1, \ldots, \succ_m) \in \Pi$ such that for each $S \in \mathcal{A}$ with $|S| \leq q$, $C(S) = S$ and for each $S \in \mathcal{A}$ with $|S| > q$, $C(S)$ is obtained by collecting the maximizers of the priority orderings in S, that is,

$$C(S) = \bigcup_{i \in \{1, \ldots, m\}} \max(S, \succ_i).$$

Next, we give two examples of well-known q-acceptant choice rules that satisfy substitutability. These choice rules have been studied particularly in the two-sided matching context (Roth and Sotomayor, 1990). It follows from Aizerman and Malishevski (1981) that these choice rules are collected maximal.

Example 1. *A choice rule C is **responsive** if there exists a priority ordering $\succ$ such that for each $S \in \mathcal{A}$, $C(S)$ is obtained by choosing the highest $\succ$-priority elements until the capacity q is reached or no element is left.*[8]

Example 2. *A choice rule C is **lexicographic** if there is a list of priority orderings $(\succ_1, \ldots, \succ_n) \in \Pi$ such that for each choice set $S \in \mathcal{A}$, $C(S)$ is obtained by choosing the highest $\succ_1$-priority alternative in S, then choosing the highest $\succ_2$-priority alternative among the remaining alternatives, and so on until q alternatives are chosen or no alternative is left. Lexicographic choice rules have been useful in designing allocation mechanisms for school choice to achieve diversity.*[9]

[8] Chambers and Yenmez (2013), in their Theorem 6, show that, a choice rule satisfies *acceptance* and *weakened weak axiom of revealed priority (WWARP)* if and only if it is responsive.

[9] See **?** for an axiomatic characterization of lexicographic choice rules in the school choice context.

3 A canonical representation of acceptant and substitutable choice rules

Although it follows from Aizerman and Malishevski (1981) that a q-acceptant choice rule C satisfies substitutability if and only if C is collected maximal, Aizerman and Malishevski (1981) tells nothing about the minimal size of the collected maximal representation. Moreover, to best of our knowledge, including Aizerman and Malishevski (1981) there is no prior study constructing the priority orderings that render a collected maximal representation of acceptant and substitutable choice rules. In this section, we attack this problem and provide a canonical representation of acceptant and substitutable choice rules.

We introduce the concept of a "prime atom" of a choice rule, which will be key in finding the minimal number of priorities needed for rendering a collected maximal representation of an acceptant and substitutable choice rule. Given a choice rule C, a choice set $T \in \mathcal{A}$ is a **prime atom** if $|T| = q$, and there exists $a \notin T$ such that a is chosen from $T \cup \{a\}$, but no longer chosen whenever any other element is added to $T \cup \{a\}$. Next, we state this definition formally.

Definition. *A choice set $T \in \mathcal{A}$ is a **prime atom** if $|T| = q$, and there exists an element $a \notin T$ such that $a \in C(S \cup \{a\})$ and for each $b \notin S \cup \{a\}$, $a \notin C(S \cup \{a, b\})$.*

The first result shows that the number of prime atoms determines the smallest size collected maximal representation.

Theorem 1. *For each choice rule C that satisfies q-acceptance and substitutability,*

i. C has a collected maximal representation of a size equal to the number of its prime atoms.

ii. C fails to have a collected maximal representation of any size smaller than the number of its prime atoms.

Proof of Theorem 1 : **Part i.** Let C be an acceptant and substitutable choice rule. We first define some notions, then introduce some lemmas, and finally present the representation result.

A choice set is called "maximal" if it includes any other choice set from which the same set of alternatives is chosen. Formally, a choice set $S \in \mathcal{A}$ is **maximal** for C if for each choice set $S' \in \mathcal{A} \setminus S$ such that $C(S') = C(S)$, we have $S' \subset S$. Let $\mathcal{M}$ denote the set of maximal choice sets for C.

We define the following binary relations on $\mathcal{M}$. For each $S, S' \in \mathcal{M}$, S is a **parent** of S', denoted by $S \to S'$, if there exists $a \in C(S)$ such that $S' = S \setminus \{a\}$. For each $S, S' \in \mathcal{M}$, S is an **ancestor** of S', denoted by $S \searrow S'$, if there exists a collection of sets in $S_1, \ldots, S_k \in \mathcal{M}$ such that $S \to S_1 \to \cdots \to S_k \to S'$. Since the binary relation $\searrow$ is transitive, $(\mathcal{M}, \searrow)$ is a partially ordered set.

Each subset of $\mathcal{M}$ that is linearly ordered according to $\searrow$ is called a **chain** in $(\mathcal{M}, \searrow$ $)$. By adopting the terminology from order theory, we call each $T \in \mathcal{M}$ such that $|T| = q$ as an **atom** of C. A choice set $S \in \mathcal{M}$ is a **prime** of C if S has a unique parent, that is, there exists a unique $S' \in \mathcal{M}$ such that $S' \to S$. Let $\mathcal{P}$ denote the set of all primes of C. A collection of primes $S_1, \ldots, S_k \in \mathcal{P}$ such that $S_1 \to \cdots \to S_k$ is called a **prime chain** from S_1 to S_k. An atom of C that is also a prime of C is called a **prime atom** of C.

Lemma 1. *For each choice set $S \in \mathcal{A}$ such that $|S| = q$, there exists a unique set $S' \in \mathcal{M}$ such that $C(S') = S$.*

Proof. Since, by acceptance, $C(S) = S$, a maximal choice set S' with $C(S') = S$ exists. Suppose that there are two such maximal sets, say S' and S''. Since they are both maximal, $S' \cup S''$ is different from and includes each of S' and S''. By path independence, $C(S' \cup S'') = S$, contradicting that S' and S'' are maximal. □

Lemma 2. *If a maximal choice set S is not an atom, then each choice set S' such that $S \to S'$ is also maximal. That is, for each choice set $S \in \mathcal{M}$ such that $|S| > q$ and each $a \in C(S)$, we have $S \setminus \{a\} \in M$.*

Proof. By contradiction suppose there exists $S' \in \mathcal{M}$ with $S \setminus \{a\} \subset S'$ and $C(S') = C(S \setminus \{a\})$. Now, consider the set $S' \cup \{a\}$. By path independence, we have $C(S' \cup \{a\}) = C(C(S') \cup \{a\})$. We have $C(S') = C(S \setminus \{a\})$ and $a \in C(S)$. Since C satisfies substitutability, $C(C(S') \cup \{a\}) = C(S)$. It follows that $C(S' \cup \{a\}) = C(S)$, contradicting that $S \in \mathcal{M}$. □

Lemma 3. *If a maximal choice set nests another, then there is a path that connects the two. That is, for each $S, S' \in \mathcal{M}$ such that $S' \subset S$, we have $S \searrow S'$.*

Proof. Let $S, S' \in \mathcal{M}$ such that $S' \subset S$. Since $S, S' \in \mathcal{M}$ and C satisfies substitutability, there exists $x_0 \in C(S) \setminus S'$. Let $S_1 = S \setminus \{x_0\}$. It follows from Lemma 2 that $S_1 \in \mathcal{M}$, and $S' \subset S$ implies there exists $x_1 \in C(S_1) \setminus S'$. By proceeding similarly we obtain a path $\{S_1, \ldots, S_k\}$ that connects S to S'. That is, $S \searrow S'$. $\square$

Lemma 4. *Let $S \in \mathcal{P}$ be a prime and let $a \in A \setminus S$. Then, $S \cup \{a\}$ is a parent of S if and only if a is no longer chosen whenever any other element is added to $S \cup \{a\}$, that is, $S \cup \{a\} \to S$ if and only if $a \in C(S \cup \{a\})$, but for each $b \notin S \cup \{a\}$, $a \notin C(S \cup \{a, b\})$.*

Proof. (*If part*) Suppose $S \cup \{a\} \in \mathcal{M}$ is the parent of S for some $a \in A$. By contradiction suppose that there exists $b \notin S \cup \{a\}$ such that $a \in C(S \cup \{a, b\})$. In what follows we argue that S has a parent other than $S \cup \{a\}$, contradicting that S is prime. To see this, first note that since $S \cup \{a\}$ is the maximal set in which the $C(S \cup \{a\})$ is chosen, $C(S \cup \{a, b\}) \neq C(S \cup \{a\})$. Now, let S' be the maximal set with $C(S') = C(S \cup \{a, b\})$. Since $S \cup \{a, b\} \subset S'$, $|S'| > |S \cup \{a\}|$. Since $a \in C(S \cup \{a\})$, $a \in C(S')$. Let $S'' = S' \setminus \{a\}$. Since $a \in C(S')$, it follows from Lemma 2 that $S'' \in \mathcal{M}$. Since $S \subset S''$, it follows from Lemma 3 that $S'' \searrow S$. Since $a \notin S''$, this implies that there exists $S^* \in \mathcal{M}$ such that $S^* \neq S \cup \{a\}$ and $S^* \to S$, contradicting that S is prime.

(*Only if part:*) Suppose $a \in C(S \cup \{a\})$ and for each $b \notin S \cup \{a\}$, $a \notin C(S \cup \{a, b\})$. It directly follows that, since C satisfies substitutability, there is no $S' \in \mathcal{A}$ with $C(S') = C(S \cup \{a\})$. Therefore $S \cup \{a\} \in \mathcal{M}$ and $S \cup \{a\} \to S$. $\square$

Lemma 5. *Each prime that is not an atom is the parent of a unique prime. That is, for each $S \in \mathcal{P}$ such that $|S| > q$, there exists a unique prime $S' \in \mathcal{P}$ such that $S \to S'$.*

Proof. Let $S \in \mathcal{M}$ with $|S| > q$, and suppose that S is prime. Let $S \cup \{b^*\}$ be the unique parent of S. Since $b^* \in C(S \cup \{b^*\})$ and C is q-acceptant, there exists $a \in C(S) \setminus C(S \cup \{b^*\})$. Consider the choice set $S \setminus \{a\}$. Clearly $S \to S \setminus \{a\}$. Since $S \in \mathcal{M}$ and $a \in C(S)$, it follows from Lemma 2 that $S \setminus \{a\} \in \mathcal{M}$. In what follows we show that $S \setminus \{a\}$ is prime. Suppose that $S \setminus \{a\}$ is not prime. Lemma 4 implies that there exists $b \notin S$ such that $a \in C(S \cup \{b\})$. Since, by our choice, $a \notin C(S \cup \{b^*\})$, $b \neq b^*$. Next, consider the choice set $S \cup \{b^*, b\}$. Since S is prime, $S \cup \{b^*\} \to S$, and $b \notin S \cup \{b^*\}$, it follows from Lemma 4 that $b^* \notin C(S \cup \{b^*, b\})$. Now, since $a \in C(S \cup \{b\})$, it follows from path independence that $a \in C(S \cup \{b^*, b\})$. But we also have $a \notin C(S \cup \{b^*\})$, contradicting that C satisfies substitutability. Thus, we obtain that $S \setminus \{a\}$ is prime and $S \to S \setminus \{a\}$. That is, S has a prime child.

To see that it is unique, by contradiction, suppose that there exist $a, a' \in C(S)$ such that $S \to S \setminus \{a\}$ and $S \to S \setminus \{a'\}$, where both $S \setminus \{a\}$ and $S \setminus \{a'\}$ are prime. Now, since A is not prime, $S \neq A$, and there exists some $x \notin S$. Next, consider $S \cup \{x\}$. It follows from Lemma 4 that $a, a' \notin C(S \cup \{x\})$. This combined with C being q-acceptant implies there exists $y \in C(S \cup \{x\}) \setminus C(S)$. This contradicts that C satisfies substitutability. $\square$

Lemma 6. *From each prime $S \in \mathcal{P}$ that is not an atom, there exists a unique prime chain connecting S to a prime atom $T \in \mathcal{P}$. Moreover, the unique prime chain from S to T is included in any chain from A to T. Formally, for each $S \in \mathcal{P}$ such that $|S| > q$, there exists a unique list $S_1, \ldots, S_k \in \mathcal{P}$ such that $S \to S_1 \cdots \to S_k \to T$ and T is a prime atom; and for each $t \in \{1, \ldots, k\}$, S_t is included in each chain from A to T.*

Proof. By Lemma 5, there exists a unique prime $S_1 \in \mathcal{P}$ such that $S \to S_1$. Applying Lemma 5 consecutively, there exists a unique list $S_1, \ldots, S_k \in \mathcal{P}$ such that $S \to S_1 \cdots \to S_k \to T$ and T is a prime atom. Now, since for each $t \in \{1, \ldots, k\}$, S_t is prime and T is prime atom, each S_t and T has a unique parent. It follows that for each $t \in \{1, \ldots, k\}$, S_t must be included in any chain that connects A to T. $\square$

Lemma 7. *For each nonprime maximal choice set S and its parent $S \cup \{a\}$, there exists a maximal choice set $S' \in \mathcal{M}$ such that $S \cup \{a\} \subsetneq S'$ and $a \in C(S')$.*

Proof. Suppose that S is a nonprime maximal choice set and $S \cup \{a\} \to S$. Since S is not a prime, there exists $b \notin S \cup \{a\}$ such that $S \cup \{b\} \to S$. Thus, $b \in C(S \cup \{b\})$. Now, consider the choice set $S \cup \{a, b\}$. By contradiction, suppose that $a \notin C(S \cup \{a, b\})$. Since C satisfies q-acceptance and substitutability, $C(S \cup \{a\}) \setminus \{a\} = C(S \cup \{b\}) \setminus \{b\} \subset C(S \cup \{a, b\})$. Since $a \notin C(S \cup \{a, b\})$, we must have $b \in C(S \cup \{a, b\})$. Therefore, $C(S \cup \{a, b\}) = C(S \cup \{b\})$, contradicting to $S \cup \{b\} \in \mathcal{M}$. Hence, $a \in C(S \cup \{a, b\})$. Now, by Lemma 1, there exists $S' \in \mathcal{M}$ such that $C(S') = C(S \cup \{a, b\})$. Since S' is maximal, $S \cup \{a, b\} \subset S'$. Since $a \in C(S \cup \{a, b\}) = C(S')$, and C satisfies substitutability, $a \in C(S')$. Thus, S' gives us the choice set with all the desired properties. $\square$

Now, we are ready to construct the set of priority orderings that renders the desired representation. For each prime atom $T \in \mathcal{P}^{\mathcal{A}}$, we will associate a priority ordering with a chain that connects A to T. Let $T \in \mathcal{P}^{\mathcal{A}}$ and let $S_1, \ldots, S_k \in \mathcal{M}$ be such that $A \to S_1 \to \cdots \to S_k \to T$ is a chain that connects A to T. Let $a_1 = A \setminus S_1$, $a_{k+1} = S_k \setminus T$, and for each $i \in \{2, \ldots, k\}$, $a_i = S_{i-1} \setminus S_i$. Note that by definition of a parent, for each $i \in \{1, \ldots, k+1\}$, a_i is well-defined. Now, let $\succ^T$ be such that for each $i, j \in \{1, \ldots, k+1\}$, $a_i \succ^T a_j$ if $i < j$, and assume that any other remaining element is ranked below a_{k+1} arbitrarily. A priority profile $(\succ^T)_{T \in \mathcal{P}^{\mathcal{A}}}$ is constructed similarly.

Note that, since each $S \in \mathcal{M}$ with $A \to S$ is prime, we obtain at least q-many priority orderings, i.e. $|\mathcal{P}^{\mathcal{A}}| \geq q$. Otherwise, by Lemma 6, there would be two different primes S and S' such that $A \to S$, $A \to S'$, and S and S' have prime chains to the same prime atom T, which would be a contradiction since it would imply that the prime atom T has at least two parents.

In what follows we show that for each $S \in \mathcal{A}$ such that $|S| > q$,

$$C(S) = \bigcup_{T \in \mathcal{P}^{\mathcal{A}}} \max(S, \succ^T) \tag{1}$$

It is sufficient to show that (1) holds for each maximal set. So, let $S \in \mathcal{M}$ be such that $|S| > q$.

First, we show that $\cup_{T \in \mathcal{P}^{\mathcal{A}}} \max(A, \succ^T) \subset C(S)$. Let $T \in \mathcal{P}^{\mathcal{A}}$. For each $a \in A$, if $a = max(S, \succ^T)$, then it follows from the construction of $\succ^T$ that there exists $S^* \in \mathcal{M}$ such that $a \in C(S^*)$ and $S \subset S^*$. Since C satisfies substitutability, $a \in C(S)$.

Next, we show that $C(S) \subset \cup_{T \in \mathcal{P}^{\mathcal{A}}} \max(S, \succ^T)$. The proof is by induction on the cardinality of S. Suppose that $|S| = n$, that is, $S = A$. Let $a \in C(S)$. Note that $S' = A \setminus \{a\}$ is a prime. If S' is a prime atom, which is the case if and only if $q = n - 1$, then $a = \max(S, \succ^{S'})$. If S' is a prime that is not an atom, then by Lemma 6, there exists a unique prime chain connecting S to a prime atom $T \in \mathcal{P}$. Moreover, again by Lemma 6, the unique prime chain from S to T is included in any chain from A to T. Hence, $a = \max(S, \succ^T)$.

Now, let k be such that $n > k > q + 2$ and assume that for each $S \in \mathcal{M}$ with $|S| \geq k$, our induction hypothesis is true, that is, $C(S) \subset \cup_{T \in \mathcal{P}^{\mathcal{A}}} \max(S, \succ^T)$. Next, we show that the hypothesis is true also for each set of cardinality $k-1$.

Let $S \in \mathcal{M}$ be such that $|S| = k-1$. Let S' be a parent of S, i.e. $S' \in \mathcal{M}$ with $S' \to S$. Let $a \in A$ and suppose that $S' = S \cup \{a\}$. Since $|S'| = k$, by induction hypothesis, we know that (1) holds for S'. Since C satisfies substitutability, we have $C(S') \setminus \{a\} \subset C(S)$ and $C(S') \setminus \{a\} \subset \cup_{T \in \mathcal{P}^{\mathcal{A}}} \max(S, \succ^T)$. Now, let $x \in C(S) \setminus C(S')$. Next, we show that there exists $T \in \mathcal{P}^{\mathcal{A}}$ such that $x \in max(S, \succ^T)$. We consider two cases.

Case 1: Suppose that S is a prime. It follows from Lemma 2 that there is a unique $S'' \in \mathcal{P}$ with $S \to S''$. Since, by acceptance, x is the only element in S that is chosen in S but not chosen in S', by Lemma 4, we have $S'' = S \setminus \{x\}$. If S'' is a prime atom, then $x = \max(S, \succ^{S''})$. If S'' is a prime that is not an atom, then by Lemma 6, there exists a unique prime chain connecting S'' to a prime atom $T \in \mathcal{P}$. Moreover, again by Lemma 6, the unique prime chain from S'' to T is included in any chain from A to T. Hence, $a = \max(S, \succ^T)$.

Case 2: Suppose that S is not a prime. Consider $S \setminus \{x\}$. By Lemma 2, $S \setminus \{x\} \in \mathcal{M}$. Suppose that $S \setminus \{x\} \in \mathcal{P}$. If $S \setminus \{x\}$ is a prime atom, then $x = \max(S, \succ^{S \setminus \{x\}})$. If $S \setminus \{x\}$ is a prime that is not an atom, then by Lemma 6, there exists a unique prime chain connecting $S \setminus \{x\}$ to a prime atom $T \in \mathcal{P}$. Moreover, again by Lemma 6, the unique prime chain from $S \setminus \{x\}$ to T is included in any chain from A to T. Hence, $a = \max(S, \succ^T)$.

Suppose that $S \setminus \{x\} \notin \mathcal{P}$. Then, by Lemma 7, there exists $S^* \in \mathcal{M}$ such that $S \subsetneq S^*$ and $x \in C(S^*)$. Since $S \subsetneq S^*$, $|S^*| \geq k+1$. It follows from the induction hypothesis that $x = max(S^*, \succ^T)$ for some $T \in \mathcal{P}^{\mathcal{A}}$. Since $S \subsetneq S^*$, $x \in S$, and C satisfies substitutability, we obtain $x = max(S, \succ^T)$.

Part ii. Let C be an acceptant and substitutable choice rule. Suppose that C has an acceptant and collected maximal representation of size $m \in \mathbb{N}$, say for the priority profile $(\succ_1, \ldots, \succ_m)$. Consider the set of prime atoms $\mathcal{P}^{\mathcal{A}}$ of C. Consider a pair of distinct prime atoms $T, T' \in \mathcal{P}^{\mathcal{A}}$. Since T and T' are prime, there exist a unique $a \notin T$ and a unique $a' \notin T'$ such that $T \cup \{a\} \to T$ and $T' \cup \{a'\} \to T'$.

Now, we show that there exist $i, j \in \{1, \ldots, m\}$ such that $\succ_i \neq \succ_j$, and $a = max(T \cup \{a\}, \succ_i)$, $a' = max(T' \cup \{a'\}, \succ_j)$. By contradiction, suppose that there is a unique $k \in \{1, \ldots, m\}$ with $a = max(T \cup \{a\}, \succ_k)$ and $a' = max(T' \cup \{a'\}, \succ_k)$. Suppose, without loss of generality, that $a \succ_k a'$ or $a = a'$. Consider the choice set $T \cup T'$. Clearly, $a = max(T \cup T', \succ_k)$. Hence, $a \in C(T \cup T')$. Since $T \neq T'$ and C satisfies substitutability, there exists $x \notin T \cup \{a\}$ such that $a \in C(T \cup \{a, x\})$, which is a contradiction since $T \cup \{a\}$ is a parent of the prime T and by Lemma 4, a is no longer chosen whenever any other element is added to $T \cup \{a\}$.

4 Collected maximal representation of responsive choice rules

A well-known example of a q-acceptant choice rule that satisfies substitutability is *responsive* choice rule that have been studied particularly in the two-sided matching context (Roth and Sotomayor, 1990). A q-acceptant choice rule C is **responsive** if there exists a priority ordering $\succ$ such that for each $S \in \mathcal{A}$, $C(S)$ is obtained by choosing the highest $\succ$-priority elements until the capacity q is reached or no element is left.[10] In Proposition 1, we show that the upper bound on the number of prime atoms is achieved by responsive choice rules. Put differently, responsive choice rules render a collected maximal representation of the largest size among all q-acceptant choice rules that satisfy substitutability.

Proposition 1. *There exists a unique m^* such that*

i. Each q-acceptant choice rule C that satisfies substitutability has a collected maximal representation of a size less than or equal to m^*.

ii. If C is responsive, then C has a collected maximal representation of size at least m^*.

Proof. First to prove (i), let C be a q-acceptant choice rule that satisfies substitutability, and consider the maximal choice sets with $q+1$ elements, i.e. $\mathcal{G} = \{S \in \mathcal{M} : |S| = q+1\}$. It follows from Lemma 5 that each $S \in \mathcal{G}$ has at most one prime child $S \setminus \{a\} \in \mathcal{P}^{\mathcal{A}}$. Therefore, the number of prime atoms is at most $|\mathcal{G}|$. Thus, we show that m^* is an upper bound on the number of prime atoms. Then, (i) directly follows from Theorem 1.

[10] Chambers and Yenmez (2013), in their Theorem 6, show that, a classical choice rule satisfies *acceptance* and *weakened weak axiom of revealed priority (WWARP)* if and only if it is responsive.

Next to prove (ii), let C be a q-acceptant choice rule that is responsive with respect to priority $\succ$. We show that there exists $a \in A$ such that for each $S \in \mathcal{G}$, $S \setminus \{a\} \in \mathcal{P}^{\mathcal{A}}$. Let $a \in A$ be the $(n-1)^{th}$-ranked and b be the $(n)^{th}$-ranked element at $\succ$. First, note that since C is responsive w.r.t. $\succ$, for each $S \in \mathcal{M}$ with $|S| = n-1$, the $q+1^{th}$-ranked element at $\succ$ is chosen from S. Proceeding similarly, we have for each $S \in \mathcal{G}$, $a \in C(S)$. Moreover, since C is responsive w.r.t. $\succ$ and b is the bottom ranked element at $\succ$, for each $S, S' \in \mathcal{G}$, we have $S \setminus C(S) = S' \setminus C(S')$={b}.

We argue that for each $S \in \mathcal{G}$, $S \setminus \{a\}$ is prime. To see this, we use Lemma 4. For each $x \notin S \cup \{a\}$, consider $S \cup \{a, x\}$. Since $x \neq b$, we have $x \succ \{a\}$. It follows that $a \notin C(S \cup \{a, x\})$. Lemma 4 implies that $S \setminus \{x\}$ is prime. Now, since for each $S, S' \in \mathcal{G}$, $S \neq S'$, we have $S \setminus \{a\} \neq S' \setminus \{a\}$. That is each $S \in \mathcal{G}$ is the parent of a prime. This combined with Theorem 1 implies that C is collected m-maximal if and only if $m = |\mathcal{G}|$. □

Our Example 3 shows that responsive choice rules are not unique in requiring the maximal number of priority orderings to get represented as a collected maximal of a priority profile. In that, although the choice rule in Example 3 is not responsive, the number of its prime atoms equals the number of its maximal choice sets with $q+1$ elements.

Example 3. *Let $A = \{1, 2, 3, 4, 5, 6\}$ and consider the priority profile $(\succ_\alpha, \succ_\beta, \succ_\gamma, \succ_\delta)$. Let C be the 2-acceptant choice rule that is collected maximal of this priority profile. The choice lattice $(\mathcal{M}, \searrow)$ associated with C is depicted in Figure 1.*

$\succ_\alpha$	$\succ_\beta$	$\succ_\gamma$	$\succ_\delta$
5	5	3	3
1	1	4	4
4	3	5	1
2	2	2	2
6	6	6	6
3	4	1	5

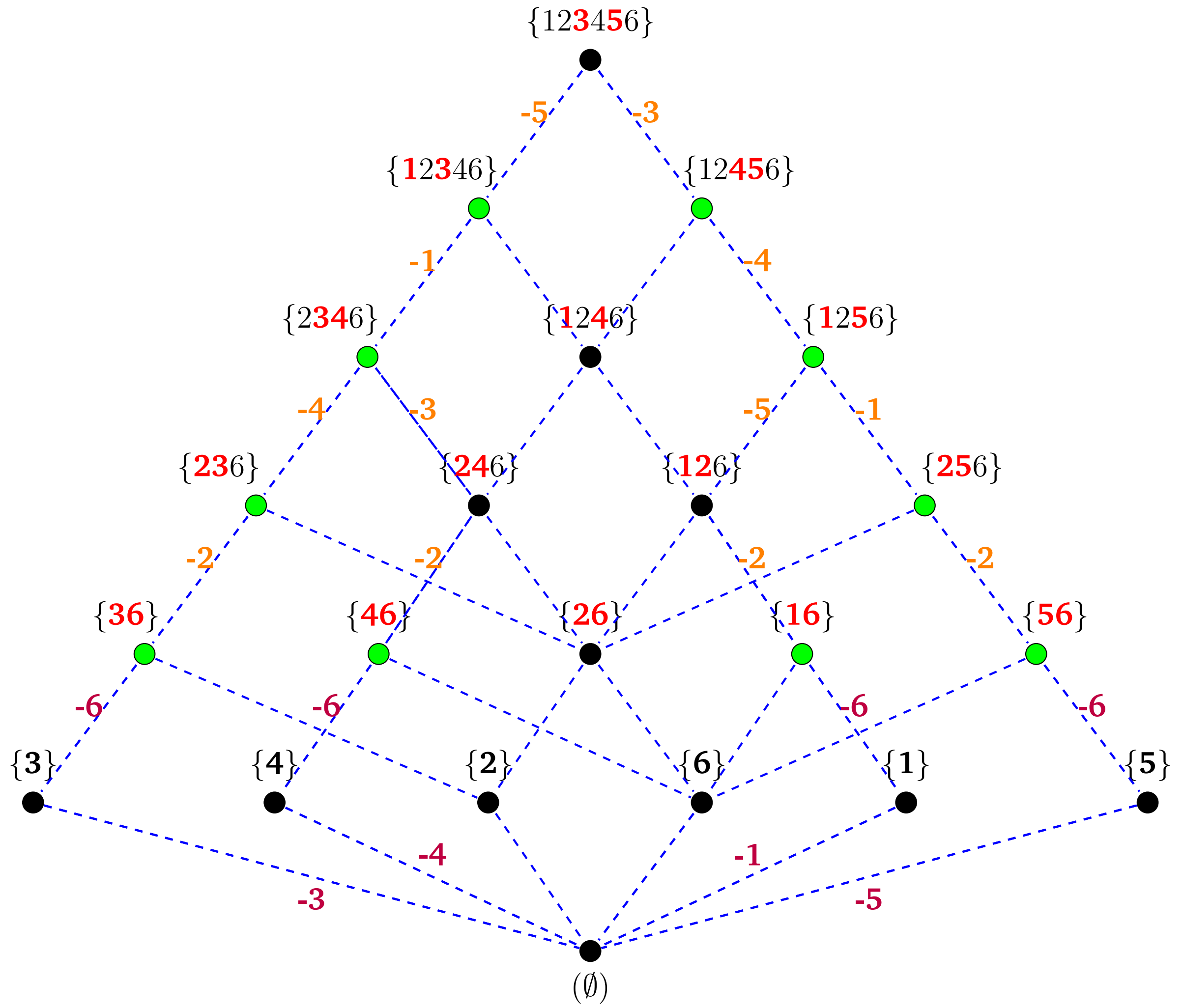

Figure 1: Lattice representation of the choice rule in Example 3

5 Collected q-maximal choice rules

Proposition 1 shows that the upper bound on the number of prime atoms is achieved by responsive choice rules. It follows that responsive choice rules have the largest size canonical collected maximal representation. On the other hand, given a universal set A with n elements and capacity q, for each q-acceptant choice rule, the minimum number of priorities that can render a collected maximal representation is at least q. In this section, we analyze choice rules that are collected q-maximal. In what follows, we first characterize q-acceptant choice rules that are collected q-maximal. It follows from this characterization that if the difference between the size of the universal set of elements and the capacity is bigger than two, then there is no q-acceptant choice rule that is collected q-maximal.

To characterize the q-acceptant choice rules that are collected q-maximal, we introduce a new property called *strong blocking*. An element a **blocks** another element b in a choice set S, if a is chosen in S, and b is not chosen in S, but b is chosen whenever a is removed from S.

Strong Blocking: For each choice set S with $|S| > q+1$, if an element a blocks another element b in S, then for each $S' \subset S$ with $a, b \in S'$ and $|S'| > q+1$, a blocks b in S'.

Theorem 2. *Let C be a q-acceptant choice rule. C is collected q-maximal if and only if C satisfies substitutability and strong blocking.*

Proof. (*Only if part*) Let C be a q-acceptant choice rule that is collected maximal of the priority profile $(\succ_1, \ldots, \succ_q)$. It is easy to see that C satisfies *substitutability*. To see that it satisfies *no blocking*, let $a, b \in A$ be such that a blocks b. Then, there exists $S \in \mathcal{A}$ and $i \in \{1, \ldots, q\}$ such that $a \succ_i b$ and for each $c \in S \setminus \{a, b\}$, $b \succ_i c$. Now, let $S' \subset S$ be such that $a \in S'$ and $|S'| > q+1$. Consider the set $S'' = S' \setminus \{a\}$. Since $|S''| \geq q+1$ and b is top ranked by $\succ_i$ in S'', $b \in C(S'')$. Since C must choose q distinct elements from S'', there cannot be any other priority that top ranks b in S''. Since $a \succ_i b$, there is no priority that top ranks b in S'. Thus, $b \notin C(S')$.

(*If part*) Let C be a q-acceptant choice rule that satisfies the substitutability and strong blocking. First, we construct a priority profile $(\succ_1, \ldots, \succ_q)$. Since C is q-acceptant, $|C(A)| = q$. Let $C(A) = \{a_{11}, \ldots, a_{q1}\}$. For each $i \in \{1, \ldots, q\}$, let $a_{i2} = C(A \setminus \{a_{i1}\})$. Similarly, for each $j \in \{2, \ldots, n-q+1\}$, let $a_{ij} = C(A \setminus \{a_{i1}, \ldots a_{ij-1}\}) \setminus C(A)$. Note that since C satisfies *substitutability*, $C(A) \setminus \{a_{i1}\} \subset C(A \setminus \{a_{i1}, \ldots a_{ij-1}\})$. Since C satisfies *q-acceptance*, $C(A \setminus \{a_{i1}, \ldots a_{ij-1}\}) \setminus C(A)$ is a singleton. Therefore, for each $j \in \{2, \ldots, n-q+1\}$, a_{ij} is well-defined. Now, for each $i \in \{1, \ldots, q\}$, define $\succ_i$ such that $a_{i1} \succ_i a_{i2} \cdots \succ_i a_{in-q+1}$. Note that we did not specify how $\succ_i$ ranks the elements in $C(A) \setminus \{a_{i1}\}$. Let $\succ_i$ rank these elements at the bottom such that $a_{in-q+1} \succ_i a_{11} \cdots a_{(i-1)1} \succ_i a_{(i+1)1} \cdots \succ_i a_{q1}$.

At each choice set $S \in \mathcal{A}$ such that $|S| \leq q$, by *acceptance*, $C(S) = S$. Let $\bar{\mathcal{A}} \subseteq \mathcal{A}$ be the set of choice sets with cardinality at least $q+1$. We show that for each $S \in \bar{\mathcal{A}}$, $C(S) = \cup_{i \in \{1,\ldots,q\}} \max(S, \succ_i)$.

We claim that for each $S \in \bar{\mathcal{A}}$ and $a \in A$, if $a = \max(S, \succ_i)$ for some $i \in \{1, \ldots, q\}$, then $a \in C(S)$. To see this suppose that a is the j^{th}-ranked element in $\succ_i$, i.e. $a = a_{ij}$. Since $a_{ij} = \max(S, \succ_i)$, for each $k \in \{1, \ldots, j-1\}$, $a_{ik} \notin S$. Thus, $S \subset X \setminus \{a_{i1}, \ldots a_{ij-1}\}$. Now, since C satisfies *substitutability* and $a_{ij} = C(X \setminus \{a_{i1}, \ldots a_{ij-1}\})$, $a \in C(S)$.

Next, we show that at each choice set $S \in \bar{\mathcal{A}}$, the maximizers in $(\succ_1, \ldots, \succ_q)$ are distinct, i.e., there is no $a \in A$ and $i, j \in \{1, \ldots, q\}$, $i \neq j$ such that $a = \max(S, \succ_i) = \max(S, \succ_j)$. By contradiction, suppose not. Let $\mathcal{F} \subseteq \bar{\mathcal{A}}$ be the collection of all choice sets which have cardinality at least $q+1$ and for which the maximizers in $(\succ_1, \ldots, \succ_q)$ are not distinct. Note that, by supposition, $\mathcal{F} \neq \emptyset$.

Let $T \in \mathcal{F}$ be maximal in $\mathcal{F}$ according to set containment, i.e., there is no $T' \in \mathcal{F}$ such that $T \subsetneq T'$. Since $T \in \mathcal{F}$, there exist $b \in A$ and $i, j \in \{1, \ldots, q\}$, $i \neq j$ such that $a = \max(S, \succ_i) = \max(S, \succ_j)$. We claim that that there cannot be a priority $\succ_k$, different than $\succ_i$ and $\succ_j$, such that $b \in max(S, \succ_k)$. Suppose not. Since for each priority, there is a distinct top ranked element in A, b can be top ranked by at most one of the priorities $\{\succ_i, \succ_j, \succ_k\}$ in A. Let $c \in A \setminus \{b\}$ be top ranked by one of the other priorities in A, and consider the set $T \cup \{c\}$. Now, note that either b or c must be top ranked by two of the priorities $\{\succ_i, \succ_j, \succ_k\}$ at $T \cup \{c\}$, contradicting that T is maximal in $\mathcal{F}$.

Now, given that there are exactly two priorities $\succ_i$ and $\succ_j$ that top rank b in S, we claim that $b \neq \max(A, \succ_i)$ and $b \neq \max(A, \succ_j)$. By contradiction suppose w.l.o.g. that $b = \max(A, \succ_i)$. Then, by the construction of $(\succ_1, \ldots, \succ_q)$, for any $d \in A$, $b \succ_j d$ only if $d \in \{a_{11}, \ldots, a_{q1}\}$. Now, since $\succ_j$ top ranks b in S, $T \subset \{a_{11}, \ldots, a_{q1}\}$, which contradicts that $|T| \geq q+1$.

Since b is top ranked only by $\succ_i$ and $\succ_j$, but not top ranked by an of them in A, consider a_{i1} and a_{j1}, which are different from b and each other. Next, consider the set $T \cup \{a_{i1}, a_{j1}\}$. First, note that since $a_{i1} = \max(T \cup \{a_{i1}, a_{j1}\}, \succ_i)$, $a_{i1} \in C(T \cup \{a_{i1}, a_{j1}\})$. However, since there is no priority that top ranks b at $T \cup \{a_{i1}, a_{j1}\}$, $b \notin C(T \cup \{a_{i1}, a_{j1}\})$, otherwise T is not maximal in $\mathcal{F}$. Moreover, since $\succ_i$ top ranks b at $T \cup \{a_{j1}\}$, $b \in$

$C(T \cup \{a_{j1}\})$. Also, we have $b \in C(T \cup \{a_{i1}\})$, since $\succ_j$ top-ranks b at $T \cup \{a_{i1}\}$. Then, since $|T \cup \{a_{i1}\}| > q + 1$, , a_{i1} blocks b at $T \cup \{a_{i1}, a_{j1}\}$.

Finally, the fact that for each $S \in \bar{\mathcal{A}}$ and $a \in A$, if $a = \max(S, \succ_i)$ for some $i \in \{1, \ldots, q\}$, then $a \in C(S)$, together with the fact that at each choice set $S \in \bar{\mathcal{A}}$, the maximizers in $(\succ_1, \ldots, \succ_q)$ are distinct and that C satisfies *acceptance* imply that C is collected maximal. $\square$

Corollary 1. *For each capacity constraint q and universal set of elements with n members, if $q > 3$ and $n > q+2$, then there is no q-acceptant choice rule C that is collected q-maximal.*

Proof. We use Theorem 2 and show that if $q > 3$ and $n > q + 2$, then there is no q-acceptant choice rule C that satisfies substitutability and strong blocking. To see this, first let $C(A) = \{a_1, a_2, \ldots, a_q\}$. Note that since $n > q + 2$, there are at least three distinct elements $\{b_1, b_2, b_3\}$ such that for each $i \in \{1, 2, 3\}$, $b_i = C(A \setminus \{a_i\}) \setminus C(A)$. Now, consider the choice set $S = C(A) \cup \{b_1, b_2, b_3\}$. Since C satisfies substitutability, $C(S) = C(A)$. Moreover, since $q > 3$, there exists $a_4 \in C(A) \setminus \{a_1, a_2, a_3\}$. Next, consider the choice set $S \setminus \{a_4\}$. Since C is q-acceptant $C(S) \cap \{b_1, b_2, b_3\} \neq \emptyset$. Assume w.l.o.g that $b_1 \in C(S \setminus \{a_4\})$. It follows that a_4 blocks b_1 at S. Now, consider the choice set $S \setminus \{a_1\}$. We have $S \setminus \{a_1\} \subset S$ and $|S \setminus \{a_1\}| > q + 1$. But, since $b_1 \in C(A \setminus \{a_1\})$, substitutability implies that $b_1 \in C(S \setminus \{a_1\})$. Thus, a_4 fails to block b_1 at $S \setminus \{a_1\}$, indicating that C violates strong blocking. Therefore, it follows from Theorem 2 that if $q > 3$ and $n > q+2$, then there is no q-acceptant choice rule C that is collected q-maximal. $\square$

Remark 1. *For each capacity constraint q and universal set of elements with n members, if $q > 2$ and $n > q + 1$, then there is no q-acceptant choice rule C that is collected q-maximal of a priority profile $(\succ_1, \ldots, \succ_q)$, which also satisfies for each $S \in \mathcal{A}$ with $|S| \leq q$, $C(S) = \bigcup_{i \in \{1,\ldots,q\}} \max(S, \succ_i)$.*

To see this, by contradiction, suppose there is such a priority profile. Let $C(A) = \{a_1, \ldots, a_q\}$, then since $n \geq q + 2$, there exists $\{a, b\} \subset A \setminus C(A)$. Now, consider the choice set $\{a, b\}$, since $C(\{a, b\}) = \{a, b\}$, there exist $i, j \in \{1, \ldots, q\}$ such that $a \succ_i b$ and $b \succ_j a$. Since $q \geq 3$, there exists $k \in \{1, \ldots, q\}$ with $k \notin \{i, j\}$. Let $a_k = max(A, \succ_k)$, and consider the choice set $S^b = (C(A) \setminus \{a_i, a_k\}) \cup \{a, b\}$, which has q-many elements. It follows that $C(S^b) = S^b$ and $b \in C(S^b)$. Since $b \notin C(A)$ and $a \succ_i b$, this is possible only if $b \succ_k a$. Next *consider the choice set $S^a = (C(A) \setminus \{a_j, a_k\}) \cup \{a, b\}$. This time, by the same reasoning, we obtain that $a \succ_k b$, which leads a contradiction.*

Moreover, to see that if $n = q + 1$, then such choice rules exist, let $x = A \setminus C(A)$, and consider a priority profile with q priorities such that $C(A)$ is top ranked, and for each priority x is second ranked. It directly follows that a choice rule that collected is q-maximal of this priority profile also chooses S from each choice set S with $|S| \leq q$.

A q-acceptant choice rule C can be collected q-maximal of two different priority profiles. In fact, such a priority profile is never unique. However, if C collected maximal of two different priority profiles, these two profiles must be similar to each other in a particular way. To explain this similarity, consider a q-acceptant choice rule C that is collected q-maximal of priority profiles $(\succ_1, \ldots, \succ_q)$ and $(\succ'_1, \ldots, \succ'_q)$. Now, for each $\succ_i$ there exists $\succ'_j$ such that both have the same maximum element, and the other elements that are chosen from A are among the q bottom-ranked elements. Moreover, in both priorities, relative orderings of the elements that are not chosen from A are the same. To state this observation formally, let $(\succ_1, \ldots, \succ_q)$ be a given priority profile. For each $i \in \{1, \ldots, q\}$, let $a_{i1} = max(A, \succ_i)$ and $A_i = (A \setminus C(A)) \cup \{a_{i1}\}$. Let $\succ_i |_{A_i}$ stand for the restriction of $\succ_i$ to A_i.

Proposition 2. *If a q-acceptant choice rule C is collected q-maximal of a priority profile $(\succ_1, \ldots, \succ_q)$, then C is collected q-maximal of another priority profile $(\succ'_1, \ldots, \succ'_q)$ if and only if for each $\succ_i$, there exists $\succ'_j$ such that*

- *i.* $C(A) \setminus \{a_{i1}\}$ *are among the q bottom-ranked elements at $\succ'_j$, and*
- *ii.* $\succ_i |_{A_i} = \succ'_j |_{A_j}$.

Proof. (*If part*:) Let C be collected q-maximal of a priority profile $(\succ_1, \ldots, \succ_q)$, and $(\succ'_1, \ldots, \succ'_q)$ be another priority profile that satisfies the required property. For each $S \in \mathcal{A}$ and $a \in C(S)$, there exists some $\succ_i$ such that $a = max(S, \succ_i)$. Now, for this $\succ_i$, consider the priority $\succ'_j$ such that that satisfies (i) and (ii)

First, note that if a is among the q bottom-ranked elements at $\succ_i$, then it follows from (i) and (ii) that a is also among the q bottom-ranked elements at $\succ'_j$. Moreover, $a = max(S, \succ_i)$ implies that $|S| \leq q$. Therefore, we do not refer to the priority profile for the choice. Now, suppose that a is not among the q bottom-ranked elements at $\succ'_j$.

Since $\succ_i |_{A_i} = \succ'_j |_{A_i}$, $a = max(S, \succ_i)$. It directly follows that $a = max(S, \succ'_j)$. Finally, if $a \in C(A) \setminus \{a_{i1}\}$, then $a = max(A, \succ_k)$ for some $k \in \{1, \ldots, q\}$ with $k \neq i$. It follows from (i) and (ii) that there exists $\succ'_k$ with $a = max(A, \succ'_k)$, so $a = max(S, \succ'_k)$. Therefore, C is also collected q-maximal of $(\succ'_1, \ldots, \succ'_q)$.

(*Only if part*:) Consider a q-acceptant choice rule C that is collected q-maximal of both priority profiles $(\succ_1, \ldots, \succ_q)$ and $(\succ'_1, \ldots, \succ'_q)$. Since C is q-acceptant, for each $a \in C(A)$, there exists a unique $i, j \in \{1, \ldots, q\}$ such that $a_{i1} = max(A, \succ_i)$ and $a_{i1} = max(A, \succ_j)$. We show that $\succ'_j$ satisfies (i) and (ii) for $\succ_i$.

First, we show that $\succ'_j$ satisfies (i). By contradiction, suppose there exists $a \in C(A)$, which is top ranked at another $\succ'_k$, but is not among the q bottom ranked elements at $\succ'_j$. That is, there exist distinct $x, y \in A \setminus C(A)$ with $a \succ'_j x$ and $a \succ'_j y$. Since a_{j1} is top ranked at $\succ'_j$, $x \neq a_{j1}$ and $y \neq a_{j1}$. Now, consider the choice set $(C(A) \setminus \{a_{j1}\}) \cup \{x, y\}$. Note that this set has $q+1$ elements. But, since $a \succ'_j x$, $a \succ'_j y$, and a is top ranked at another $\succ'_k$, collected maximization of $(\succ'_1, \ldots, \succ'_q)$ gives the set $C(A) \setminus \{a_{j1}\}$ with $q-1$ elements, contradicting that C is q-acceptant.

Next, we show that $\succ'_j$ satisfies (ii), i.e. $\succ_i |_{A_i} = \succ'_j |_{A_i}$. For each $i \in \{1, \ldots, q\}$, let $\mathcal{A}_i$ stand for the collection of all nonempty subsets of A_i. Now, we show that there is a unique way to specify $\succ_i |_{A_i}$ To see this, first define the choice function $c_i : \mathcal{A}_i \to A_i$ such that for each $S \in \mathcal{A}_i$, $c_i(S) = C(S) \setminus (C(A) \setminus \{a_{i1}\})$. Since C satisfies substitutability, $C(A) \setminus \{a_{i1}\} \subset C(S)$. Since C is also q-acceptant, it follows that $c_i(S)$ is single valued. Moreover, since C satisfies *substitutability*, c_i also satisfies *substitutability*. Therefore, the preference relation $\succ_i^*$ that satisfies for each $S \in \mathcal{A}_i$, $c_i(S) = \max\{S, \succ_i^*\}$ is unique. It follows that we must have $\succ_i |_{A_i} = \succ'_j |_{A_i} = \succ_i^*$. □

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