Choice Regularities: Relative identification of choice theories*

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Abstract

For a given choice function, **a first order regularity** is an "*if...then*" requirement of the form: *if some alternative a_1 is chosen from some choice set S_1, then some alternative a_2 should be chosen from some choice set S_2*. This definition is extended to k^{th}*-order regularity* by having k many choice requirements in the precedent part of the statement. A choice theory is k^{th}*-order regular* if one can find a collection of k^{th}-order regularities that identifies this theory. We use the order of regularity to discipline the complexity of a choice axiom. In our analysis, first we establish a formal account of the contrast between extending rational choice theory and having simple axiomatic characterizations. As our main result, we consider a family of boundedly rational choice theories, and characterize all the second order regularities that might be satisfied by any of these theories. We observe that second order regularities are sufficient to observationally disentangle each theory from other theories in this family. **J.E.L. codes**: DO

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1 Introduction

In most economic models it is assumed that decision makers are rational, in that they choose as if maximizing a preference relation. Samuelson's classical result (Samuelson (1938)) characterizes rational choice theory in terms of a simple axiom, called *weak axiom of revealed preference* (WARP). The simplicity of WARP provides normative appeal and ease of identification from the observables. On the other hand, recent and growing body of literature on boundedly rational choice theory seeks to accommodate choice behavior that a rational decision maker does not exhibit. In this literature, choice theories that are consistent with such choice behavior are proposed. Analysis of these choice theories provide axiomatic characterizations used to identify the proposed choice theory from the observables. However, the used axioms turn out to be not as simple as WARP.

In this paper, we propose a novel approach to identify choice theories from observed choice behavior. We call this approach *relative identification*. To motive relative identification, consider an outside observer who has in mind a set of choice theories that may be consistent with the observed choices of a decision maker. Put differently, the observer starts with a set of choice theories that may or may not span the set of all possible choice functions. Now, the problem is to identify which theories are consistent with the observed choices. For each theory, one can answer this question by verifying the axioms that characterizes the theory. Our view is that, in such a problem instead of going through the characterization axioms for each theory, one can find the answer by verifying some simple behavioral differences that distinguishes each theory. If for each pair of theories in consideration, there is a simple condition that is satisfied by one of the theories but not by the other, then for this set of theories we achieve *full relative identification*.

To achieve relative identification for a given set of choice theories, we seek to find out some *simple* behavioral differences that distinguishes each model. This task requires a measure for the *simplicity* of a choice axiom. We propose the notion of *regularities* to measure the simplicity of a choice axiom. First, we establish a formal account of the contrast between extending rational choice theory to accommodate choice anomalies and having simple axiomatic characterizations. Next, we consider a family of boundedly rational choice theories to conduct a relative identification exercise. We investigate if we can observationally disentangle each theory from other theories in this family via simple axioms.

We formulate the notion of regularity in the revealed preference framework in which there is a finite *alternative set*, any non-empty subset of which is a *choice set*. A *choice function* singles out an alternative from each choice set. We refer to any collection of choice functions as a *choice theory*. Given a choice function c, **a first order regularity** is a statement of the form:

$$\textbf{if } a = c(S_1), \ \textbf{then } b = c(S_2)$$

for some pair of alternatives a, b and choice sets S_1, S_2.[1] A choice theory is **first order regular** (**1-regular**) if one can find a collection of first order regularities that identifies the theory. Put differently, a choice function satisfies all of these regularities if and only if this choice function belongs to the theory. For example, consider the weak axiom of revealed preference (WARP):

WARP: For each choice sets S_1, S_2, such that $S_2 \subset S_1$ and alternative $a \in S_2$,

$$\text{if } a = c(S_1), \ \text{then } a = c(S_2).$$

WARP condition is a collection of first order regularities. Since a choice function c is rational if and only if c satisfies WARP, rational choice theory is first order regular. Next, we ask if one can formulate another theory that is first order regular and that nests rational choice. We observe that if a theory that nests rational choice satisfies a 1-regularity, then this 1-regularity must be in the form of WARP. Roughly speaking, it follows that any choice theory that nests rational choice and accommodates choice anomalies is not first order regular. This observation formally clarifies the contrast

[1] Inspiration comes from Glazer and Rubinstein (2014).

between extending rational choice theory to accommodate systematic choice anomalies and the simplicity of the choice axioms used in characterizations.

Given the pessimistic result regarding first order regularity, we focus on *second order regularities*. Similarly, given a choice function c, **a second order regularity** is a statement of the form:

$$\textbf{if } a = c(S_1) \ \textbf{ and } \ b = c(S_2), \ \textbf{then } c = c(S_3)$$

for some alternatives a, b, c and choice sets S_1, S_2, S_3. A choice theory is **second order regular (2-regular)** if one can find a collection of second order regularities that identifies the theory. We observe that there exist second order regular choice theories[2] that nests rational choice. However, there are several interesting boundedly rational choice theories that are not 2-regular. For the main result of this paper, we focus on a particular set of boundedly rational choice theories, namely Rationalization via Game Trees (Xu and Zhou (2007)), Sequentially Rational Choice (Manzini and Mariotti (2007), Apesteguia and Ballester (2013)), Revealed Attention (Masatlioglu et al. (2012)), and List Rational Choice (Yildiz (2012)). We characterize the set of second order regularities that at least one of these theories satisfy. As a corollary to this result, we show that second order regularities provide sufficient statistics for relative identification of the theories in our set. More precisely, suppose in addition to observing the choice function of a decision maker, we know that this choice behavior would fit into at least one of the theories in our set. Now, instead of going through the characterization axioms for each theory, one can find the answer via possible separations emanating from second order regularities. For example, we show that *list rational choice* satisfies our *weak path independence* condition but *revealed attention* does not. Indeed, we observe that for each pair of theories in our set, there is at least one second order regularity that is satisfied by one of the theories but not by the other, hence for this set we achieve full relative identification via second order regularities.

[2] Rational Shortlist Method (Manzini and Mariotti (2007)), Categorize then Choose (Manzini and Mariotti (2012)), and Rationalization (Cherepanov et al. (2013)).

In the rest of the paper, we formulate and analyze the regularity notion in its full generality. We observe that any arbitrary choice theory can be identified once we allow the highest order of regularity, which is $2^n - n - 2$ where n is the number of alternatives. This observation indicates that if we can verify enough complex statements pertaining to observed choice behavior, then we can identify any choice theory. The number $2^n - n - 2$ can be taken as an index for the complexity of any choice theory for which there is no characterization. Now, one can argue that the characterization of rational choice is particularly appealing since it shows the theory is 1-regular as opposed to being $(2^n - n - 2)$-regular. Similarly, one can measure the "normative appeal" or "ease of identification" obtained out of axiomatic characterizations via the gains in order of regularities. For such an exercise, we consider two choice theories: Revealed Attention (Masatlioglu et al. (2012)) and List Rational Choice Theory (Yildiz (2012)). We show that axioms used in their characterizations consist of $2n-1$ order of regularities where n is the number of alternatives.

2 Model

Let A be a fixed non-empty alternative set with n alternatives. Let Ω denote the collection of all subsets of A with at least 2 elements. A choice function is a non-empty valued mapping $c : \Omega \to A$ such that for each $S \in \Omega$, $c(S) \in S$. A choice theory τ is a collection of choice functions. By this definition two choice procedures with possibly different formulations are considered as equivalent as far as these models are observationally indistinguishable in the revealed preference framework.

For a choice function c, **a first order regularity** (**1-reg**) is a statement of the form:

$$\textbf{if } a = c(S_1), \ \textbf{then } b = c(S_2)$$

for some $a, b \in A$ and $S_1, S_2 \in \Omega$.

Definition 1 *A choice theory τ is **first order regular (1-regular)** if one can find a* *collection of* ***first order regularities*** *that identifies τ, i.e. a choice function c satisfies all of these regularities if and only if $c \in \tau$.*

WARP: For each $S_1, S_2 \in \Omega$, such that $S_2 \subset S_1$ and $a \in S_2$,

$$\text{if } a = c(S_1), \text{ then } a = c(S_2).$$

Since, a choice function c is rational if and only if c satisfies WARP, rational choice theory is 1-regular. At a general level, this result characterizes a choice procedure in terms of a simple axiom. The simplicity of such an axiom provides normative appeal for the choice theory, and facilitates the identification of the choice theory from the observables. Given the notion of first order regularity, a natural question is: Can we formulate another theory that is 1-regular and nests (generalizes) rational choice. Next, we show that this is not possible. This observation formally clarifies the contrast between accommodating choice anomalies and the simplicity of the choice axioms used to characterize a choice theory.

Observation 1 *Let τ be any choice theory that nests rational choice. If τ satisfies a 1-regularity q then q must be in the form of WARP.*

Proof. Let τ be any theory such that rational choice theory $\tau^{RC} \subset \tau$. Consider any 1-reg q that τ satisfies:

$$\textbf{if } a = c(S_1), \textbf{ then } b = c(S_2)$$

for some $a, b \in A$ and $S_1, S_2 \in \Omega$.

Step 1: We show that one must have $S_2 \subset S_1$. Suppose not, i.e. $S_2 \setminus S_1 \neq \emptyset$.

Case 1: Suppose $a = b$. Let $d \in S_2 \setminus S_1$. Next, we show that there is a choice function $c \in \tau$ that fails to satisfy q. Consider a choice function c rationalized by a preference relation such that: d is first-ranked and a is second-ranked. Note that $c(S_1) = a$, but $c(S_2) = d$.

Case 2: Suppose $a \neq b$, then consider a choice function c rationalized by a preference relation such that a is first ranked and b is bottom ranked. Note that $c(S_1) = a$, but $c(S_2) \neq b$ since there is at least one other alternative in S_2 that is preferred to a. It follows that we must have $S_2 \subset S_1$ at q.

Step 2: Now, we must have $a = b$, otherwise any rational choice function that chooses a from S_1 would not satisfy q. Hence the conclusion follows. ■

For a choice function c, a **second order regularity** (**2-reg**) is a statement of the form:

$$\textbf{if } a = c(S_1) \textbf{ and } b = c(S_2), \textbf{ then } c = c(S_3)$$

for some $a, b, c \in A$ and $S_1, S_2, S_3 \in \Omega$.

Definition 2 *A theory τ is* ***second order regular (2-regular)*** *if one can find a collection of* ***second order regularities*** *that identifies τ, i.e. a choice function c satisfies all of these regularities if and only if $c \in \tau$.*

Given our pessimistic observation regarding first order regularities, next we consider second order regularities. We ask if there is any known choice theory that is second order regular and nests rational choice. Answer is in the affirmative; for example consider the weak WARP condition formulated by Manzini and Mariotti (2007).

wWARP: For each $S_1, S_2 \in \Omega$ such that $S_2 \subset S_1$, and $a, b \in S_2$,

$$\text{if } a = c(S_1) \text{ and } b = c(S_2), \text{ then } b = c(a, b).$$

Note that wWARP is a collection of second order regularities. Since Rationalization (Cherepanov et al. (2013)) and Categorize then Choose Procedures (Manzini and Mariotti (2012)) are characterized by wWARP condition, these are 2-regular choice theories that nest rational choice theory.[3]

[3] Similarly, one can also easily observe that Rational Shortlist Method (Manzini and Mariotti (2007)) is also 2-regular, since wWARP and expansion axioms constitute a collection of second order regularities.

3 Relative Identification via 2-regularities

Our previous results indicates the contrast between formulating a theory that accommodates choice anomalies and having axiomatizations in terms of first order regularities. In this section, we consider a particular set of boundedly rational choice theories. Next, we provide a brief description for each theory that we will consider.

Rationalizability by Game Trees (Xu and Zhou (2007)) The primitive of this choice procedure is an extensive form game G. Each player has a preference relation $\succ_i$ over the outcomes of the game.[4] Each alternative appears as an end node of the associated game tree only once. For each choice set S, consider the reduced game $G|S$ derived from G by retaining the paths that only lead to the terminal nodes having outcomes in S. The decision maker chooses from each choice set S, the *subgame perfect Nash equilibrium* outcome of the game $G|S$.

Choice with Limited Attention (Masatlioglu et al. (2012)) This choice procedure has two primitives an attention filter Γ and a welfare preference $\succ$. From each choice set S, a decision maker first commits to the alternatives in $\Gamma(S)$ and then chooses the $\succ$-best alternative among these, where Γ is such that for each choice set S and $z \notin \Gamma(S)$, $\Gamma(S \setminus z) = \Gamma(S)$.[5]

Sequential rationalizability by binary rationales (Manzini and Mariotti (2007), Apesteguia and Ballester (2013)): The primitive of this procedure is a set of binary rationales such that each rationale compares only a pair of alternatives. decision maker remove inferior alternatives by sequentially applying this set of binary rationales according to a fixed order.

List Rationalizable Choice (Yildiz (2012)): LRC procedure has two primitives; an

[4]From a decision theoretic perspective each player can be interpreted as a different self of the same decision maker, concentrating on the different aspects of the alternatives

[5] Salant and Rubinstein (2008) present a choice procedure that has similar features in Section 5.

ordering of the alternatives, namely a *list*, and a *binary relation*[6] used to compare pairs of alternatives. A *list rational* decision maker chooses from a choice set as follows. First, he orders the alternatives according to the list. Then, by using the binary relation, he compares the first and second alternatives in the list and records the winner to be compared to the next alternative. This process of carrying the current winner to the next round continues until the last alternative in the list is compared to the winner from the previous round. Winner of the last round is the alternative chosen from the entire set. A list rational decision maker uses the same non-observable list and the same binary relation to make a choice from each choice set.

3.1 Result

As our main result we aim to identify all second order regularities that can be satisfied by these theories. Put differently, we provide the set of second order regularities such that each one is satisfied by at least one of these theories. Let us consider the following second order regularities which are not only simple to verify, but also might be normatively appealing. As for ease of identification, note that only WPI and NPC conditions are pertaining to choice sets with possibly more than two alternatives.

Weak Path Independence (WPI): For each $S \in \Omega$, and $a, b \in S$,

$$\text{if } a = c(S) \text{ and } b = c(S \setminus \{a\}), \text{ then } a = c(a, b).$$

Never Pairwise Chosen (NPC): For each $a, b, c \in A$, and $S \in \Omega$ such that $a, b \in S$,

$$\text{if } a = c(a, b, c) \text{ and } b = c(S), \text{ then } a = c(a, c).$$

Rival Monotonicity (RM): For each distinct $a, b \in A$ and $S, T \in \Omega$ such that $S \subset T$ and $b \in S$,

$$\text{if } a = c(S) \text{ and } b = c(T), \text{ then } a = c(S \setminus \{b\}).$$

[6]We assume it is complete and asymmetric.

Binary Expansion (BE): For each $a, b, c \in A$,

$$\text{If } a = c(a,b) \text{ and } a = c(a,c), \text{ then } a = c(a,b,c).$$

Path Existence (PE): For each $a, b, c \in A$,

$$\text{If } a = c(a,b,c) \text{ and } b = c(a,b), \text{ then } c = c(b,c).$$

Proposition 1 *Consider the following boundedly rational choice theories: Rationalization via Game Trees, Sequentially Rational Choice , Revealed Attention, and List Rational Choice. If any of these theories satisfies a second order regularity, then it must be in the form of WPI, NPC, RM, BE or PE. In the following table, we show the second order regularities that each theory satisfies.*

	WPI	*NPC*	*BE*	*RM*
GT	$+$	$+$	$+$	$-$
SRC	$-$	$+$	$+$	$-$
RA	$-$	$-$	$-$	$-$
LRC	$+$	$+$	$+$	$+$

Discussion: The following are possible insights from this exercise:

1. We obtain a better comparison among these models by observing simple behavioral differences that distinguishes each model. These differences might be difficult to infer from the characterization results.

2. One can not find an axiomatization of any of these choice theories such that the used axioms are second order regularities. Put differently, any other behavioral difference among these models can only be represented in terms of a third or higher order of regularity.

3. Given that the observed choice behavior is consistent with one of these theories, second order regularities provide sufficient statistics for relative identification.

More precisely, suppose in addition to observing the choice data of a decision maker, we know that his choice behavior would fit into at least one of the theories in our set. Now, the problem is to find out which ones are those. Instead of going through the characterization axioms for each theory, one can find the answer via possible separations emanating from second order regularities. For example, list rational choice satisfies weak path independence but revealed attention does not. Indeed, as shown in the table, for each pair of theories in our collection, there is at least one second order regularity that is satisfied by one of the theories but not by the other, hence for this set we achieve full relative identification via second order regularities.

4 k^{th}-order regularities

In this section, we formulate and analyze the regularity notion in its full generality: For any choice function c, a k^{th}**-order regularity** (**k-reg**) is a statement of the form:

$$\text{If } a_1 = c(S_1) \text{ and } a_2 = c(S_2) \cdots \text{ and } a_k = c(S_k), \text{ then } a_{k+1} = c(S_{k+1})$$

for some $a_1, \ldots, a_{k+1} \in A$ and $S_1, \ldots S_{k+1} \in \Omega$.

Definition 3 *A theory τ is **k-regular** if one can find a collection of k-regs that identifies τ, i.e. a choice function c satisfies all of these regularities if and only if $c \in \tau$.*

Our next result shows that any arbitrary choice theory can be identified once we allow any arbitrary degree of regularity. This observation indicates that the notion of regularity might be rich enough to serve as a complexity measure for the classification of choice theories.

Observation 2 *Let A be an alternative set with n elements. Any choice theory is $(2^n - n - 2)$-regular.*

Proof. Let τ be any choice theory. First, fix any distinct $a, b \in A$, and let $\Omega \setminus \{a, b\} = \{S_1, \ldots, S_m\}$, where $m = 2^n - n - 2$. Next, consider any choice function $c' \notin \tau$.

Assume w.l.o.g. that $a = c'(\{a, b\})$. Now, formulate a $2^n - n - 2$-regularity $q_{c'}$ such that:

$$c'(S_1) = c(S_1) \wedge c'(S_2) = c(S_2) \cdots \wedge c'(S_m) = c(S_m) \Rightarrow b = c(\{a, b\})$$

Finally, consider the collection of these regs $\{q_{c'}\}_{c' \notin \tau}$. We argue that $\{q_{c'}\}_{c' \notin \tau}$ identifies τ. To see this, first note that for each $c' \notin \tau$, by construction, c' does not satisfy $q_{c'}$. Next, we argue that for each $c \in \tau$ and $c' \notin \tau$, c satisfies $q_{c'}$. Suppose not, then it follows that there exists $c' \notin \tau$ such that c satisfies the precedent part of the statement $q_{c'}$, but not the consequent part. By the construction of $q_{c'}$, this means that for each $S \in \Omega$, $c(S) = c'(S)$. Hence we obtain a contradiction. ■

It follows from this result that if we can verify enough complex statements regarding choice data, then any choice theory can be identified. The number $2^n - n - 2$ can be taken as an index for the complexity of any choice theory for which there is no characterization. Now, one can argue that the characterization of rational choice is particularly appealing since it shows the theory is 1-regular as opposed to being $(2^n - n - 2)$-regular. Similarly, we can measure the "normative appeal" or "ease of identification" obtained out of axiomatic characterizations in terms of the gain in regularities. For such an exercise, we consider two choice theories: Revealed Attention (Masatlioglu et al. (2012)) and List Rational Choice Theory (Yildiz (2012)). We show that characterization of these models provides identification via $2n - 1$ regularities.

Observation 3 *Let A be an alternative set with n elements. The list rational choice theory, and the theory of revealed attention are $(2n - 1)$-regular.*

Proof. It follows from the characterization result in Masatlioglu et al. (2012) that a choice function c belongs to the theory of revealed attention if and only if the following binary relation P is acyclic.

$$x \; P \; y \text{ if there exists } T \in \Omega \text{ such that } c(T) = x \text{ and } x \neq c(T \setminus y).$$

Note that acyclicity condition in particular requires for each $x, y \in A$, if $x\ P\ y$, then $y \not P\ x$. Let $S_1, S_2 \in \Omega$ be given, then this requirement can be obtained via a collection of 3-regs $\{q_z\}_{z \in S_1 \setminus x}$ such that for each $z \in S_1 \setminus x$, q_z is written as:

$$x = c(S_1) \wedge z = c(S_1 \setminus y) \wedge y = c(S_2) \Rightarrow y = c(S_2 \setminus x).$$

First note that each cycle can be decomposed into non-intersecting cycles. Further, the length of a such a cycle is bounded by the number of alternatives, n. To describe each $x\ P\ y$ we add two regs. It follows that to rule out the the longest cycle we can form a collection $(2n - 1) - regs$.

Similarly, it follows from Yildiz (2012) that a choice function c is list rational if and only if together with the above relation, another similarly defined binary relation is acyclic. Derivation of the result for list rational choice theory is similar. ■

5 Proof of Proposition 1

It follows from Yildiz (2012) that list rational choice theory is nested by all the other three theories we consider. Hence, it would suffice to find out the 2-regularities satisfied by list rational choice theory. Consider any 2-reg, q, satisfied by τ^{LRC}: if $a = c(S_1)$ and $b = c(S_2)$, then $c = c(S_3)$ for some $a, b, c \in A$, and distinct $S_1, S_2, S_3 \in \Omega$.

Lemma 1 *If $a = b$ then q is in the form of BE, i.e. we have $S_1 = \{a, b\}, S_2 = \{a, c\}, S_3 = \{a, b, c\}$ for some $c \in A \setminus \{a, b\}$.*

Proof. First we show it is impossible to have $a = b \neq c$. Suppose not, then consider the following list rational, c: $c < .... < a$. Since for each $x \in S_1 \cup S_2 \cup S_3$, we have $x < a$, we obtain $a = c(S_1) = c(S_2)$. But since S_3 contains an alternative other than c, that one eliminates c, so we have $c \neq c(S_3)$.

Next, we show that it is possible to have $a = b = c$, only if q is in the form of BE. We first show that $S_3 \subset S_1 \cup S_2$. Suppose there is $x \in S_3 \setminus (S_1 \cup S_2)$, then consider the list rational $c : ...a < x$. For any choice of $S_1, S_2, S_3 \in \Omega$, we have $a = c(S_1) = c(S_2)$, but $c \neq c(S_3)$.

Second, we show that $S_3 \setminus S_1 \neq \emptyset$ and $S_3 \setminus S_2 \neq \emptyset$. Suppose $S_3 \subset S_1 \cap S_2$. Let $x \in S_1 \setminus S_3$, $y \in S_2 \setminus S_3$, and $z \in S_3 \setminus \{a\}$. Since $S_1 \neq S_2$, we can assume that $x \neq y$. Now, consider the list rational $c : ...z < y < x < a$, and also $z > a,\ a > y,\ x > z$. Note that in S_1 or S_2, z can eliminate a. However, x eliminates z in S_1, y eliminates z in S_2. Hence, we have $a = c(S_1) = c(S_2)$. But, since $x, y \notin S_3$, z is compared to a in the final round and eliminates a. So, we have $a \neq c(S_3)$.

Now, we know that there is $c \in S_3 \setminus S_1$ and $b \in S_3 \setminus S_2$, such that $b \neq c$, $b \in S_1$, $c \in S_2$. Finally we show that $S_1 \cup S_2 \subset \{a, b, c\}$. Suppose there is $z \in S_1 \setminus \{a, b, c\}$. Next, consider the list rational $c : ...\ z < c < b < a$, and also $z > b,\ a > z,\ a > c$. Since $c \notin S_1$ and $z > b$, z is compared to a in the final round and we have $a = c(S_1)$. Since $b \notin S_2$ and c eliminates z, c is compared to a in the final round and we have $a = c(S_2)$. But, since $b, c \in S_3$, c eliminates z, b eliminates c and then eliminates a in the final round. Hence, we have $b = c(S_3)$.

Since $S_3 \subset S_1 \cup S_2 \subset \{a, b, c\}$ and $a, b, c \in S_3$, we have $S_3 = \{a, b, c\}$, $S_1 = \{a, b\}$ and $S_2 = \{a, c\}$. It follows that q is in the form of BE. ■

For the next three lemmas consider any 2-reg q' satisfied by τ^{LRC} which requires: if $a = c(S_1)$ and $b = c(S_2)$, then $a = c(S_3)$ for some distinct $a, b \in A$ and distinct $S_1, S_2, S_3 \in \Omega$.

Lemma 2 *We have* $b \in S_1$.

Proof. Suppose $b \notin S_1$. As in the previous case first suppose there is $x \in S_3 \setminus S_1$, and let us pretend as if $x \neq$ and consider the $c : ...b > a < x$ and $b > x$. Note that irrespective of $x = b$ or not, we have $a = c(S_1)$ and $b = c(S_2)$, but since x eliminates a in S_3, $a \neq c(S_3)$. So, there is $y \in S_1 \setminus S_3$. Now, let $z \in S_3 \setminus \{a\}$. If $z = b$, then consider $c : ...y < a < b$, and also $b > y$. If $z \neq b$, then consider $c : ...z < y < a < b$, and also $z > a,\ b > \{y, z\}$. In both cases we have $a = c(S_1)$ and $b = c(S_2)$, but since b and z

respectively eliminates a in S_3, $a \neq c(S_3)$. ■

Lemma 3 *We have $S_3 \subset S_1$.*

Proof. Suppose not, i.e. there exists $y \in S_3 \setminus S_1$. Since $b \in S_1$, $y \neq b$. First suppose $S_1 \setminus S_2 \neq \emptyset$, and let $x \in S_1 \setminus S_2$. If $x \neq a$, then consider the list rational c: ... $b < x < a < y$, and $b > \{a, y\}$. Since $x \in S_1$, x eliminates b, and compared to a in the final round, so we have $a = c(S_1)$. Since $x \notin S_2$, and b eliminates a and y, we have $b = c(S_2)$. But since y eliminates a, we have $a \neq c(S_3)$.

If $x = a$, then consider the list rational c: ... $b < a < y$, and $b > y$. Since $y \notin S_1$, we have $a = c(S_1)$. Since $a \notin S_2$, and b eliminates y, we have $b = c(S_2)$. But since y eliminates a, we have $a \neq c(S_3)$.

Next, suppose $S_2 \setminus S_1 \neq \emptyset$, and let $z \in S_2 \setminus S_1$. Now, consider the list rational c: ... $a < z < b > y$, and also $a > b,\ y > a$. Since $z, y \notin S_1$, we have $a = c(S_1)$. Since z eliminates a in S_2, and b eliminates z and y, we have $b = c(S_2)$. But since y eliminates a, we have $a \neq c(S_3)$. ■

Lemma 4 *If q' is not in the form of NPC or RM, then we have $S_2 \subset S_1$.*

Proof. Suppose q' is not in the form of NPC or RM but there exists $x \in S_2 \setminus S_1$. By Lemma 2, we have $b \in S_1$, so $x \neq b$. Since, by Lemma 3, $S_3 \subset S_1$ and $S_1 \neq S_3$, there is $y \in S_1 \setminus S_3$. Since $x \notin S_1$, we have $x \neq y$.

A. Suppose there is $c \in S_3$ with $c \notin \{a, b\}$. Since $S_3 \subset S_1$ and $x \notin S_1$, we have $c \neq x$. Since $y \notin S_3$, $c \neq y$ either. Now, we are left with two possibilities. If $y \neq b$, then a, b, x, y, c are all distinct. If we must have $y = b$, then it follows that $S_1 = S_3 \cup \{b\}$.

A1. Suppose $y \neq b$, so a, b, x, y, c are all distinct. Now, consider the list rational c: ...$c < y < a < x < b$, also $c > a,\ a > b$, and $\{x, b\} > \{c, y\}$. Since $y \in S_1$, y eliminates c. For $c(S_1)$, since $x \notin S_1$, a eliminates b, and we have $a = c(S_1)$. For $c(S_2)$, note that x is compared to b in the final round, so we have $b = c(S_2)$. For $c(S_3)$, since $y \notin S_3$, c

always eliminates a.

A2. Now, suppose $S_1 = S_3 \cup \{b\}$. First, suppose that $S_3 = \{a, c\}$. Then $S_1 = \{a, b, c\}$. Since q' is not in the form of NPC, we have $a \notin S_2$. Now, consider the list rational c: $...c < b < a$, and also $c > a$. We have $a = c(S_1)$ and $b = c(S_2)$, but since c eliminates a, $a \neq c(S_3)$.

Second, suppose that there is $z \in S_3 \setminus \{a, c\}$. Since $S_3 \subset S_1$ and $x \notin S_1$, $z \neq x$. Since $b \notin S_3$, $z \neq b$. So, we know that a, b, c, x, z are all distinct. Moreover, we have $S_3 = S_1 \setminus \{b\}$. Since q' is not in the form of RM, there must be $w \in S_1 \setminus S_2$. Now, there are two possibilities $w = a$ or $w \neq a$. First suppose $w = a$ and consider c^l : $...c < x < b < a$, also $c > \{a, w\}$, and $\{x, b\} > c$. For $c(S_1)$, b eliminates c, a eliminates b, and we get $a = c(S_1)$. For $c(S_2)$, x eliminates c, b eliminates x. Since $a \notin S_2$, we have $b = c(S_2)$. For $c(S_3)$, since $x, b \notin S_3$ and c eliminates a, $a \neq c(S_3)$.

Next, suppose $w \neq a$. Now we might have $w \in S_3$, but either $w \neq c$ or $w \neq z$. Assume w.l.o.g. that $w \neq c$ and consider c^l: $...c < x < b < w < a$, also $c > \{a, w\}$ $b > a$, and $\{x, b\} > c$. For $c(S_1)$, b eliminates c, w eliminates b, and we get $a = c(S_1)$. For $c(S_2)$, x eliminates c, b eliminates x and a. Since $w \notin S_2$, we have $b = c(S_2)$. For $c(S_3)$, since $x, b \notin S_3$ and c eliminates w and a, $a \neq c(S_3)$.

B. Suppose $S_3 = \{a, b\}$. Since we assume that $S_3 = \{a, b\}$ and $x \notin S_1$, we have $y \notin \{a, b, x\}$. So, we know that a, b, x, y are all distinct. Now, consider the list rational c: $...y < x < b > a$, and also $y > b,\ a > y$. For $c(S_1)$, y eliminates b and compared to a in the final round. Since a eliminates y, we have $a = c(S_1)$. For $c(S_2)$, since x eliminates y and b eliminates x and a, we have $b = c(S_2)$. But for $c(S_3)$, since b eliminates a, $a \neq c(S_3)$. ■

Now, let us consider any 2-reg satisfied by τ^{LRC}. We focus on two cases separately First we consider the case where at least one of S_1, S_2, S_3 contains more than three alternatives.

Lemma 5 *For each 2-reg, q, satisfied by τ^{LRC}, if at least one of S_1, S_2, S_3 contains more* *than three alternatives, then q is in the form of WPI, NPC or RM.*

Proof. Consider any 2-reg, q, that requires: if $a = c(S_1)$ and $b = c(S_2)$, then $c = c(S_3)$ for some $a, b, c \in A$ and distinct $S_1, S_2, S_3 \in \Omega$ where at least one of them contains more three alternatives. Since at least one of the sets contains more three alternatives, q is not BE. It follows from Lemma 1 that $a \neq b$. In the rest of the proof we show that if q is not in the form of NPC or RM, then q must be in the form of WPI.

Step 1: We show that $c \in \{a, b\}$. Suppose this is not true. Since $S_1 \neq S_2$, assume w.l.o.g. that $S_1 \setminus S_2 \neq \emptyset$. Now, there are two possibilities:

A. Suppose there is $x \in S_1 \setminus S_2$ such that $x \neq c$. Now, there are two possibilities: $x = a$ or $x \neq a$. If $x = a$, then consider c^l : $...c < b < a$. If $x \neq a$, then consider c^l : $...c < b < x < a$, and also $a > x$, $b > a$. Note that in both cases we have $a = c(S_1)$ and $b = c(S_2) = b$, but $c \neq c(S_3)$.

B. Suppose $S_1 \setminus S_2 = \{c\}$. Since for some $i \in \{1, 2, 3\}$, S_i has more than three alternatives, there exists $d \in S_i$ such that $d \notin \{a, b, c\}$.

B1. Suppose $d \in S_3$. Then, consider c^l : $...c > b > d < a$, also $a > c$, and $b > a$. Regardless of which other choice set might also contain d, and whether $b \in S_1$ or not: For $c(S_1)$, first c eliminates b, then a eliminates c and d, so we have $a = c(S_1)$. For $c(S_2)$, since $c \notin S_2$, b eliminates d and a, so we have $b = c(S_2)$. For $c(S_3)$, since d eliminates c, $c \neq c(S_3)$.

B2. Suppose there is no such $d \in S_3$, but $d \in S_2$. It follows that $S_3 \subset \{a, b, c\}$

i. If $d \notin S_1$, then consider c^l : $...a < d < b$, and $a > b$. For $c(S_1)$, since $d \notin S_1$ and a eliminates b, we have $a = c(S_1)$. For $c(S_2)$, first d eliminates a, then b eliminates d and we have $b = c(S_2)$. For $c(S_3)$, since $S_3 \subset \{a, b, c\}$ and both a and b eliminate c, $c \neq c(S_3)$.

ii. If $d \in S_1$, then consider c^l: $...d < c < a > b$ also $d > a$, $b > c$ and $b > d$ For $c(S_1)$, since $c \in S_1$, first c eliminates d, then a eliminates c and possibly b, so we have

$a = c(S_1)$. For $c(S_2)$, since $c \notin S_2$ and $d \in S_2$, first d eliminates a, then b eliminates d, so we have $b = c(S_2)$. For $c(S_3)$, since $S_3 \subset \{a, b, c\}$ and both a and b eliminate c, $c \neq c(S_3)$.

B3. Suppose there is no such $d \in S_3 \cup S_2$, but $d \in S_1$. It It follows that $S_2 \cup S_3 \subset \{a, b, c\}$ Now consider $c^l : ...b < d < a$, and $b > a$. For $c(S_1)$, since $d \in S_1$, first d eliminates b, then a eliminates d, and we get $c(S_1) = a$. For $c(S_2)$, since $d \notin S_2$, b eliminates a, and we get $c(S_2) = b$. For $c(S_3)$, since $S_3 \subset \{a, b, c\}$ and both a and b eliminate c, $c \neq c(S_3)$.

Finally, we can conclude that $c \in \{a, b\}$. For the rest assume w.l.o.g. that $c = a$, where $a = c(S_1)$.

Step 2: Since q is not in the form of NPC or RM, it follows from Lemma 3 and Lemma 4 that $S_2 \cup S_3 \subset S_1$.

Step 3: We show that $S_3 = \{a, b\}$. Suppose not, i.e. there is $c \in S_3 \setminus \{a, b\}$. By the previous step, we know that $S_2 \subset S_1$ and $S_3 \subset S_1$.

A. Suppose there is $x \in S_1 \setminus (S_2 \cup S_3)$, so $x \neq c$. Since $b = c(S_2)$ and $a = c(S_3)$, $x \notin \{a, b\}$ either. It follows that a, b, c, x are all distinct. Now, consider $c^l : ...b < x > c > a$, also $b > \{c, a\}$ and $a > x$. For $c(S_1)$, since $x \in S_1$, first x eliminates b and c, then a eliminates x, and we get $a = c(S_1)$. For $c(S_2)$, since $x \notin S_2$, b eliminates a and c, and we get $b = c(S_2)$. But, for $c(S_3)$ since $x \notin S_2$ and both b and c eliminate a, we get $a \neq c(S_3)$.

B. Suppose $S_1 = S_2 \cup S_3$. Let $x \in S_1 \setminus S_2$ and $y \in S_1 \setminus S_3$. We know that $x \neq y$, $x \in S_3$ and $y \in S_2$. It follows that $x \neq b$ and $y \notin \{a, c\}$. Moreover since S_1 must have at least four elements we can not both have $x = a$ and $y = b$.

i. Suppose that $x = a$ but $y \neq b$, and consider $c^l : ...c < y > b < a$ and $c > a$. For $c(S_1)$, first y eliminates c and b, then a eliminates y and we get $a = c(S_1)$. For $c(S_2)$, since $y \in S_2$ and $a \notin S_2$, first y eliminates the rest, then b eliminates y and we get

$b = c(S_2)$. But, for $c(S_3)$ since c eliminates b and a, we get $a \neq c(S_3)$.

ii. Suppose that $x \neq a$ but $y = b$, and consider $c^l : ...c < b < x < a$, also $c > \{a, x\}$, and $b > a$ For $c(S_1)$, we clearly have $a = c(S_1)$. For $c(S_2)$, since $x \notin S_2$ and b eliminates c and a, and we have $b = c(S_2)$. But, For $c(S_3)$ since $b \notin S_3$ and c eliminates both x and a, we have $a \neq c(S_3)$.

iii. Finally suppose both $x \neq a$ and $y \neq b$, and consider $c^l : ...c < y > b < x < a$, also $c > a,\ c > x,\ b > a$, and $c > b$. For $c(S_1)$, note that x is the alternative compared to a in the last round. Since a eliminates x, we have $a = c(S_1)$. For $c(S_2)$, first y eliminates c and b eliminates y, then since $x \notin S_2$, b eliminates a and we get $b = c(S_2)$. But, for $c(S_3)$ since $y \notin S_3$ and c eliminates a, b and x, we have $a \neq c(S_3)$.

Step 4: We show that $S_2 = S_1 \setminus \{a\}$. We already know that $S_2 \subset S_1$. Suppose there is $x \in S_1 \setminus S_2$ such that $x \neq a$. Consider $c^l : ...b < x < a$ and $b > a$. We clearly have $a = c(S_1)$, $b = c(S_2)$. For $c(S_3)$, since by Step 3 $S_3 = \{a, b\}$, b eliminates a and we get $a \neq c(S_3)$. It follows that $S_1 = S_2 \cup \{a\}$. Since $S_1 \neq S_2$, we have $S_2 = S_1 \setminus \{a\}$. ■

Lemma 6 *For each 2-reg, q, satisfied by τ^{LRC}, if S_1, S_2 and S_3 contains at most three alternatives, then q must be in the form of WPI , NPC, PE or BE.*

Proof. First suppose that $a = b$. It follows from Lemma 1 that q is in the form of BE. Next, let us consider any 2-reg, q where $a \neq b$. Now, suppose $c \in \{a, b\}$. Assume w.l.o.g. that $c = a$. By Lemma 2, we have $S_2 \cup S_3 \subset S_1$. Since S_1 contains at most three alternatives, we have $S_1 = \{a, b, c\}$ for some $c \in A \setminus \{a, b\}$. Moreover, since S_1, S_2, S_3 are distinct, there are two possibilities, either $S_2 = \{a, b\}$ and $S_3 = \{a, c\}$ or $S_2 = \{b, c\}$ and $S_3 = \{a, b\}$. For the first possibility, we obtain a 2-reg in the form of NPC, for the second we obtain a 2-reg in the form of PE.

Next, suppose $c \notin \{a, b\}$. We show that q must be in the form of PE. We first show that $a \in S_2$. Suppose not, then consider the list rational c: $c < ... < b < a,\ a > c$. Clearly we have $a = c(S_1)$. Since, $a \notin S_2$, we have $b = c(S_2)$. But since each alternative can eliminate c, $c \neq c(S_3)$. Next, we show that $b \in S_1$. Suppose not, then consider the choice function above with $b > a$. Similarly, we obtain the desired contradiction.

Next, we show that either $S_1 \subset S_2$ or $S_2 \subset S_1$. Suppose not, and let $x \in S_1 \setminus S_3$ and $y \in S_2 \setminus S_1$. Since each set has at most three alternatives, we have $S_1 = \{a, b, x\}$ and $S_2 = \{a, b, y\}$. If $c \neq x$, then consider the list rational $c : c < ... < b < x < a,\ b > a$. If $c = x$, then consider the list rational $c : c < ... < a < y < b,\ a > b$. By similar reasoning, we have $a = c(S_1)$ and $b = c(S_2)$, but $c \neq c(S_3)$.

Assume w.l.o.g. that $S_2 \subset S_1$. It follows that there is $x \in S_1 \setminus \{a, b\}$. Next, we show that $x = c$. Suppose not, then consider the list rational $c : c < ... < b < x < a,\ b > a$. Once more we get a contradiction. Hence, we obtain $S_1 = \{a, b, c\}$ and $S_2 = \{a, b\}$

Finally, we show that $S_3 \subset S_1$. Suppose not and let $x \in S_3 \setminus S_1$. Then consider the list rational $c : ...x < b < c < a,\ b > a,\ x > c$. Once more we get a contradiction. Hence, we obtain $S_3 \subset \{a, b, c\}$. Now, we can have $S_3 = \{a, c\}$ or $S_3 = \{b, c\}$. If the former holds then q contradicts NPC. It follows that $S_3 = \{b, c\}$ and q is in the form of PE. ■

Proof of Theorem 1. Consider any 2-reg, q, satisfied by τ^{LRC} that requires: If $a = c(S_1)$ and $b = c(S_2)$, then $c = c(S_3)$ for some $a, b, c \in A$, and distinct $S_1, S_2, S_3 \in \Omega$. First suppose that at least one of S_1, S_2, S_3 contains more than three alternatives, then, it follows from Lemma 4 that q is in the form of WPI, NPC or RM. Next suppose each of S_1, S_2 and S_3 contains at most three alternatives, then it follows from Lemma 6 that q is in the form of WPI, NPC, PE or BE. ■

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