Consistent Indices *

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Abstract

In many decision problems, agents base their actions on a simple *objective index*, a single number that summarizes the available information about objects of choice independently of their particular preferences. This paper proposes an axiomatic approach for deriving an index which is *objective* and, nevertheless, can serve as a guide for decision making for decision makers with different preferences. Unique indices are derived for five decision making settings: the Aumann and Serrano [2008] index of riskiness (additive gambles), a novel generalized Sharpe ratio (for a standard portfolio allocation problem), Schreiber's [2013] index of relative riskiness (multiplicative gambles), a novel index of delay embedded in investment cashflows (for a standard capital budgeting problem), and the index of appeal of information transactions [Cabrales et al., 2014]. All indices share several attractive properties in addition to satisfying the axioms. The approach may be applicable in other settings in which indices are needed.

1 Introduction

In many decision problems, agents base their actions on a simple *objective index*, a single number that summarizes the available information about objects of choice and does not depend on the agent's particular preferences.[1] Agents might choose to do this due to difficulties in attaining and interpreting information, or due to an overabundance of useful information. For example, the *Sharpe ratio* [Sharpe, 1966], the ratio between the expected net return and its standard deviation, is frequently used as a performance measure for portfolios [Welch, 2008, Kadan and Liu, 2014].

This paper proposes an axiomatic approach for deriving an index that is *objective* and, nevertheless, can serve as a guide for decision making for decision makers with different preferences. The approach is unifying and may be used in a variety of decision making settings. I present

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[1] As shown by Luca [2011], for the case of online restaurant star ratings.

five applications: for a setting of *additive gambles*, which like lottery tickets change the baseline wealth of the owner independently of its level (an *index of riskiness*); for a standard *portfolio allocation* problem (a *generalized Sharpe ratio*); for a setting of *multiplicative gambles*, which change the wealth of the owner proportionally to its baseline level (an *index of relative riskiness*); for a standard *capital budgeting* problem (an index of the *delay* embedded in investment cashflows); and for a setting of *information acquisition* by investors in an Arrovian [Arrow, 1972] environment (an *index of appeal of information transactions*). In each of the settings I study, a unique index emerges that is theoretically appealing and often improves upon commonly used indices. The approach may be applicable in other settings in which indices are needed.[2]

In my setting, agents choose whether to accept or reject a transaction (a gamble, a cashflow, etc.). The starting point of this paper is a given decision problem and the requirement that (at least) small decisions can be made based on the index. This is the content of the *local consistency* axiom. The axiom states, roughly, that all agents can make acceptance and rejection decisions for small, "local," transactions using the index and a *cut-off* value (which is the only parameter that depends on their preferences), without knowing other details about the transaction, so that the outcomes of their decisions will mirror the outcomes they would achieve by optimizing when possessing detailed knowledge about the transaction.

Even though transactions are complex and multidimensional, I show that a numeric, single dimensional, index can summarize all the decision-relevant information for small transactions. I thus view local consistency as a minimal requirement for an index to be a useful guide for decision making, and, as I show, it is indeed satisfied by many well-known indices in various decision making problems. However, while this property is desirable, I show that many indices that have it also have normatively undesirable properties.[3] The Sharpe ratio, for example, has such property outside the domain of normal distributions. As shown in Example 8, the Sharpe ratio is not, in general, monotonic with respect to first order stochastic dominance outside that domain.[4]

A second criterion for assessing the validity of an index, *global consistency*, is therefore suggested. Global consistency extends local consistency by making restrictions over large transactions, but it is actually quite a weak restriction. Nevertheless, the combination of local and global consistency turns out to be powerful. In the various decision making problems which are discussed below, it pins down a unique order over transactions that has several desirable properties in addition to local

[2]A particular setting which seems promising in this regard is the measurement of inequality, which has many similarities to the setting of risk [Atkinson, 1970].

[3]As stated here, the result follows trivially given the existence of one locally consistent index, as one could change the values of large transactions without changing those of small, local, ones. The exact statement makes further technical requirements which disqualify such indices.

[4]This undesirable property is related to the fact that this index depends only on the first two moments of the distribution. These moments are sufficient statistic for a normal distribution, and therefore basing an index on them solely may be reasonable if returns are assumed (or known) to be normally distributed. This assumption, however, is often rejected in empirical tests in settings where the Sharpe ratio is used in practice [e.g. Fama, 1965, Agarwal and Naik, 2000, Kat and Brooks, 2001]. Moreover, a large body of literature documents the importance of higher order moments for investment decisions [e.g. Kraus and Litzenberger, 1976, Kane, 1982, Harvey and Siddique, 2000, Barro, 2006, 2007, Gabaix, 2008].

and global consistency.[5] Since I use results from the setting of additive gambles in my treatment of other decision making environments, I begin by reviewing this setting and cover it in detail in order to illustrate the general concepts.

The approach I take is different from the standard decision theoretic approach. I start with a given *objective index* – a function that assigns to each transaction some number, independently of any agent specific characteristics. In the case of additive gambles, a higher number is associated with a higher level of riskiness. As different functions induce different orders, for a given index Q, I refer to the Q-riskiness of a gamble. Only then I define the aversion to Q-riskiness. I define the relation *locally at least as averse to Q-riskiness* as follows: Agent u with wealth w is locally at least as averse to Q-riskiness as agent v with wealth w' if, for all gambles with small support (defined precisely in Section 3),[6] when u at w accepts any small gamble with a certain level of Q-riskiness, v at w' accepts all small gambles which are significantly less Q-risky. This definition assumes a certain kind of consistency between the index and the aversion to the property it evaluates, as it implies that agents that are less Q-riskiness averse would accept Q-riskier gambles. This approach is the dual of the standard approach, since instead of starting with an ordering over preferences and asserting that risk is *"what risk-averters hate"* [Machina and Rothschild, 2008], I start with an ordering over the objects of choice (an index of riskiness Q) and derive from it judgments on preferences (Q-riskiness aversion).

In Section 3, I show that if Q is a locally consistent index which satisfies an additional mild condition, then the relation "at least as averse to Q-riskiness" induces the same order as the classic *coefficient of absolute risk aversion* [ARA, Pratt, 1964, Arrow, 1965, 1971]. This property is shown to be satisfied by several well-known indices. However, it is also satisfied by many other indices, including ones that are not monotonic with respect to first order stochastic dominance [Hanoch and Levy, 1969, Hadar and Russell, 1969, Rothschild and Stiglitz, 1970].

As local consistency is insufficient for pinning down normatively acceptable indices, a second criterion, *global consistency*, is suggested. I say that one agent is *globally at least as averse to Q-riskiness* as another agent, if he is locally at least as averse to Q-riskiness at any two arbitrary wealth levels. In the additive gambles setting, global consistency requires that if two agents can be compared using this partial order, then the more Q-riskiness averse agent rejects gambles which are riskier than ones rejected by the other agent. Note that the partial order on preferences which is used to make this requirement of consistency is defined using the index Q, and not based on preexisting notions of risk aversion. Global consistency is a weak requirement, in the sense that it imposes no restriction for the (common) case of a pair of agents who cannot be compared using this partial order. In Section 4, I show that with additional mild conditions, the Aumann and Serrano [2008] index of riskiness, which is monotonic with respect to stochastic dominance, is the unique index that satisfies local consistency and global consistency.

[5] To be precise, additional mild conditions are required as well.

[6] The need to restrict attention to small supports is nicely illustrated by a discussion Samuelson [1963] describes having with with Stanislaw Ulam. Samuelson [1963] quotes Ulam as saying *"I define a coward as someone who will not bet when you offer him two-to-one odds and let him choose his side,"* to which he replied *"You mean will not make a sufficiently small bet (so that the change in the marginal utility of money will not contaminate his choice)."*

In Section 5, I show that the global consistency axiom can be replaced by a requirement that involves only one agent at a time (not pairs of agents) - the *generalized Samuelson property*. An index of riskiness has this property when no agent accepts a large gamble of a certain degree of riskiness if he rejects small ones of the same degree of riskiness at any wealth level, and no agent rejects a large gamble of a certain degree of riskiness if he accepts small ones of the same degree of riskiness at any wealth level. I also show that no agent whose risk tolerance (the inverse of the coefficient of absolute risk aversion) is always higher than the AS riskiness of g will reject g, and no agent whose risk tolerance is always lower will accept it. Given an empirical range of the degrees of risk aversion in a population, the model provides advice to individuals and policy makers based on the index. It also allows researchers a simple way to estimate bounds on the degree of risk aversion in the population from observations of acceptance and rejection of different gambles.

Section 6 addresses the ranking of performance of a market portfolio in the presence of a risk free asset. One well known index of performance is the Sharpe ratio [Sharpe, 1966], the ratio between the expected net return and its standard deviation. My approach suggests a *generalized Sharpe ratio*, where the role of the standard deviation is taken by the Aumann-Serrano (AS) index. This index of performance coincides with the Sharpe ratio on the domain of normal distributions but differs from it in general.[7] Unlike the Sharpe ratio, it is monotonic with respect to stochastic dominance, even when the risky return is not normally distributed, and it satisfies other desirable properties.

Section 7 covers the setting of multiplicative gambles. The results are quite analogous to those of the additive gambles setting. The role of ARA is replaced by the *coefficient of relative risk aversion* (RRA). I show that with mild conditions, the index of relative riskiness of Schreiber [2013] is the unique index which satisfies local consistency and global consistency (or the generalized Samuelson property).

Section 8 considers a capital budgeting setting. Agents are proposed *investment cashflows*, opportunities of investment for several periods with return at later times. I label indices for this setting as indices of *delay*. Paralleling results in previous sections, I show that local consistency, combined with additional mild conditions, ensures that the local aversion to delay, as defined by an index, is ordinally equivalent to the instantaneous discount rate. Adding the requirement of global consistency (or the generalized Samuelson property) is then shown to pin down a novel index for the delay embedded in investment cashflows. The index is continuous and monotonic with respect to *time dominance* [Bøhren and Hansen, 1980, Ekern, 1981], a partial order on cashflows in the spirit of stochastic dominance.

Section 9 treats the setting of information acquisition by investors facing a standard investment problem [Arrow, 1972]. I show that the local taste for informativeness, as defined by the index, coincides with the inverse of ARA for any index which satisfies local consistency and another mild condition. These include Cabrales et al. [2013] and Cabrales et al. [2014], but also indices which have a normatively undesirable property: they are not monotonic with respect to Blackwell's [1953]

[7] The index is increasing in odd distribution moments and decreasing in even ones.

partial order.[8] I then show that the index of Cabrales et al. [2014] is the unique index which satisfies the additional requirement of global consistency.

1.1 Relation to the Literature

Apart from serving as input in decision making processes, indices are also used to limit the discretion of agents by regulators [Artzner, 1999] or when decision rights are being delegated [Turvey, 1963]. For example, a mutual fund manager may be required to invest in bonds that are rated AAA. Similarly, credit decisions are frequently based on a credit rating, a number that is supposed to summarize relevant financial information about an individual. Indices are also used in empirical studies in order to evaluate complex, multidimensional, attributes. Examples include the cost of living [Diewert, 1998], segregation [Echenique and Fryer Jr., 2007], academic influence [Palacios-Huerta and Volij, 2004, Perry and Reny, 2013], market concentration [Herfindahl, 1950], the upstreamness of production and trade flows [Antràs et al., 2012], contract intensity in production [Nunn, 2007], centrality in a network [Bonacich, 1987], inequality [Yitzhaki, 1983, Atkinson, 1983], poverty [Atkinson, 1987], risk and performance [Sharpe, 1966, Artzner et al., 1999], political influence [Shapley and Shubik, 1954, Banzhaf III, 1964], and corruption perceptions [Lambsdorff, 2007].

Although indices are used extensively in economic research and in practice, in many cases the index is not carefully derived from theory. Even in cases where they make theoretical sense in a specific setting, they are often used in larger domains. For example, risk has been evaluated using numerous indices including the standard deviation of returns, the Sharpe ratio, value at risk (VaR), variance over expected return and the coherent measures of Artzner et al. [1999].[9] Some of these indices, like the Sharpe ratio, suffer from a severe normative drawback: they are not monotonic with respect to first order stochastic dominance outside specific domains.[10] That is, increasing a gamble's value in every state of the world does not necessarily lead the index to deem it less risky. Different indices have other undesirable properties. For example, some indices are not continuous, which makes them hard to estimate empirically. Some indices, like VaR, are independent of outcomes in the tails. Finally, and key to this paper, some of the indices are not locally consistent,[11] so they may not be used to guide decisions. My approach is to consider fairly general settings, and concentrate on consistency.

This paper contributes to the growing literature, pioneered by Aumann and Serrano [2008], which identifies objective indices for specific decision making problems. For additive gambles, Aumann and Serrano present an objective index of riskiness, based on a small set of axioms, including

[8] One information structure dominates the other in the sense of Blackwell if it is preferred to the other by all decision makers for all decision making problems.

[9] Even though all of the above indices are meant to measure "*risk*," they were derived with different decision making problems in mind: some take the point of view of a regulator, and others of an investor; some assume the existence of a risk free asset and others do not; some allow agents to adjust their level of investment, and others assume indivisible assets.

[10] See Example 8.

[11] See Example 1.

centrally a "duality axiom," which requires a certain kind of consistency. Roughly speaking, it asserts that (uniformly) less risk-averse individuals accept riskier gambles.[12] Importantly, their definition of risk aversion takes the traditional view, and does not refer to risk as defined by the index. Foster and Hart [FH, 2009] present a different index of riskiness with an operational interpretation.[13] Their index identifies for every gamble the critical wealth level below which it becomes "risky" to accept the gamble.[14] Schreiber [2013] uses insights from this literature to develop an *index of relative riskiness* for multiplicative gambles. Cabrales et al. [2013] and Cabrales et al. [2014] treat the setting of information acquisition and the *appeal* of different *information transactions* for investors.

My approach provides a unifying framework for the decision making problems mentioned above, and it can also be applied to new settings. It provides the first axiomatization for the index of delay and for the generalized Sharpe ratio. All of the indices share several desirable properties, such as monotonicity (e.g. with respect to stochastic dominance) and continuity. The generalized Sharpe ratio, one of the two novel indices presented here, is monotonic with respect to *stochastic dominance in the presence of a risk free asset* [Levy and Kroll, 1978], the analogue of stochastic dominance, of the first and second degree. The index of delay is monotonic with respect to time dominance [Bøhren and Hansen, 1980, Ekern, 1981], the analogous partial order on cashflows.

The index of delay is closely related to a well-known measure of delay which is used in practice: the *internal rate of return* (IRR). I discuss this relation as well as the close connection of the index to the AS index of riskiness. Like the generalized Sharpe ratio, this index treats a decision making environment which has not yet been treated by the recent literature on indices for decision problems. These applications therefore underscore a strength of the proposed approach: indices emerge from the same requirements in different decision making environment.

This paper also contributes to the strand of the literature which attempts to extend the partial order of Blackwell by restricting the class of decision problems and agents under consideration [e.g. Persico, 2000, Athey and Levin, 2001, Jewitt, 2007]. Both Cabrales et al. [2014] and Cabrales et al. [2013] treat an investment decision making environment with a known, common and fixed prior. The order induced by their indices depends on this prior; there exists pairs of information transactions which are ranked differently depending on the prior selected. But an analyst cannot always observe the relevant prior. Subsection 9.5 asks whether the index I derive has prior-free implications for the way information transactions are ranked, which go beyond monotonicity in Blackwell's order and in price. The answer is shown to be positive: there exist pairs of information structures such that neither dominates the other in the sense of Blackwell, and when priced identically, one is ranked higher than the other by the index of appeal of information transactions for any prior distribution. A similar result is shown by Peretz and Shorrer [2014] for the index of Cabrales et al. [2013].

[12] Agent i uniformly no less risk-averse than agent j if whenever i accepts a gamble at some wealth, j accepts that gamble at any wealth.

[13] Homm and Pigorsch [2012a] provide an operational interpretation of the Aumann–Serrano index of riskiness.

[14] Hart [2011] later demonstrated that both indices also arise from a comparison of acceptance and rejection of gambles.

2 Preliminaries

In this section I provide some notation which will be required for the next sections.

A *gamble* g is a real-valued random variable with positive expectation and some negative values (i.e., $E[g] > 0$ and $P[g < 0] > 0$); for simplicity, I assume that g takes finitely many values. $\mathcal{G}$ is the collection of all such gambles. For any gamble $g \in \mathcal{G}$, $L(g)$ and $M(g)$ are respectively the maximal loss and gain from the gamble that occur with positive probability. Formally, $L(g) := \max supp(-g)$ and $M(g) := \max supp(g)$.

$\mathcal{G}_\epsilon$ is the class of gambles with support contained in an ϵ-ball around zero:

$$\mathcal{G}_\epsilon := \{g \in \mathcal{G} : \max\{M(g), L(g)\} \leq \epsilon\}.$$

$[x_1,p_1; x_2, p_2...; x_n, p_n]$ represents a gamble which takes values $x_1, x_2, ..., x_n$ with respective probabilities of $p_1, p_2, ..., p_n$.[15]

An *index of riskiness* is a function from the collection of gambles to the positive reals, $Q : \mathcal{G} \to \mathbb{R}_+$. Note that an index of riskiness is *objective*, in the sense that its value depends only on the gamble and not on any agent-specific attribute. An index of riskiness Q is *homogeneous* (of degree k) if $Q(tg) = t^k \cdot Q(g)$ for all $t > 0$ and all gambles $g \in \mathcal{G}$.

$Q^{AS}(g)$, the *Aumann-Serrano index of riskiness* of gamble g, is implicitly defined by the equation

$$E\left[\exp\left(-\frac{g}{Q^{AS}(g)}\right)\right] = 1.$$

$Q^{FH}(g)$, the *Foster-Hart measure of riskiness* of g,[16] is implicitly defined by the equation

$$E\left[\log\left(1 + \frac{g}{Q^{FH}(g)}\right)\right] = 0.$$

Note that both Q^{AS} and Q^{FH} are homogeneous of degree 1. Additionally, these indices are monotone with respect to first and second order stochastic dominance;[17] namely, if g is stochastically dominated by g' then $Q^{AS}(g) > Q^{AS}(g')$ and also $Q^{FH}(g) > Q^{FH}(g')$ [Aumann and Serrano, 2008, Foster and Hart, 2009].

Value at Risk (VaR) is a family of indices commonly used in the financial industry [Artzner, 1999, Aumann and Serrano, 2008]. VaR indices depend on a parameter called the *confidence level.* For example, the VaR of a gamble at the 95 percent confidence level is the largest loss that occurs with probability greater than 5 percent.

In this paper, a *utility function* is a von Neumann–Morgenstern utility function for money. I assume that utility functions are strictly increasing, strictly concave and twice continuously differ-

[15] This notation will not be used when it is important to distinguish between random variables and distributions.

[16] I also refer to Q^{FH} as an index of riskiness.

[17] A gamble g *first order stochastically dominates* h iff for every weakly increasing (not necessarily concave) utility function u and every $w \in \mathbb{R}$, $E[u(w+g)] \geq E[u(w+h)]$, with strict inequality for at least one such function. A gamble g *second order stochastically dominates* h iff for every weakly concave utility function u and every $w \in \mathbb{R}$, $E[u(w+g)] \geq E[u(w+h)]$, with strict inequality for at least one such function.

entiable unless otherwise mentioned. The *Arrow-Pratt coefficient of absolute risk aversion* (ARA), ρ, of u at wealth w is defined

$$\rho_u(w) := -\frac{u''(w)}{u'(w)}.$$

The *Arrow-Pratt coefficient of relative risk aversion* (RRA), ϱ, of u at wealth w is defined

$$\varrho_u(w) := -w\frac{u''(w)}{u'(w)}.$$

Note that $\rho_u(\cdot)$ and $\varrho_u(\cdot)$ are utility specific attributes and that both ρ and ϱ yield a complete order on utility-wealth pairs. That is, the risk aversion, as measured by ρ (or ϱ), of any two agents with two given wealth levels can be compared.

A gamble g is *accepted* by u at wealth w if $E\left[u(w+g)\right] > u(w)$, and is *rejected* otherwise. Given an index of riskiness Q, a utility function u, a wealth level w and $\epsilon > 0$:

Definition 1. $R_Q^\epsilon(u,w) := \sup\left\{Q(g)|\, g \in \mathcal{G}_\epsilon \text{ and } g \text{ is accepted by } u \text{ at } w\right\}$

Definition 2. $S_Q^\epsilon(u,w) := \inf\left\{Q(g)|\, g \in \mathcal{G}_\epsilon \text{ and } g \text{ is rejected by } u \text{ at } w\right\}$

$R_Q^\epsilon(u,w)$ is the Q-riskiness of the *riskiest* accepted gamble according to Q, restricting the support of the gambles to an ϵ-ball. $S_Q^\epsilon(u,w)$ is the Q-riskiness of the *safest* rejected gamble according to Q, again restricting the support of the gambles to an ϵ-ball.

Definition. u at w is *(locally) at least as averse to Q-riskiness* as v at w' if for every $\delta > 0$ there exists $\epsilon > 0$ such that $S_Q^\epsilon(v,w') \geq R_Q^\epsilon(u,w) - \delta$.

The interpretation of u at w being at least as averse to Q-riskiness *as* v at w' is that, at least for small gambles, if u at w accepts *any* small gamble with a certain level of Q-riskiness, v at w' accepts *all* small gambles which are significantly (by at least δ) less Q-risky. Alternatively, if v at w' rejects *any* small gamble with a certain level of Q-riskiness, u at w rejects *all* small gambles which are significantly (by at least δ) Q-riskier.

The following definitions will also prove useful:

Definition 3. $R_Q(u,w) := \lim_{\epsilon\to 0^+} R_Q^\epsilon(u,w)$

Definition 4. $S_Q(u,w) := \lim_{\epsilon\to 0^+} S_Q^\epsilon(u,w)$.[18]

Roughly speaking, $R_Q(u,w)$ is the Q-riskiness of the Q-riskiest "local gamble" that u accepts at w, and $S_Q(u,w)$ is the Q-riskiness of the Q-safest "local gamble" that is rejected by u at w. The inverse of R_Q and S_Q is a natural measure of the aversion to Q-riskiness.[19] The reason is that R_Q is high for utility-wealth pairs in which Q-risky gambles are accepted, so a reasonable Q-riskiness aversion measure should imply that the aversion to Q-riskiness at such utility-wealth

[18]The existence of the limit in the wide sense is guaranteed by the fact that the suprema (infima) in the definitions of R^ϵ (S^ϵ) are taken on nested supports.

[19]For our purposes, $0 = \infty^{-1}$and $\infty = 0^{-1}$.

is low. Similarly, S_Q is low at a given utility-wealth pair when Q-safe gambles are rejected, so the measure of local aversion to Q-riskiness must be high in this case.

The *coefficient of local aversion to Q-riskiness of u at w* is therefore defined as

$$A_Q(u,w) := \frac{1}{R_Q(u,w)},$$

noting that unless otherwise mentioned, all of the results would hold for $\frac{1}{S_Q(u,w)}$ as well. As is shown below, this definition makes it possible to discuss the ordinal equivalence of the coefficient of local aversion to Q-riskiness, which depends both on agents behavior and on the properties of the index Q, with orders such as ARA or RRA, which depend on the preferences exclusively, and are independent of the index.

3 Local Aversion to Q-Riskiness

Since no restrictions on Q were made (other then possibly homogeneity), at this point coefficients of local aversion to Q-riskiness might look like a class of arbitrary orderings over (u,w) pairs. However, I claim that its members are connected to the standard concepts of local risk aversion. One reason is that they induce orderings which *refine* the following *natural partial order [Yaari, 1969]*: u at w is locally no less risk averse than v at w' (written $(u,w) \gtrdot (v,w')$) if and only if there exists $\epsilon > 0$ such that for every $g \in \mathcal{G}_\epsilon$, if u accepts g at w then so does v at w'. An order O refines the natural partial order if for all g and h, $g \gtrdot h \implies gOh$.

Lemma 1. For every index of riskiness Q, the order induced by A_Q refines the natural partial order.

Proof. Assume that $(u,w) \gtrdot (v,w')$. Then there exists $\epsilon' > 0$ such that for every $g \in \mathcal{G}_{\epsilon'}$ if u accepts g at w then so does v at w'. As in the definition of R_Q we have $\epsilon \to 0^+$, disregarding all $\epsilon \geq \epsilon'$ will not change the result. Note that for every $\epsilon < \epsilon'$

$$\{Q(g)|\, g \in \mathcal{G}_\epsilon \text{ and } g \text{ is accepted by } u \text{ at } w\} \subseteq \left\{Q(g)|\, g \in \mathcal{G}_\epsilon \text{ and } g \text{ is accepted by } v \text{ at } w'\right\}.$$

This means that for every $\epsilon < \epsilon'$, $R^\epsilon(u,w) \leq R^\epsilon(v,w')$ as the suprema in the definition of $R^\epsilon_Q(v,w')$ are taken on a superset of the corresponding sets in the definition of $R^\epsilon_Q(u,w)$. The result follows as weak inequalities are preserved in the limit. □

Next, I show that the coefficient of local aversion to AS (FH) riskiness gives rise to a complete order which coincides with the one implied by the Arrow-Pratt ARA coefficient.

Lemma 2. For every utility function u and every w, $R_{Q^{AS}}(u,w) = S_{Q^{AS}}(u,w)$ and $A_{Q^{AS}}(u,w) = \rho_u(w)$.

Proof. First, observe that if u and v are two utility functions and there exists an interval $I \subseteq R$ such that $\rho_u(x) \geq \rho_v(x)$ for every $x \in I$ then for every wealth level w and lottery g such that $w+g \subset I$, if g is rejected by v at w it is also rejected by u for the same wealth level. Put differently, if g is accepted by u at w it is also accepted by v at the same wealth level. The reason is that the condition implies that in this domain, u is a concave transformation of v [Pratt, 1964], hence by Jensen's inequality $u(w) \leq \mathbb{E}\left[u\left(w+g\right)\right]$ implies that $v(w) \leq \mathbb{E}\left[v\left(w+g\right)\right]$.

Keeping in mind that $u'(x) > 0$ we have that $\rho_u(x)$ is continuous. Specifically,

$$\forall\, \delta > 0\ \exists\, \epsilon > 0 \text{ s.t } x \in (w-\epsilon, w+\epsilon) \Rightarrow |\rho_u(x) - \rho_u(w)| < \delta \tag{3.0.1}$$

Recall that a CARA utility function with ARA coefficient of α rejects all gambles with AS riskiness greater than $\frac{1}{\alpha}$ and accepts all gambles with AS riskiness smaller than $\frac{1}{\alpha}$ [Aumann and Serrano, 2008]. Given an ϵ-environment of w in which $\rho_u \in (\rho_u(w)-\delta, \rho_u(w)+\delta)$, taking the CARA functions with ARA of $\rho_u(w)+\delta$ and $\rho_u(w)-\delta$,[20] and applying the first observation (where I is $(w-\epsilon, w+\epsilon)$ completes the proof. □

Lemma 2 essentially shows that every utility function may be approximated locally using CARA functions, which are well-behaved with respect to the AS index. Given the ARA of u at a given wealth level, I take two CARA utility functions, one with slightly higher ARA, and the other with slightly lower ARA. For small environments around the given wealth level, ρ_u is almost constant, so the two CARA functions "sandwich" the utility function in terms of ARA. This implies that for small gambles, one CARA function accepts more gambles than u, and the other less gambles, in the sense of set inclusion. Since CARA functions accept and reject exactly according to an AS riskiness cutoff, and since cutoffs are close for similar ARA values, it follows that the coefficient of local aversion to AS-riskiness is pinned down completely.

Lemma 3. *For every utility function u and every w, $R_{Q^{FH}}(u,w) = S_{Q^{FH}}(u,w)$ and $A_{Q^{FH}}(u,w) = \rho_u(w)$.*

Proof. According to Statement 4 in Foster and Hart [2009]:

$$-L(g) \leq Q^{AS}(g) - Q^{FH}(g) \leq M(g). \tag{3.0.2}$$

Therefore, if $g \in \mathcal{G}_\epsilon$ then:

$$\left|Q^{AS}(g) - Q^{FH}(g)\right| \leq \epsilon. \tag{3.0.3}$$

From Statement 3.0.3 one can deduce that $R_{Q^{FH}}(u,w) = R_{Q^{AS}}(u,w)$ and $S_{Q^{FH}}(u,w) = S_{Q^{AS}}(u,w)$. Lemma 2 completes the proof. □

The result of Lemma 3 is not surprising in light of Lemma 2, as Foster and Hart [2009] already noted that the Taylor expansions around 0 of the functions that define Q^{FH} and Q^{AS} differ only

[20] In some cases, a smaller δ may be required to ensure that $\rho_u(w)-\delta$ is positive. This could be achieved by looking at a smaller environment of w.

from the third term on. Roughly speaking, this means that for gambles with small supports Q^{AS} and Q^{FH} are close.

Theorem 1 summarizes the results of Lemmata 1-3.

Theorem 1. *(i) For any index of riskiness Q, A_Q refines the natural partial order. (ii)* For every utility function u and every w, $A_{Q^{AS}}(u,w) = A_{Q^{FH}}(u,w) = \rho_u(w)$. Furthermore, $R_{Q^{AS}}(u,w) = S_{Q^{AS}}(u,w)$ and $R_{Q^{FH}}(u,w) = S_{Q^{FH}}(u,w)$.

Corollary 1. *For $Q \in \{Q^{AS}, Q^{FH}\}$ u at w is at least as averse to Q-riskiness as v at w' iff $\rho_u(w) \geq \rho_v(w')$.*

Note that part (i) of Theorem 1 states that the order induced by A_Q refines the weak, no-less risk averse, partial order, and not the strict one. The strict version of this statement is not correct as the following example demonstrates. The example also shows that it is not the case that for all popular risk indices the coefficient of local aversion is equal to ρ or refines the order it induces, and that the same is true for the relation at least as averse to Q-riskiness.

Example 1. For any confidence level $\alpha \in (0,1)$, for all agents and wealth levels, the coefficient of local aversion to $Q(\cdot) := \exp\{\mathrm{VaR}_\alpha(\cdot)\}$ is equal to 1, and any agent at any wealth level is at least as averse to Q-riskiness as any other agent.[21]

It is noteworthy that the example would go through with the exponent of any *coherent risk measure* [Artzner et al., 1999]. The fact that these indices are not well suited for the task of comparing agents' preferences is not surprising. These indices are motivated by the problem of setting a minimal reserve requirements for investors in a given position [Artzner, 1999], and so they take the point of view of a regulator, not the investor.

Up until this point, I showed that the local aversion to AS and FH riskiness induces the same order as the ARA coefficient, the standard measure of local risk aversion, and that the coefficient of local aversion to AS and FH riskiness is in fact equal to the ARA coefficient. This means that one can start with a small set of axioms, namely Aumann and Serrano's [2008] or Foster and Hart's [2013], and define a complete order of riskiness over gambles. Then, the coefficient of local aversion of agents to riskiness can be derived, and it will be equal to the well-known Arrow-Pratt coefficient. The relation at least as averse to AS (FH)-riskiness will also induce the same order. Hence, both AS and FH satisfy the desirable property that less risk averse agents according to ARA accept riskier gambles according to AS or FH.

Theorem 1 and Corollary 1 might be interpreted as evidence that AS and FH were "well-chosen" in some sense. However, Theorem 2 shows that while according to the previous results AS and FH satisfy the desirable properties mentioned above, there are other indices which satisfy the same properties. Moreover, some of these indices are not "reasonable" in the sense that they are not monotone with respect to first order stochastic dominance, in clear violation of the requirement that an index of riskiness should judge as riskier the alternative risk-averse individuals less prefer.

[21] The exponent is only used to assure that the index is positive and has no ordinal effect.

Theorem 3 further identifies sufficient conditions on Q under which the coefficient of local aversion to Q-riskiness and the relation at least as averse to Q-riskiness yield the same order as the Arrow-Pratt (local) absolute risk aversion.

Axiom. *Homogeneity. Q is homogeneous of degree k for some $k > 0$.*

The homogeneity axiom has both cardinal and ordinal content. For the case $k = 1$, its cardinal interpretation is that doubling the stakes doubles the riskiness. The ordinal content is that doubling the stakes increases the riskiness. When taking the point of view of an agent, not a regulator setting a minimal reserve requirement, the cardinal part is not necessarily desirable. In what follows, I assume it for its simplicity and since homogeneity of degree 1 appears in the original axiomatic characterization of the AS index, but later I remove this axiom.

Axiom. *Local consistency.* $\forall u\ \forall w\ \exists \lambda > 0\ \forall \delta > 0\ \exists \epsilon > 0\ R_Q^\epsilon(u, w) - \delta < \lambda < S_Q^\epsilon(u, w) + \delta.$

Local consistency says that small gambles that are significantly Q-safer than some cut-off level are always accepted, and that ones significantly riskier than the cutoff are always rejected. Lemma 6 in the appendix shows that whenever homogeneity is satisfied, local consistency implies that $0 < S_Q(u, w) = R_Q(u, w) < \infty$. This means, that for "small" gambles Q is sufficient information to determine an agent's optimal behavior. In other words, the decisions of agents are consistent with the index, on small domains.

Definition. *Reflexivity.* The relation at least as averse to Q-riskiness is *reflexive* if for all u and w, u at w is at least as averse to Q-riskiness as u at w.

Proposition. *If Q satisfies local consistency, then the relation locally at least as averse to Q-riskiness is reflexive.*

Definition. *Ordinally equivalent.* Given an index of riskiness Q, A_Q is *ordinally equivalent* to the coefficient of absolute risk aversion ρ, if $\forall u, v\ \forall w, w'\ A_Q(u, w) > A_Q(v, w') \iff \rho_u(w) > \rho_v(w')$.

Theorem 2. *(i)* There exists a continuum of locally consistent, homogeneous of degree 1, riskiness indices for which the coefficient of local aversion equals the Arrow-Pratt coefficient.[22] (ii) Moreover, some of these indices are not monotone with respect to first order stochastic dominance.

(i) is proved in the appendix using the observation that for every $a > 0$ any combination of the form $Q_a(\cdot) := Q^{FH}(\cdot) + a \cdot \left|Q^{FH}(\cdot) - Q^{AS}(\cdot)\right|$ is an index of riskiness for which the coefficient of local aversion equals the coefficient of local aversion to Q^{FH}. The reason this holds is that for small supports, the second element in the definition is vanishingly small by Inequality 3.0.3, and so Q_a and Q^{FH} should be close. (ii) follows from Example 2.

[22]Omitting the homogeneity of degree 1 requirement would yield a trivial statement as, for example, an arbitrary change of the values of Q^{AS} for gambles taking values larger than some $M > 0$ will result in a valid index. The requirement that that the local aversion to the index coincides with the Arrow-Pratt coefficient, and not just with the order it implies, is a normalization that rules out, for example, the use of positive multiples of Q^{AS}.

Example 2. Take $Q_1(\cdot) := Q^{FH}(\cdot) + \left|Q^{FH}(\cdot) - Q^{AS}(\cdot)\right|$ and $g = [1, \frac{e}{1+e}; -1, \frac{1}{1+e}]$. $Q^{AS}(g) = 1$ and $Q^{FH}(g) \approx 1.26$, hence $Q_1(g) < 1.6$. Now take $g' = [1, 1-\epsilon; -1, \epsilon]$. For small values of ϵ, $Q^{AS}(g') \approx 0$ but $Q^{FH}(g') > 1$, so $Q_1(g') > 1.6$. Therefore, while g' first order stochastically dominates g, $Q_1(g) < Q_1(g')$.

Theorem 3. *If Q satisfies local consistency and homogeneity of degree $k > 0$, then A_Q is ordinally equivalent to ρ, and the relation at least as averse to Q-riskiness induces the same order as ρ.*[23]

The proof is in the appendix. It extends the reasoning of Lemma 1.

Remark 1. Both axioms in Theorem 3 are essential: omitting either admits indices for which the coefficient of local aversion is not ordinally equivalent to ρ, and the relation at least as averse to Q-riskiness does not induce the same order as ρ.

Proof. Follows from following examples. □

Example 3. $Q(\cdot) \equiv 5$ satisfies local consistency, but it does not satisfy homogeneity of degree $k > 0$. The local aversion to this index induces the trivial order and $A_Q \equiv \frac{1}{5}$.

Example 4. $Q(\cdot) = E[\cdot]$ is homogeneous of degree 1, but it violates local consistency. The local aversion to this index induces the trivial order and $A_Q \equiv \infty$.

In the later part of the next section, homogeneity will no longer required. It will be replaced by a requirement of continuity (which will be precisely defined later) and monotonicity with respect to first order stochastic dominance. For completeness, I present an example of a locally consistent index which satisfies continuity and monotonicity with respect to first order stochastic dominance but does not possess the ordinal content of homogeneity.

Example 5. $Q(\cdot) = \exp\left\{Q^{AS}(\cdot) - E[\cdot]\right\}$ inherits its positivity from the exponent, it is continuous and monotonic with respect to first order stochastic dominance as both $Q^{AS}(\cdot)$ and $-E[\cdot]$ are. Q satisfies local consistency as for small supports it is almost equal to $\exp\left\{Q^{AS}(\cdot)\right\}$, which is locally consistent. Finally, for g such that $Q^{AS}(g) < E[g]$ and $\lambda > 1$, $Q(\lambda g) < Q(g)$. For small $\epsilon > 0$, gambles of the form $g = \left[-\epsilon, \frac{1}{2}; 1, \frac{1}{2}\right]$ satisfy the required inequality.

4 Global Consistency

Theorem 3 identifies conditions under which the coefficient of local aversion to Q-riskiness and the relation at least as averse to Q-riskiness induce the same order as the Arrow-Pratt ARA. But according to Theorem 2 and Example 5 this property is not enough to characterize a "reasonable" index of riskiness. These findings call for additional requirements from an index of riskiness.

[23] To be precise, this statement means that u at w is at least as averse to Q-riskiness as v at w' if and only if $\rho_u(w) \geq \rho_v(w')$.

Definition. *Globally more averse to Q-riskiness.* Let Q be an index of riskiness. u is globally *at least as averse to Q-riskiness as* v is (written $u \succsim_Q v$) if, for every w and w', u at w is at least as averse to Q-riskiness as v at w'. u is *globally more averse to Q-riskiness than* v (written $u \succ_Q v$) if $u \succsim_Q v$ and not $v \succsim_Q u$.[24]

Axiom. *Global consistency. For every pair of utilities u and v, for every w and every g and h in $\mathcal{G}$, if $u \succ_Q v$, u accepts g at w, and $Q(g) > Q(h)$, then v accepts h at w.*

The axiom of global consistency is a weak requirement, in the sense that it imposes no restriction for pairs of utilities which cannot be compared using the partial order globally more averse to Q-riskiness. It is inspired by the duality axiom of AS. For small gambles, it follows immediately from local consistency. In fact, local consistency could have been stated in a very similar way, had it been assumed that the relation at least as averse to Q-riskiness is reflexive. It would state that if u at w is at least as averse to Q-riskiness as v at w' is, then there exists $\lambda > 0$ such that for all $\delta > 0$ there exists $\epsilon > 0$ with $R^{\epsilon}_Q(u,w) - \delta < \lambda < S^{\epsilon}_Q(v,w') + \delta$. Roughly, it states that if the risk averse agent accepts a small gamble with a certain level of riskiness, the less risk averse agent will accept small gambles which are Q-safer. The content of the axiom of global consistency comes from the fact that it places no restriction on the support of gambles, so that when two agents that can be compared by the partial order "globally more averse to Q-riskiness," the axiom requires that the less averse agent accepts Q-riskier gambles, and the requirement applies not only for small gambles.

Theorem 4. $\left(Q^{AS}\right)^k$ *is the unique index of riskiness that satisfies local consistency, global consistency and homogeneity of degree $k > 0$, up to a multiplication by a positive number.*

Proof. Let Q be homogeneous of degree 1. From Theorem 3, A_Q is ordinally equivalent to ρ, and the relation at least as averse to Q-riskiness induces the same order as ρ. The AS duality axiom states that if u is uniformly more averse to risk than v, u accepts g at w, and $Q(g) > Q(h)$, then v accepts h at w. That the relation at least as averse to Q-riskiness induces the same order as ρ means that u is globally more averse to Q-riskiness than v if and only if u is uniformly more risk averse than v. With global consistency, this implies the duality axiom. But the only indices that satisfy homogeneity of degree 1 and the duality axiom are positive multiples of Q^{AS} [Aumann and Serrano, 2008]. If Q is homogeneous of degree $0 < k \neq 1$, $Q' = (Q)^{\frac{1}{k}}$ is homogeneous of degree 1, and still satisfies the other properties,[25] so Q' must equal $C \cdot Q^{AS}$ for some $C > 0$, and so Q is equal to $C^k \cdot \left(Q^{AS}\right)^k$. Finally, Theorems 1 and 3 and the discussion above imply that for all $k > 0$, $\left(Q^{AS}\right)^k$ satisfies the axioms,[26] and the same holds for its positive multiples. □

Corollary 2. Q^{FH}*, the FH index of riskiness, does not satisfy global consistency.*

[24] The above definition is different from the AS definition of uniformly more risk-averse. It is derived directly from the index Q and the utility function u. However, if the relation at least as averse to Q-riskiness induces the same order as ρ the two definitions are equivalent.

[25] To verify this, note that $f(x) = x^{\frac{1}{k}}$ is continuous, and Q and Q' are ordinally equivalent.

[26] In fact, this was shown only for the case $k = 1$, but it is clear that the other cases are implied by this case.

Example 6. Consider a gamble $g = [1, \frac{e}{1+e}; -1, \frac{1}{1+e}]$, $Q^{AS}(g) = 1$ and $Q^{FH}(g) \approx 1.26$, and a gamble $g' = [2, 1-\epsilon; -2, \epsilon]$. For small values of ϵ, $Q^{AS}(g') \approx 0$ but $Q^{FH}(g') > 2$. Hence $Q^{AS}(g) > Q^{AS}(g')$ yet $Q^{FH}(g) < Q^{FH}(g')$. Since the local aversion to FH-riskiness is equal to the local aversion to AS-riskiness by Theorem 1, any two CARA utility functions with different ARA between $\frac{1}{Q^{AS}(g)}$ and $\frac{1}{Q^{AS}(g')}$ together with the two gambles violate global consistency.

As was discussed previously, the cardinal content of the homogeneity axiom is not necessarily appealing for general indices of riskiness. In what follows, this axiom will be removed and replaced with less demanding conditions: monotonicity with respect to first order stochastic dominance and continuity. Example 7 will show that these axioms will not suffice for assuring that the coefficient of local aversion to Q-riskiness is non-degenerate, or even to ensure that the index is monotonic with respect to second order stochastic dominance, and so I will require a slightly stronger version of global consistency. On the other hand, the combination of strong global consistency, monotonicity, continuity and reflexivity of the relation locally at least as averse to Q-riskiness implies local consistency, and so the local consistency requirement could be replaced with the weaker requirement of reflexivity.

Definition. *Continuity.* An index of riskiness Q is continuous if $Q(g) = \lim_{n\to\infty} Q(g_n)$ whenever g_n are uniformly bounded gambles which converge to g in probability.

Example 7. Let $Q(\cdot) = \exp\{-E[\cdot]\}$. It is positive, continuous, monotonic with respect to first order stochastic dominance and locally consistent. Additionally, every u is globally at least as averse to Q-riskiness as any v. Hence, no agent is globally more averse to Q-riskiness than another, and so global consistency in satisfied. The coefficient of local aversion to Q-riskiness is equal to 1 identically. Finally, mean preserving spreads do not change the value of the index.

Axiom. *Strong global consistency. For every pair of utilities u and v, for every w and every g and h in $\mathcal{G}$, if $u \succsim_Q v$, u accepts g at w, and $Q(g) > Q(h)$, then v accepts h at w.*

The difference between the two axioms is that the weak version uses $\succ_Q$ while the strong one uses $\succsim_Q$. The strong version, therefore, requires more, as it has a bite for more pairs of utilities. Note that this axiom is violated by the index from Example 7. To see this, observe that any two agents u and v satisfy both $u \succsim_Q v$ and $v \succsim_Q u$, so Q must be degenerate in order to satisfy the axiom, but it is not.

Theorem 5. *If Q is a continuous index of riskiness that satisfies monotonicity with respect to first order stochastic dominance and strong global consistency, and the relation at least as averse to Q-riskiness is reflexive, then Q is ordinally equivalent to Q^{AS}.*

Corollary. *If Q is a continuous index of riskiness that satisfies monotonicity with respect to first order stochastic dominance and strong global consistency and the relation at least as averse to Q-riskiness is reflexive, then Q satisfies local consistency and A_Q is ordinally equivalent to ρ.*

Remark 2. The monotonicity requirement in the theorem could be replaced by each of the following conditions:

(a) Monotonicity with respect to mean-preserving spreads

(b) Satisfying the ordinal content of homogeneity

(c) Monotonicity with respect to increases in the lowest value of the gamble, leaving the rest unchanged

In such case, monotonicity with respect to first order stochastic dominance will be a result, not an assumption.[27]

5 The Aversion to AS-Riskiness and the Demand for Gambles

Samuelson [1960] shows that *"if you would always refuse to take favorable odds on a single toss, you must rationally refuse to participate in any (finite) sequence of such tosses"* [Samuelson, 1963]. But Samuelson [1963] also warns against undue extrapolation of his theorem saying *"It does not say that one must always refuse a sequence if one refuses a single venture: if, at higher income levels the single losses become acceptable, and at lower levels the penalty of losses does not become infinite, there might well be a long sequence that it is optimal."* The following propositions shows that AS has a properties which generalize the property discussed by Samuelson.

Proposition 1. *A gamble g with $Q^{AS}(g) = c$ is rejected by u at w only if there exist some w' such that small gambles with Q^{AS} of c are rejected. A gamble g with $Q^{AS}(g) = c$ is accepted by u at w only if there exist some w' such that small gambles with Q^{AS} of c are accepted.*

Proof. Omitted. □

Corollary. *If $Q^{AS}(g) > \sup_w A^{-1}_{Q^{AS}}(u,w) = \sup_w \rho_u^{-1}(w)$ then u rejects g at any wealth level. If $Q^{AS}(g) < \inf_w A^{-1}_{Q^{AS}}(u,w) = \inf_w \rho_u^{-1}(w)$ then u accepts g at any wealth level.*

The corollary suggests a partition of the class of gambles into three: "risky" gambles, which the agent never accepts, "safe" gambles which are always accepted, and gambles whose acceptance is subject to wealth effects. Knowing the distribution of preferences in a given population, the intersection of the relevant "risky" and "safe" segments yields a partition which is mutually agreed upon. Such a partition could be used as a simple tool for evaluating policies, as I will show in the next section. It may also be used as a simple tool for providing bounds on risk attitudes, as illustrated in the following example.

Example. Say that a population of agents are observed making acceptance and rejection decisions on gambles. Say that A is the set of gambles rejected by some agent, and B is the set of gambles accepted by some agent. Then if, for some $g \in B$ for all u, $Q^{AS}(g) > \sup_w \rho_u^{-1}(w)$, a contradiction would be implied. So, for some u $\max_{g\in B} Q^{AS}(g) \leq \sup_w \rho_u^{-1}(w)$ and similarly $\min_{g\in A} Q^{AS}(g) \geq \inf_w \rho_v^{-1}(w)$.

[27] The continuity assumption could also be relaxed, for example, by requiring continuity in payoffs for fixed probabilities.

The next result shows a property of the index which is in the spirit of Samuelson's argument, and in fact implies Samuelson's theorem. It shows that the sets of "risky" and "safe" gambles are closed under compounding of independent gambles.

Definition 5. *Compound gamble property.* An index Q has the *compound gamble* property if for every compound gamble of the form $f = g + \mathbf{1}_A h$, where $\mathbf{1}$ is an indicator, A is an event such that g is constant on A ($g|_A \equiv x$ for some x) and h is independent of A, $\max\{Q(g), Q(h)\} \geq Q(f) \geq \min\{Q(g), Q(h)\}$.

Proposition 2. *Q^{AS} satisfies the compound gamble property. Thus, if g, $h \in \mathcal{G}$ are independent, and $\min\left\{Q^{AS}(g), Q^{AS}(h)\right\} > \sup_w \rho_u^{-1}(w)$, then a compound gamble of g and h will also satisfy the inequality. Additionally, if g, $h \in \mathcal{G}$ are independent, and $\max\left\{Q^{AS}(g), Q^{AS}(h)\right\} < \inf_w \rho_u^{-1}(w)$, then a compound gamble of g and h will also satisfy the inequality.*

Proof. See appendix. □

To complete the discussion, I propose a *generalized Samuelson property* and show that it could replace global consistency.

Axiom. *Generalized Samuelson property. $\forall u$, w' $S_Q^{\infty}(u, w') \geq \inf_w S_Q(u, w)$ and $R_Q^{\infty}(u, w') \leq \sup_w R_Q(u, w)$.*

The axiom says that no agent accepts a large gamble of a certain degree of riskiness if he rejects small ones of the same degree of riskiness at any wealth level, and no agent rejects a large gamble of a certain degree of riskiness if he accepts small ones of the same degree of riskiness at any wealth level.

Theorem 6. *If Q satisfies the generalized Samuelson property, reflexivity, monotonicity with respect to first order stochastic dominance and continuity then Q is ordinally equivalent to Q^{AS}.*

Proof. Let Q be as in the statement. Take some CARA function, u, and an arbitrary wealth level w_0, and observe that

$$S_Q^{\infty}(u, w_0) \geq \inf_w S_Q(u, w) = S_Q(u, w_0) \geq R_Q(u, w_0) = \sup_w R_Q(u, w) \geq R_Q^{\infty}(u, w_0).$$

The equalities follow from the lack of wealth effects in CARA functions acceptance and rejection decisions, and the middle inequality follows from reflexivity.

The inequality suggests that all rejected gambles are (weakly) Q-riskier than all accepted ones. Using monotonicity, and continuity of u, for each accepted gamble there exists $\epsilon > 0$ small enough such that if reduced from all the realizations of the gamble, the resulting gamble will still be accepted. Hence, the ranking is in fact strict.

Iterating the above argument with all other possible (C)ARA values proves that Q refines the order that Q^{AS} yields (recall that CARA functions accept or reject according to a Q^{AS} riskiness cutoff, which is the inverse of their ARA coefficient). Finally, continuity implies that the index must induce the same order as Q^{AS}. That Q^{AS} satisfies the properties follows from the discussion above. $\square$

Remark. Continuity was only used to show that Q does not strictly refine the order induced by Q^{AS}.

Remark. The generalized Samuelson property is closely related to global consistency. It implies that if $\inf_{w'} S_Q(v,w') \geq \sup_{w} R_Q(u,w)$ then $S_Q^{\infty}(v,w_0) \geq R_Q^{\infty}(u,w_1)$. Note, however, that the generalized Samuelson property is stated for a single agent and does not place (directly) any simultaneous restrictions on pairs of agents.

6 A Generalized Sharpe Ratio

This section considers an investor facing the problem of asset allocation between a risk free asset, with return r_f and a market portfolio.[28] Fixing r_f, a *market return* r is a real-valued random variable such that $r - r_f \in \mathcal{G}$. In particular, the *net return,* $r - r_f$ has a positive expected value and a positive probability to be negative. For each value of r_f, let $\mathcal{R}^{r_f}$, or simply $\mathcal{R}$ when there is no risk of confusion, denote the class of all such market returns. An index of performance is a collection of functions $Q_{r_f} : \mathcal{R}^{r_f} \to \mathbb{R}_+$, one for each possible value of the risk free rate.

One well known index of performance is the *Sharpe ratio*, the ratio between the expected net return and its standard deviation.[29] This measure of "risk adjusted returns," or "reward-to-variability" [Sharpe, 1966], is frequently used as a performance measure for portfolios [Welch, 2008, Kadan and Liu, 2014]. Formally, it is defined by:

$$Sh_{r_f}(r) = \frac{E[r - r_f]}{\sigma(r - r_f)}.$$

The validity of this measure relies critically on several assumptions on the distribution of returns as well as on agents' preferences [Meyer, 1987]. In particular, for general distributions, the Sharpe ratio is not monotonic with respect to first order stochastic dominance: portfolio r_1 may have returns that are always higher than portfolio r_2 and yet it will be ranked lower according to the index. This normatively undesirable property of the Sharpe ratio is illustrated by the following example, which is based on an example from Aumann and Serrano [2008]:

Example 8. Let $r_1 = [-1, .02; 1, .98]$, $r_2 = [-1, .02; 1, .49; 2, .49]$ and $r_f = 0$.

$$E[r_1 - r_f] = .96, \;\; \sigma(r_1 - r_f) = .28,$$

hence,

$$Sh_{r_f}(r_1) = \frac{.96}{.28} \approx 3.43.$$

[28] r_f may be negative, but greater than -1.

[29] Note that $\sigma(r - r_f) \neq 0$ from the assumption that $r - r_f \in \mathcal{G}$.

But,

$$E[r_2 - r_f] = 1.45, \;\; \sigma(r_2 - r_f) = \frac{7\sqrt{3}}{20},$$

hence,

$$Sh_{r_f}(r_2) = \frac{1.45 \times 20}{7\sqrt{3}} \approx 2.39.$$

The result will continue to hold if we add some small $\epsilon > 0$ to all of the payoffs of r_2.

This undesirable property of the Sharpe ratio is related to the fact that it depends only on the first two moments of the distribution. These moments are sufficient statistic for a normal distribution, and therefore basing an index on them solely may be reasonable under the assumption of normally distributed returns. This assumption is, however, often rejected in settings where the Sharpe ratio is often used [e.g. Fama, 1965, Agarwal and Naik, 2000, Kat and Brooks, 2001]. Moreover, a large body of literature documents the importance of higher order moments for investment decisions [e.g. Kraus and Litzenberger, 1976, Kane, 1982, Harvey and Siddique, 2000, Barro, 2006, 2007, Gabaix, 2008].

Recognizing these limitations of the Sharpe ratio as a measure of performance, Kadan and Liu [2014] propose a reinterpretation of the inverse of the AS index of riskiness as a performance measure, and show that it may be more favorable than the Sharpe ratio in an empirical setting. Homm and Pigorsch [2012b] propose a different index, which was mentioned originally in AS, the expected net return divided by the AS riskiness. The index is not derived from first principles, but is motivated by a "reward-to-risk" reasoning, where the AS riskiness takes the place of σ in the Sharpe ratio. This section asks which of these indices, if any, does the consistency motivated approach suggest?

The findings of this section support the latter alternative, which coincides with the Sharpe ratio on the domain of normally distributed returns. The index possesses other desirable properties, importantly monotonicity with respect to stochastic dominance and with respect to *stochastic dominance in the presence of a risk free asset* [Levy and Kroll, 1978],[30] of the first and second degree.

6.1 Preliminaries

Definition 6. A *market transaction* is a pair, $(q, r) \in \mathbb{R}_+ \times \mathcal{R}$. Denote by $\mathcal{T}$ the class of all market transactions.

Say that an agent with utility function u and initial wealth w accepts a market transaction if

$$E[u((w-q)(1+r_f) + q(1+r))] > u(w(1+r_f)),$$

and rejects it otherwise.

[30] r_1 first (second) order *stochastically dominates* r_2 *in the presence of a risk free asset* r_f if for every $\alpha \geq 0$ there exists $\beta \geq 0$ such that $\alpha r_2 + (1-\alpha) r_f$ is first (second) order stochastically dominated by $\beta r_1 + (1-\beta) r_f$.

I assume that it is only the net return that matters for the index. That is, by shifting r_f and all the possible values of r by a constant, the performance does not change. This is a standard assumptions which makes is possible to compare market returns under different risk-free rates. All the results will continue to hold without this assumption, fixing r_f.

Axiom. *Translation invariance.* $\forall\lambda > 0 \ \forall r_f > -1 \ \forall r \in \mathcal{R}^{r_f} \quad Q_{r_f+\lambda}(r+\lambda) = Q_{r_f}(r)$.[31]

The next axiom could be interpreted as saying that if the price of a unit of the market portfolio decreases, but it continues to yield the same proceeds, the market performs better. This intuitive notion is the ordinal content of the axiom T of Artzner et al. [1999].

Axiom. *Monotonicity.* $\forall r_f > -1 \ \forall r \in \mathcal{R}^{r_f} \ \forall\lambda > 0$, *if* $r_f + \lambda \in \mathcal{R}^{r_f}$ *then* $Q_{r_f}(r+\lambda) > Q_{r_f}(r)$.

With translation invariance, monotonicity is equivalent to the requirement that the same market return should be considered as better performing in the face of a lower risk free rate.

To motivate the next axiom, assume for a moment that the risk free rate is 0, and that agents are free to allocate their resources between the market and a risk free asset. A reasonable requirement is that an index of performance be homogeneous of degree 0, since any portfolio that could be achieved with market return r could be mimicked when the return is λg for any $\lambda > 0$ by scaling the amount of investment by $\frac{1}{\lambda}$. This reasoning clearly extends to the net return, $r - r_f$, for any r_f and r.

Axiom. *Homogeneity.* $\forall\lambda > 0, \ \forall r_f > -1 \ \forall r \in \mathcal{R}^{r_f}, \quad Q_{r_f}(\lambda \cdot (r - r_f) + r_f) = Q_{r_f}(r)$.

The Sharpe ratio is an example for a performance index that satisfies this property. Note that unlike in the other settings presented in this paper, here, the homogeneity axiom is *ordinal* and has no cardinal implications.

Remark 3. A continuous index which satisfies translation invariance and monotonicity but fails to satisfy homogeneity of degree 0 is not monotonic with respect to stochastic dominance in the presence of a risk free asset.[32]

Proof. For some $\lambda > 0$, say $Q_{r_f}(\lambda \cdot (r - r_f) + r_f) > Q_{r_f}(r)$. From translation invariance, $Q_0(\lambda \cdot (r - r_f)) > Q_0(r - r_f)$. From continuity, it will also be the case that $Q_0(\lambda \cdot (r - r_f)) > Q_0(r - r_f + \epsilon)$ for some small $\epsilon > 0$. But $r - r_f + \epsilon$ first order stochastically dominates $\lambda \cdot (r - r_f)$ in the presence of a risk free asset with 0 rate of return, as discussed in the argument motivating the homogeneity axiom. □

Corollary. *The index of performance used by Kadan and Liu [2014] violates monotonicity with respect to stochastic dominance in the presence of a risk free asset.*

Example 9. Let r be a market return with $E[r] = 1$ and let $r_f = 0$. The index proposed by Kadan and Liu [2014] equals to $\frac{1}{Q^{AS}(r)} > 0$. Their index for $\frac{1}{2}r$, under the same conditions, is $\frac{2}{Q^{AS}(r)}$. From the continuity of their index, this implies that for small $\epsilon > 0$, $\frac{1}{2}r - \epsilon$ performs better than r in the Kadan-Liu sense.

[31] If r is in $\mathcal{R}^{r_f}$ then $r + \lambda$ is in $\mathcal{R}^{r_f+\lambda}$.

[32] A precise definition of continuity appears later in this section.

For $c \geq 0$ and $r_f > -1$, define $\mathcal{R}_c^{r_f} := \{r \in \mathcal{R}^{r_f} \,|\, E[r] = r_f + c\}$, the class of market returns with expected net return of c. If Q satisfies homogeneity, it is completely characterized by the restriction of Q_{r_f} to $\mathcal{R}_{r_f+1}$. If Q further satisfies translation invariance then there is no loss of generality in writing $Q(r - r_f) := Q_0(r - r_f) = Q_{r_f}(r)$. This means that it is sufficient to consider the case that $r_f = 0$ and to characterize $Q : \mathcal{R}_1 \to \mathbb{R}_+$. From this point on, unless specifically mentioned, attention will be restricted to this case.

Denote, $\mathcal{T}_\epsilon := \{(q, r) \in \mathcal{T} \,|\; \max\{qr\} - \min\{qr\} < \epsilon,\, r \in \mathcal{R}_1\}$, the class of "local" market transaction.[33]

Definition 7. Given a performance index Q, say that u at w is *locally at least as inclined to invest in $Q-$performers* as v at w' if there exists $\bar{q}$, such that for all for all $\bar{q} > q > 0$ and $\delta > 0$ there exists $\epsilon > 0$ with

$$0 \leq \sup_{(q,r)\in\mathcal{T}_\epsilon} \{Q(r) \,|\, (q,r) \text{ is rejected by } u \text{at } w\} \leq \inf_{(q,r)\in\mathcal{T}_\epsilon} \left\{Q(r) \,|\, (q,r) \text{ is accepted by } v \text{at } w'\right\} + \delta.$$

The interpretation is as follows: for transactions with expected net return of $q > 0$, if v at w' is willing to invest in some local transaction, u at w is willing to invest in any local transaction that performs significantly (by δ) better according to Q.

Next, I require that the relation locally at least as inclined to invest in Q-performers is reflexive.

Axiom. *Reflexivity.* For all u and w, u at w is locally at least as inclined to invest in Q-performers *as u at w.*

Definition 8. u is *globally inclined to invest in $Q-$performers at least as* v if for all w, w', u is locally inclined to invest in $Q-$performers at wealth w at least as v at wealth w'.

Axiom. *Strong global consistency.* For every $w \in \mathbb{R}$, $q > 0$, for every u and v, and every $r, r' \in \mathcal{R}_1$, if u is inclined to invest in Q-performers at least as v, v accepts (q, r) at w, and $Q(r') > Q(r)$, then u accepts (q, r') at w.

The axiom roughly says that if an agent that cares less about Q-performance is willing to invests q in a market, it must be the case that an agent who cares more about Q-performance would be willing to invest the same amount when the market performs better.

6.2 Results

Definition 9. The *generalized Sharpe ratio* is defined as

$$P_{r_f}^{AS}(r) := P^{AS}(r - r_f) = \frac{E[r - r_f]}{Q^{AS}(r - r_f)}.$$

[33]The requirement that $r \in \mathcal{R}_1$ will be important in this setting due to the assumption of homogeneity of degree 0, since for any r with expected positive net returns, $q > 0$, and any agent, there exists a small enough $\lambda > 0$ such that $(q, \lambda \cdot r)$ will be accepted.

Definition. *Continuity.* An index Q is continuous if for all $r_f > -1$, $Q_{r_f}(r_n) \to Q_{r_f}(r)$ whenever $\{r_n\}$ and r are uniformly bounded market returns, and r_n converge to r in probability.

Theorem 7. *Q is a continuous index of performance that satisfies global consistency, reflexivity, translation invariance, monotonicity and homogeneity iff it is a continuous increasing transformation of $P^{AS}(\cdot)$.*

Proof. See appendix. □

Remark. On the domain of normally distributed market returns, P^{AS} is ordinally equivalent to the Sharpe ratio.

Remark. P^{AS} is increasing is increasing in odd distribution moments, and decreasing in even distribution moments.

Proposition 3. *P^{AS} is monotonic with respect to stochastic dominance in the presence of risk free asset.*

Proof. If r_1 dominates r_2 in the presence of r_f, then there exist $\alpha, \beta > 0$ such that $\alpha r_1 + (1-\alpha) r_f$ stochastically dominates $\beta r_2 + (1-\beta) r_f$. There is no loss of generality in assuming that $r_f = 0$ and $E[r_1] = E[r_2]$. With this assumption, the above implies αr_1 stochastically dominate βr_2. The monotonicity of Q^{AS} thus implies that $Q^{AS}(\alpha r_1) < Q^{AS}(\beta r_2)$, and stochastic dominance implies $E[\alpha r_1] \geq E[\beta r_2]$. Altogether, these results imply

$$P_0^{AS}(r_1) = \frac{E[r_1]}{Q^{AS}(r_1)} = \frac{E[\alpha r_1]}{Q^{AS}(\alpha r_1)} > \frac{E[\beta r_2]}{Q^{AS}(\beta r_2)} = \frac{E[r_2]}{Q^{AS}(r_2)} = P_0^{AS}(r_2)$$

as required. □

Corollary. *Q is a continuous index of performance that satisfies global consistency, reflexivity, translation invariance, and monotonicity with respect to stochastic dominance in the presence of risk free asset iff it is a continuous increasing transformation of $P^{AS}(\cdot)$.*

Proof. Follows from Remark 3 and Theorem 7. □

6.3 The Demand for Market Transactions

The next proposition proposes a partition of market transactions into three: "attractive," "unattractive" and ones about which the decision depends on wealth effects.

Proposition 4. *If $\frac{q}{P^{AS}(g)} > \sup_w \rho_u^{-1}(w)$ then u rejects (q, g) at any wealth level. If $\frac{q}{P^{AS}(g)} < \inf_w \rho_u^{-1}(w)$ then u accepts g at any wealth level.*

Next, I show that diversification makes transactions more desirable and that a property analogous to compound gambles holds.

Proposition 5. *Fix r_f, and let $g, h \in \mathcal{R}_{r_f+1}^{r_f}$ be such that (q, g) and (q, h) are accepted by u at any wealth level, then u is accepts $(q, \alpha g + (1-\alpha) h)$ for all $\alpha \in (0,1)$ at any wealth level.*

Proof. From proposition 4 as $P^{AS}(\alpha g + (1-\alpha) h) \geq \min\left\{P^{AS}(g), P^{AS}(h)\right\}$, by the properties of Q^{AS}. □

Proposition 6. *Fix r_f, and let $g, h \in \mathcal{R}_{r_f+1}^{r_f}$ be such that (q, g) and (q, h) are accepted by u at any wealth level, then if g and h are independent then u accepts $(2q, \frac{1}{2}g + \frac{1}{2}h)$ at any wealth level.*

Proof. From proposition 4 as $P^{AS}\left(\frac{1}{2}g + \frac{1}{2}h\right) \geq 2 \cdot \min\left\{P^{AS}(g), P^{AS}(h)\right\}$, by the properties of Q^{AS}. □

This proposition implies the analog to Samuelson's theorem for the case where a risk free asset exists.

Example. *(The demand for market portfolios).* Cabrales et al. [2014] use the estimates of risk aversion from Dohmen et al. [2011] to deduce that for relevant wealth levels a large fraction of the developed world population (importantly, not the very poor or the very rich) could be characterized by $1.8 \cdot 10^{-6} < \rho_u < 5 \cdot 10^{-4}$. Kadan and Liu [2014] use historical monthly return data from the American market and estimate $E[r - r_f]$ by .406 and $\frac{1}{R^{AS}}$ by .038 suggesting an estimated value of $\frac{.406}{.038} \approx 10.69$ for P^{AS}. Based on these estimates, a policy maker may inform individuals that if they do not invest in the market they will (probably) be better-off by purchasing a well diversified portfolio with expected return of q where $\frac{q}{10.69} < \left(5 \cdot 10^{-4}\right)^{-1} = 2,000$, or, approximately, $q < 20,000$. Finally, using the estimate for expected net return, this bound suggests that an exposure of less then $\frac{20,000}{.406} \approx \$50,000$ to a well diversified portfolio of American shares is better than holding just risk free assets. An upper bound can also be suggested: investing more than $\frac{\left(1.8 \cdot 10^{-6}\right)^{-1} \cdot 10.69}{.406} \approx \$13,800,000$ is dominated by opting out of the market.[34]

Example. In the same setting, consider a policy maker who considers levying a tax on risky investment. Using the above estimates for risk aversion, and recalculating P^{AS} for the after tax return, the policy maker can derive an upper bound over possible tax revenues.

7 A Consistent Index of Relative Riskiness

This section presents an application for the setting of multiplicative gambles.

Define $\mathcal{U} := \{u : \mathbb{R}_+ \to \mathbb{R} |\, \varrho_u(w) > 1\, \forall w > 0\}$, the set of (twice continuously differentiable) utility functions with relative risk aversion higher than the logarithmic utility function. Additionally, let $\mathcal{H} := \left\{g \in \mathcal{G} | Q^{FH}(g) < 1\right\}$ be the set of gambles with FH riskiness smaller than 1. The following is a result of FH:

Fact 1. $Q^{FH}(g) < 1 \iff \prod(1+g_i)^{p_i} > 1 \iff E[\log(1+g)] > 0$.

[34] For the upper bound I make the standard assumption that utilities present (weakly) decreasing absolute risk aversion.

In what follows I will consider multiplicative gambles, so that now u *accepts* g at w if $u(w+gw) > u(w)$, and *rejects* g otherwise.[35] The interpretation of $Q^{FH}(g) < 1$ is that gambles of the form wg are accepted by a logarithmic utility function at wealth w. Repeatedly accepting independent gambles with $Q^{FH}(g) > 1$ would lead to bankruptcy with probability 1.

Adjusting the previous axioms to the current setting yields the following axioms for an index of (relative) riskiness $Q : \mathcal{H} \to \mathbb{R}_+$:

Axiom. *Scaling.* $\forall \alpha > 0 \ \forall g \in \mathcal{H}, \ Q\left((1+g)^\alpha - 1\right) = \alpha \cdot Q(g)$.[36]

Similar to the homogeneity axiom, the scaling axiom embodies a cardinal interpretation.

Definition. *Ordinally equivalent.* Given an index of riskiness Q, A_Q is *ordinally equivalent* to the coefficient of relative risk aversion ϱ if $\forall u, v \in \mathcal{U} \ \forall w, w' > 0, \ A_Q(u,w) > A_Q(v,w') \iff \varrho_u(w) > \varrho_v(w')$.[37]

Theorem 8. *If local consistency and scaling hold, then A_Q is ordinally equivalent to ϱ, and the relation at least as averse to Q-riskiness induces the same order as ϱ.*

Proof. omitted. □

Axiom. *Global consistency. For every u and v in $\mathcal{U}$, for every $w > 0$ and every g and h in $\mathcal{H}$, if $u \succ_Q v$, u accepts g at w, and $Q(g) > Q(h)$, then v accepts h at w.*

Lemma 4. *For any $g \in \mathcal{H}$ there is a unique positive number $S(g)$ such that* $E\left[(1+g)^{-\frac{1}{S(g)}}\right] = 1$.

Proof. See appendix. □

Definition. The *index of relative riskiness* S of gamble $g \in \mathcal{H}$ is implicitly defined by the equation $E\left[(1+g)^{-\frac{1}{S(g)}}\right] = 1$.

Theorem 9. *S is the unique index of riskiness that satisfies local consistency, global consistency and scaling, up to a multiplication by a positive number.*

Proof. See appendix. □

As before, scaling is not always a desirable property. In what follows I omit this requirement.

Axiom. *Strong global consistency. For every u and v in $\mathcal{U}$, for every $w > 0$ and every g and h in $\mathcal{H}$, if $u \succsim_Q v$, u accepts g at w, and $Q(g) > Q(h)$, then v accepts h at w.*

Theorem 10. *If Q is a continuous index of relative-riskiness that satisfies monotonicity with respect to first order stochastic dominance and strong global consistency, and the relation at least as averse to Q-riskiness is reflexive, then Q is ordinally equivalent to S.*

[35] g can be interpreted as the return on some risky asset.

[36] Importantly, note that for every $\alpha > 0$ if $g \in \mathcal{H}$ then $(1+g)^\alpha - 1 \in \mathcal{H}$ by fact 1.

[37] Whenever the adaptation of a definition from the previous sections is clear, I omit it for brevity.

Proof. See appendix. □

Corollary. *If Q is a continuous index of relative-riskiness that satisfies monotonicity with respect to first order stochastic dominance and strong global consistency, and the relation at least as averse to Q-riskiness is reflexive, then Q satisfies local consistency and A_Q is ordinally equivalent to ϱ.*

Remark. The monotonicity and continuity requirements could be replaced by other conditions as in Remark 2.

Proposition 7. *A gamble g with $S(g) = c$ is rejected by u at w only if there exist some w' such that small gambles with S-riskiness of c are rejected. A gamble g with $S(g) = c$ is accepted by u at w only if there exist some w' such that small gambles with S-riskiness of c are accepted.*

Proof. omitted. □

Corollary. *If $S(g) > \sup_{w>0} A_S^{-1}(u,w) = \sup_{w>0} \varrho_u^{-1}(w)$ then u rejects g at any wealth level. If $S(g) < \inf_{w>0} A_S^{-1}(u,w) = \inf_{w>0} \varrho_u^{-1}(w)$ then u accepts g at any wealth level.*

Proof. omitted. □

Definition 10. *Compound gamble property.* An index Q has the *compound gamble* property if for every compound gamble of the form $f = (1+g)(1+\mathbf{1}_A h)-1$, where $\mathbf{1}$ is an indicator, A is an event such that g is constant on A ($g|_A \equiv x$ for some x) and h is independent of A, $\max\{Q(g), Q(h)\} \geq Q(f) \geq \min\{Q(g), Q(h)\}$.

Proposition 8. *S satisfies the compound gamble property. Thus, if g, $h \in \mathcal{H}$ are independent, and $\min\{S(g), S(h)\} > \sup_w \varrho_u^{-1}(w)$, then a compound gamble of g and h will also satisfy the inequality. Additionally, if g, $h \in \mathcal{H}$ are independent, and $\max\{S(g), S(h)\} < \inf_w \varrho_u^{-1}(w)$, then a compound gamble of g and h will also satisfy the inequality.*

Proof. omitted. □

Axiom. *Generalized Samuelson property.* $\forall u, w > 0$ $S_Q^{\infty}(u,w) \geq \inf_{w>0} S_Q(u,w)$ *and* $R_Q^{\infty}(u,w) \leq \sup_{w>0} R_Q(u,w)$

Theorem 11. *If Q satisfies the generalized Samuelson property, reflexivity, monotonicity with respect to first order stochastic dominance and continuity then Q is ordinally equivalent to S.*

Proof. omitted. □

8 Consistent Index of Delay

Similar to gambles, comparing cashflows which pay (require) different sums of money over several points in time is not a simple undertaking. Some pairs of cashflows may be compared using the partial order of *time-dominance* [Bøhren and Hansen, 1980, Ekern, 1981], which is the analogue of stochastic dominance in this setting. A cashflow c is *first-order time dominated* by c' if at any point in time the sum of money generated by c up to this point is lower then the sum that was generated by c'.[38] Bøhren and Hansen [1980] show that if c is first-order time dominated by c' then every agent with positive time preferences prefers c' to c. Positive time preferences mean that the agent prefers a dollar at time s to a dollar at time $s + \Delta$ for all $\Delta > 0$. They also show that if c is *second-order time dominated* by c' then every agent with a decreasing and convex discounting function prefers c' to c.[39]

Time dominance is, however, a partial order. In this section I use the consistency motivated approach to derive a novel index for the delay embedded in an investment cashflow. The index I derive is new to the literature but it is related to the well-known internal rate of return. The index possesses several desirable properties similar to those of the AS index of riskiness. In particular, it is monotone with respect to time dominance.

8.1 Preliminaries

An *investment cashflow* is a sequence of outflows (investment) followed by inflows (return), and a sequence of times when they are conducted. Denote by $c = (x_n, t_n)_{n=1}^{N}$ such a cashflow.[40] When x_n is positive the cashflow pays out x_n at time t_n, and when it is negative, an investment of $|x_n|$ is required at t_n. Assume, without loss of generality, that $t_1 < t_2 < ... < t_N$. Further, assume that $x_1 < 0$ and $\sum x_n > 0$, so that some investment is required, and the (undiscounted) return is greater than the investment. This property implies that an agent that does not discount the future will accept any investment cashflow, while a sufficiently impatient agent will reject it. Let $\mathcal{C}$ denote the collection of such cashflows, and $\mathcal{C}_{t,\epsilon}$ be the collection of cashflows with $t_1 \leq t \leq t_N$, and $t_N - t_1 < \epsilon$.

An *index of delay* is a function $T : \mathcal{C} \to \mathbb{R}_+$ from the collection of cashflows to the positive reals. A cashflow c is said to be more T*-delayed* then c' if $T(c) > T(c')$.

I consider a capital budgeting setting in which agent i discounts using a smooth schedule of positive instantaneous discount rates, $r_i(t)$.[41,42] Similar to ρ in the risk setting, r induces a complete order on all agent and time-point pairs.[43] The *net present value (NPV)* of an investment cashflow

[38] The sum may be negative, representing a required investment.

[39] As the definition of second-order time domination requires some notation, I choose to omit it, noting that it is analogous to second order stochastic dominance from the risk setting.

[40] To keep notation simple, I avoid making the dependence of N on c explicit.

[41] An alternative interpretation may be a social planner with such time preferences [Foster and Mitra, 2003].

[42] For a discussion of this condition see Bøhren and Hansen [1980] and references provided there.

[43] Importantly, r is not a common *interest rates* path as in Debreu [1972].

$c = (x_n, t_n)_{n=1}^{N}$ for the agent i at time t is

$$NPV(c,i,t) := \sum_{n} \mathrm{e}^{-\int_{t}^{t_n} r_i(s)ds} x_n.$$

If $NPV(c,i,t) > 0$ for some t, this inequality holds for any t. Agent i *accept* cashflow c (at time t) if $NPV(c,i,t) > 0$ and *rejects* it otherwise. c could be thought of as a suggested shift to a baseline cashflow.

The following two definitions are crucial for applying the consistency motivated approach from the previous sections in order to present axioms for an index of delay. Given an index of delay T, an agent i, a time t, and $\epsilon > 0$:

Definition 11. $R_T^{\epsilon}(i,t) := \sup\{T(c) |\, c \in \mathcal{C}_{t,\epsilon} \text{ and } c \text{ is accepted by } i\}$

Definition 12. $S_T^{\epsilon}(i,t) := \inf\{T(c) |\, c \in \mathcal{C}_{t,\epsilon} \text{ and } c \text{ is rejected by } i\}$

$R_T^{\epsilon}(i,t)$ is the T-delay of the *most delayed* cashflow according to T that i is willing to accept, restricting the support of the cashflows to an ϵ-ball around t. $S_T^{\epsilon}(i,t)$ is the T-delay of the *least delayed* cashflow according to T which i rejects, again restricting the support of the cashflows to an ϵ-ball around t.

Definition. i at t is *at least as averse to T-delay* as j at t' if for every $\delta > 0$ there exists $\epsilon > 0$ such that $S_Q^{\epsilon}(j,t') \geq R_Q^{\epsilon}(i,t) - \delta$.

The interpretation of i at t being at least as averse to T-delay *as* j at t' is that, at least for cashflows with a short horizon, if i accepts *any* short-horizon cashflow concentrated around t with a certain level of T-delay, j accepts *all* short-horizon cashflows which are significantly (by at least δ) less delayed according to T and are concentrated around t'. Alternatively, if j rejects *any* short-horizon cashflow that is concentrated around t' and has a certain level of T-delay, i rejects *all* short horizon cashflows which are significantly (by at least δ) more T-delayed and are concentrated around t.

The following definitions will also prove useful:

Definition 13. $R_T(i,t) := \lim_{\epsilon \to 0^+} R_T^{\epsilon}(i,t)$

Definition 14. $S_T(i,t) := \lim_{\epsilon \to 0^+} S_T^{\epsilon}(i,t)$

Roughly speaking, $R_T(i,t)$ is the T-delay of the most T-delayed short-horizon cashflow that is concentrated around t and accepted by i, and $S_T(i,t)$ is the T-delay of the least T-delayed short-horizon cashflow that is concentrated around t and rejected by i at t. As before, the *coefficient of local aversion to T-delay of i at* t is therefore defined as

$$A_T(i,t) := \frac{1}{R_T(i,t)},$$

noting that all of the results would hold for $\frac{1}{S_T(i,t)}$ as well.

8.2 The Index

The following axioms are an adaptation of the axioms used in Theorem 3 for the current setting. They are used for presenting the analogue of this theorem, as well as the analogue of Theorem 2. Theorem 12 provides conditions under which there is only one order of local aversion to delay and it corresponds to the instantaneous discount rate.

Axiom. *Translation invariance.* $T\left((x_n, t_n + \lambda))_{n=1}^{N}\right) = T\left((x_n, t_n)_{n=1}^{N}\right)$ *for any cashflow and any* $\lambda > 0$.

Translation invariance of T means that T-delay is a time expression, like "in a week" or "a year before," and it does not depend on the start date. In contrast, the interpretation of expressions such as "this Tuesday" depends critically on whether they are said on Friday or Monday. This will be the only "new" requirement in the current setting; all other axioms are adaptions of the axioms from the risk settings to the current one.

Axiom. *Homogeneity (of degree k in dates). For any cashflow with $t_1 = 0$, for any $\lambda > 0$,* $T\left((x_n, \lambda \cdot t_n)_{n=1}^{N}\right) = \lambda^k \cdot T\left((x_n, t_n)_{n=1}^{N}\right)$ for some $k > 0$.

Homogeneity of degree 1 in dates, when combined with translation invariance, represents the notion that if each payment in the cashflow is conducted twice as late relative to the first period of investment, then the entire cashflow is twice as delayed relative to that time. This is a strong cardinal assumption and I later discuss its removal.

Axiom. *Local consistency.* $\forall i\ \forall t\ \exists \lambda > 0\ \forall \delta > 0\ \exists \epsilon > 0\ R_T^{\epsilon}(i,t) - \delta < \lambda < S_T^{\epsilon}(i,t) + \delta.$

Local consistency says that cashflows which are "local" with respect to t that are significantly less T-delayed than some cut-off level are always accepted by i, and that ones significantly more T-delayed than the cutoff are always accepted. Lemma 17 in the appendix shows that whenever homogeneity is satisfied, local consistency implies that $0 < S_T(i,t) = R_T(i,t) < \infty$. This means, that for "local" cashflows T is sufficient information to determine an agent's optimal behavior. In other words, the decisions of agents are consistent with the index, on small domains.

Definition. *Reflexivity.* The relation at least as averse to T-delay is *reflexive* if for all i and t, i at t is at least as averse to T-delay as i at t.

Proposition. *If T satisfies local consistency, then the relation locally at least as averse to T-delay is reflexive .*

Definition. *Ordinally equivalent.* Given an index of delay T, A_T is *ordinally equivalent* to the instantaneous discount rate r if $\forall i, j,\ \forall t, t'\ A_T(i,t) > A_T(j,t') \iff r_i(t) > r_j(t')$.

Theorem 12. *If T satisfies local consistency, homogeneity and translation invariance, then A_T is ordinally equivalent to r, and the relation at least as averse to T-delay induces the same order as r.*

Proof. See appendix. □

Remark 4. All axioms in Theorem 12 are essential: omitting any admits indices to which the coefficient of local aversion is not ordinally equivalent to r, and the relation at least as averse to T-delay does not induce the same order as r.

Proof. The proof follows from the following examples. □

Example 10. $T \equiv 5$ satisfies local consistency and translation invariance, but it does not satisfy homogeneity of degree $k > 0$. The local aversion to this index induces the trivial order and $A_T \equiv \frac{1}{5}$.

Example 11. $T := t_2 - t_1$ satisfies homogeneity and translation invariance, as $\lambda t_2 - \lambda \cdot 0 = \lambda (t_2 - 0)$ and $t_2 - t_1 = (t_2 + \lambda) - (t_1 + \lambda)$. Local consistency is, however, violated. The local aversion to this index induces the trivial order and $A_T \equiv \infty$.

Example 12 demonstrates that without translation invariance the inference is not necessarily correct. The following two definitions prove useful for the example as well as for the statement and proof of Theorem 14.

Definition. The *Internal rate of return* (IRR) of an investment cashflow $c = (x_n, t_n)_{n=1}^N$, written $\alpha(c)$, is the unique positive solution to the equation $\sum_n e^{-\alpha t_n} x_n = 0$.

Existence and uniqueness follow from Lemma 16 which generalizes the result of Norstrøm [1972] who had shown that investment cashflows have a unique positive IRR in the discrete setting. For general cashflows, multiple solutions to the equation defining the internal rate of return may exit.[44]

Definition. For a cashflow c, $D(c) := \frac{1}{\alpha(c)}$ is the inverse of the IRR of the cashflow.

Example 12. Consider the index of delay

$$T(c) = \begin{cases} D(c) & , if\ t_1 < 3\ or\ 5 < t_N \\ (t_1 - 2) \cdot D(c) & , if\ 3 \leq t_1 \leq 4 \\ (6 - t_1) \cdot D(c) & , if\ \ 4 \leq t_1 \leq 5. \end{cases}$$

It is homogeneous since it coincides with D on the relevant domain. It is locally consistent since D is, a fact which will be proved later, and since for any t, in small environments of t the index is approximately equal to $C \cdot D(\cdot)$ for some $C = C(t)$. Now, consider an agent, i, with a constant discount rate $r_i(t) \equiv r$. For $t = 4$, the coefficient of T-delay aversion of the agent is not equal to the coefficient of T-delay aversion for the same agent at at $t = 1$. But $r_i(\cdot)$ is constant by construction. It is also the case that i at $t = 4$ is not at least as averse to T-delay as i at $t = 1$.

[44] In addition, phenomena with the flavor of reswitching might arise [Levhari and Samuelson, 1966], as discussed in Footnote 46.

Theorem 13. *(i) There exists a continuum of translation invariant, locally consistent, homogeneous of degree 1 indices of delay to which the local aversion equals to r. (ii) Moreover, some of these indices are not monotone with respect to first order time dominance.*[45]

Proof. See appendix. □

Definition. *Globally more T-delay averse.* i is *Globally at least as T-delay averse* as j (denoted $j \underset{T}{\precsim} i$) if for every t and t', i at t is at least as averse to T-delay as j at t'. i is *globally more T-delay averse* than j (denoted by $j \prec_T i$) if $j \precsim_T i$ and not $i \precsim_T j$.

This definition generates a partial order over agents, based on their preferences and on the index of delay. As before, global consistency is an important part of the approach.

Axiom. *Global consistency.* If $j \prec_T i$, $T(c) < T(c')$, and i accepts c', then j accepts c.[46]

Theorem 14. $D^k(\cdot)$ *is the unique index of delay that satisfies local consistency, global consistency, homogeneity of degree $k > 0$ and translation invariance, up to a multiplication by a positive number.*

Proof. See appendix. □

The homogeneity axiom is not necessarily appealing in the current setting. In what follows, it will be removed and replaced with less demanding conditions: monotonicity with respect to first order time dominance and continuity. As in previous sections, Example 13 shows that these conditions are not enough to pin down desirable indices. Hence, I will require a slightly stronger version of global consistency but, as before, will replace the local consistency requirement with the weaker requirement of reflexivity.

Definition. *Continuity.* An index of delay is continuous if $T(c_n) \longrightarrow T(c)$ whenever $\{c\} \cup \{c_n\} \subset \mathcal{C}$, random variables with distribution $\left(\frac{|x_i^n|}{\sum\limits_{i|x_i>0}|x_i^n|}, t_i^n\right)$ and $\left(\frac{|x_i^n|}{\sum\limits_{i|x_i\leq 0}|x_i^n|}, t_i^n\right)$ converge in probability to $\left(\frac{|x_i|}{\sum\limits_{i|x_i>0}|x_i|}, t_i\right)$ and $\left(\frac{|x_i|}{\sum\limits_{i|x_i\leq 0}|x_i|}, t_i\right)$ respectively if all random variables are uniformly bounded and $\sum x_i^n$ converges to $\sum x_i$.

Example 13. Consider the index

$$T(c) := 1 + \sum_{j|x_j>0} \frac{|x_j|\, t_j}{\sum\limits_{i|x_i>0}|x_i|} - \sum_{j|x_j\leq 0} \frac{|x_j|\, t_j}{\sum\limits_{i|x_i\leq 0}|x_i|}.$$

It is well-defined and positive as the first summation is a weighted average of greater numbers and both summations are non-degenerate, by the definition of investment cashflow. It is translation

[45] T satisfies monotonicity with respect to first order time dominance if $T(c) < T(c')$ whenever c time dominates c'.

[46] The use of acceptance and rejection allows me to avoid the *reswitching* problem of the famous *Cambridge capital controversy* (See Cohen and Harcourt [2003] for an extensive review). In contrast to choices between two cashflows, which, in general, may not be monotonic in the discount rate, acceptance and rejection decisions of investment cashflows are monotonic in these rates. This is shown in Lemma 16 in the appendix.

invariant since adding t to all t_i's increases both summations by t. Continuity follows directly from the definition of continuity. Homogeneity of degree 0 in payoffs holds as well, since weights are not changed when all x_i's are multiplied by a positive number. local consistency holds since both summations converge to t, when considering smaller and smaller environments of t, and so $R_T \equiv S_T \equiv 1$. Hence, the coefficient of local aversion to T-delay is identically equal to 1, and every i is globally at least as averse to T-delay as any j. Thus, the relation more averse to T-delay is empty and global consistency is automatically satisfied.

Axiom. *Strong global consistency.* If $j \precsim_T i$, $T(c) < T(c')$, and i accepts c', then j accepts c.

Theorem 15. *If T is a continuous index of delay that satisfies monotonicity with respect to first order time dominance, translation invariance and strong global consistency, and the relation at least as averse to T-delay is reflexive, then T is ordinally equivalent to D.*

Proof. See appendix. □

Corollary. *If T is a continuous index of delay that satisfies monotonicity with respect to first order time dominance, translation invariance and strong global consistency, then T is locally consistent and A_T is ordinally equivalent to r.*

Remark. The monotonicity requirement in the theorem could be replaced by each of the following conditions:

(a) Satisfying the ordinal content of homogeneity

(b) Monotonicity with respect to delaying the first investment period, leaving the rest unchanged

In such case, monotonicity with respect to first order time dominance will be a result, not an assumption.[47]

8.3 D-Delay Aversion and the Demand for Investment Cashflows

Proposition 9. *A cashflow $c = (x_n, t_n)_{n=1}^N$ with $D(c) = b$ is rejected by i only if there exist some $t \in [t_1, t_N]$ such that small cashflows with D of b are rejected. A cashflow $c = (x_n, t_n)_{n=1}^N$ with $D(c) = b$ is accepted by i only if there exist some $t \in [t_1, t_N]$ such that small cashflows with D of b are accepted.*

Proof. See appendix. □

Corollary. *If $D(c) > \sup_t A_D^{-1}(i,t) = \sup_t r_i^{-1}(t)$ then i rejects any translation of c. If $D(c) < \inf_t A_D^{-1}(i,t) = \inf_t r_i^{-1}(t)$ then i accepts any translation of c.*

Similar to results in previous sections, the corollary suggests a partition of the class of cashflows into three: ones which the agent never accepts, ones which are always accepted, and ones whose acceptance or rejection may not be determined.

[47]The continuity assumption could also be relaxed.

Definition 15. *Compound cashflow property.* An index T has the *compound cashflow* property if for every compound cashflow of the form $f = c + c'$,[48] where c,c' and f are investment cashflows $\max\{T(c), T(c')\} \geq T(f) \geq \min\{T(c), T(c')\}$.

Proposition 10. *D satisfies the compound cashflow property. Thus, if c, c', $c + c' \in \mathcal{C}$ and $\min\{D(c), D(c')\} > \sup_t r_i^{-1}(t)$, then $c + c'$ also satisfies the inequality, and if c, c', $c + c' \in \mathcal{C}$ and $\max\{D(c), D(c')\} < \inf_t r_i^{-1}(t)$ then $c + c'$ also satisfies the inequality.*

Proof. See appendix. □

Axiom. *Generalized Samuelson property. $\forall i$ $S_T^{\infty}(i) \geq \inf_t S_T(i,t)$ and $R_T^{\infty}(i) \leq \sup_t R_T(i,t)$.*

Theorem 16. *If T satisfies the generalized Samuelson property, translation invariance, reflexivity, monotonicity with respect to first order time dominance and continuity then T is ordinally equivalent to D.*

Proof. Omitted. □

8.4 Other Properties of D and a Comparison with Q^{AS}

This section discusses some properties of the index of delay D and demonstrates the close connection it has with the AS index of riskiness. The IRR is a counterpart of *the rate of return over cost* suggested by Fisher [1930] as a criterion for project selection almost a century ago. Later, some economists dismissed this criterion, arguing that the NPV was superior in comparing pairs of cashflows. Yet, others mentioned that this criterion has the benefit of objectivity, in that it does not require the value judgment of setting the future discount rates [Turvey, 1963]. For example, Stalin and Nixon would agree on the IRR of an investment even though they might disagree on its NPV.[49]

Just like the AS-riskiness of a gamble depends *"on its distribution only—and not on any other parameters, such as the utility function of the decision maker or his wealth"* [Aumann and Serrano, 2008], D depends solely on the cashflow, and not on any agent specific properties. In this sense, D is an objective measure of delay. In particular, D is independent of the date when the cashflow is considered. That is, the D-delay embedded in an investments cashflow is independent of the time when it is considered.

D is homogeneous of degree 0 in payoffs and unit free. This means, for example, that the D-delay of two cashflows denominated in different currencies may be compared without knowledge of the exchange rate. This stands in contrast to the AS index of riskiness which is homogeneous of degree 1 in payoffs, but does not depend on timing. The property is analogous to the property of Q^{AS}, according to which "diluted" gambles inherit the riskiness of the original gamble. For

[48] The interpretation of $c + c'$ is that all of the payoffs which are dictated by each of the cashflows takes place at the times they dictate. If both require a payoff at the same time point, the payoffs are added up.

[49] This resembles the point made by Hart [2011] that in general there are many pairs of agents and pairs of gambles such that each agent accepts a different gamble and rejects the other – our axioms only compare very specific pairs of agents.

$p \in (0,1)$ a *p-dilution* of the gamble g takes the value of the gamble with probability p and 0 with probability $1-p$, independently of the gamble. The reason why this analogy is correct is that in the current setting, times are the parallel of payoffs from the risk setting, while payoffs are the parallel of probabilities, as demonstrated by the remark at the end of this section.

Another property that D and Q^{AS} share is monotonicity. Q^{AS} is monotonic with respect to first and second order stochastic dominance. The analogous property for cashflows is *time-dominance* [Bøhren and Hansen, 1980, Ekern, 1981]. Proposition 3 of Bøhren and Hansen [1980] implies that D is monotonic with respect to time-dominance of any order.

There are other similarities between the measurement of delay and risk. *Value at Risk* (VaR) is a family of indices commonly used in the financial industry [Aumann and Serrano, 2008]. VaR indices depend on a parameter called the *confidence level.* For example, the VaR of a gamble at the 95 percent confidence level is the largest loss that occurs with probability greater than 5 percent. Unlike the AS index, VaR is unaffected by tail events or rare-disasters, extremely negative outcomes that occur with low probability. In the context of project selection, *Turvey [1963]* mentions that *"the Pay-off Period, the number of years which it will take until the undiscounted sum of the gains realized from the investment equals its capital cost,"* was used by practitioners in the West and in Russia. He adds that *"[p]ractical men in industries with long-lived assets have perforce been made aware of the deficiencies of this criterion and have sought to bring in the time element."* The pay-off period criterion, unlike the index of delay, suffers from deficiencies similar to those of VaR. For example, shifting early or late payoffs does not change its value. In fact, recalling that times in the current setting are the parallel of payoffs in the risk setting, the lesson learned by the investors in long-lived assets should apply to investors in risky assets with distant tail events.

Q^{AS} is much more sensitive to the loss side of gambles than it is to gains. Analogously D is more sensitive to early flows than it is to later ones. This follows from the properties of the exponential function in the definition of the IRR. Additionally, both D and Q^{AS} are continuous in their respective spaces.

Finally, to clarify the analogies I made between probabilities and payoffs, and between payoffs and times, I present a reinterpretation of the AS index of riskiness in terms of the delay embedded in a (non-investment) cashflow.

Remark. Given a gamble $g := (g_j, p_j)$, a cashflow which requires an investment of one dollar at $t = 0$ and pays-out p_j at time g_j has a unique positive IRR whose inverse equal to $Q^{AS}(g)$.[50]

To see this, recall that for a cashflow $c = (x_n, t_n)_{n=1}^N$ the (unique) positive IRR is the (unique) positive solution to the equation $\sum_n \mathrm{e}^{-\alpha t_n} x_n = 0$, when it exists. Noting that at $t = 0$, $e^{-\alpha t} = 1$ and that the above cashflow requires an investment of one dollar at $t = 0$, the corresponding equation could be written as

$$-1 + \sum_n \mathrm{e}^{-\alpha g_n} p_n = 0,$$

[50]This is not the unique IRR as 0 is also a solution of the defining equation.

which could be expressed as

$$\mathbb{E}\left[e^{-\alpha g}\right] = 1.$$

But $Q^{AS}(g)$ is the inverse of the unique positive α which solves the equation.

For general cashflows, multiple solutions to the equation defining the internal rate of return may exit. Interestingly, both Arrow and Pratt took interest in finding simple conditions that would rule out this possibility [Arrow and Levhari, 1969, Pratt and Hammond, 1979]. A corollary of the previous remark is that cashflows of the above form have a unique positive IRR.

9 A Consistent Index of the Appeal of Information Transactions

Similar to the previous settings, generating a sensible complete ranking of information structures is an illusive undertaking. In some settings, certain information may be vital, while in others it will not be very important. The implication is that it is not possible to rank all information structures so that higher ranked structures are preferred to lower ranked ones by all agents at every decision making problem. Some pairs of information structures may, however, be compared in this manner. Blackwell's [1953] seminal paper shows that one information structure is preferred to another by all agents in all settings if and only if the latter is a garbling of the prior.[51] That is, if one is a noisy version of the other. But this order is partial and cannot be used to compare many pairs of information structures.

The difficulty in generating a complete ranking which is independent of agents' preferences is discussed by Willinger [1989] in his paper which studies the relation between risk aversion and the value of information. Willinger [1989] discusses his choice of using the *expected value of information* (EVI) or *"asking price"* which was defined by LaValle [1968]. The EVI measures a certain decision maker's willingness to pay for certain information, and so, *"... the difficulty of defining a controversial continuous variable representing the 'amount of information' can be avoided."*

Cabrales et al. [2013] tackle this difficulty using an approach in the spirit of Hart [2011]. They restrict attention to a decision problem of information acquisition by investors in a model a la Arrow [1972], and define an order which they name *uniform investment dominance*, which turns out to be a complete order over all *information structures*. In a separate paper, these authors take an approach in the spirit of AS, and axiomatically derive a different index for the appeal of *information transactions* [Cabrales et al., 2014]. Both approaches lead to orders which refine the order suggested by Blackwell [1953], however, they depend on the (unique, fixed, common) prior of the decision makers which are considered.

In this section, I study a problem of information acquisition by investors using the same techniques as in previous sections. I show that the coefficient of local taste for Q-informativeness is equal to the inverse of ARA when Q is one of these two prominent indices, and that the unique index which satisfies local consistency, global consistency, and a homogeneity axiom is the index of appeal of information transactions [Cabrales et al., 2014]. As always, an ordinal version of this

[51] A simple proof is provided in Leshno and Spector [1992].

result, which does not assume homogeneity, is provided. The section ends with a discussion of the prior-free implications of the index of appeal of information transactions (which is prior-dependent).

9.1 Preliminaries

This section follows closely Cabrales et al. [2014]. I consider agents with concave and twice continuously differentiable utility functions who have some initial wealth and face uncertainty about the state of nature. There are $K \in \mathbb{N}$ states of nature, $\{1, ..., K\}$,[52] over which the agents have the prior $p \in \Delta(K)$ which is assumed to have a full support.

The set of *investment opportunities* $B^* = \left\{ b \in \mathbb{R}^K | \sum_{k \in K} p_k b_k \leq 0 \right\}$, consists of all no arbitrage assets. In particular it includes the option of inaction. The reference to the members of B^* as no arbitrage investment opportunities attributes to p_k an additional interpretation as the price of an Arrow-Debreu security that pays 1 if the state k is realized and nothing otherwise. Hence, p plays a dual role in this setting. When an agent with initial wealth w chooses investment $b \in B^*$ and state k is realized, his wealth becomes $w + b_k$.

Before choosing his investment, the agent has an opportunity to engage in an *information transaction* $a = (\mu, \alpha)$, where $\mu > 0$ is the cost of the transactions, and α is the information structure representing the information that a entails. To be more precise, α is given by a finite set of signals S_α and probability distributions $\alpha_k \in \Delta(S_\alpha)$ for every $k \in K$. When the state of nature is k, the probability that the signal s is observed equals $\alpha_k(s)$. Thus, the information structure may be represented by a stochastic matrix M_α, with K rows and $|S_\alpha|$ columns, and the total probability of the signals is given by the vector $p_\alpha := p \cdot M_\alpha$. For simplicity, assume that $p_\alpha(s) > 0$ for all s, so that each signal is observed with positive probability. Further, denote by q_k^s the probability the agent assigns to state k conditional on observing the signal s, using Bayes' law. Note that although my notation does not indicate it, $(q_k^s)_{k=1}^K = q^s \in \Delta(K)$ depends on α and the prior p.

The transaction a is said to be *excluding* if for every s there exists some k such that $q_k^s = 0$. This means that for every signal the agent receives, he knows that some states will not be realized (allowing him to generate arbitrarily large profits with certainty). Throughout, I will assume that information transactions are not excluding.

Agents are assumed to optimally choose an investment opportunity in B^* given their belief, q. Therefore, the expected utility of an agent with utility u, initial wealth w and beliefs q is

$$V(u, w, q) := \sup_{b \in B^*} \sum_k q_k u(w + b_k).$$

In case that the agent acquires no information, his beliefs are given by the prior p. Since the agent is risk averse, in such case his optimal choice is inaction. Hence,

$$V(u, w, p) = u(w).$$

[52] With a slight abuse of notation, I also denote $\{1, ..., K\}$ by K. The meaning of K should be clear from the context.

Accordingly, an agent *accepts* an information transaction if

$$\sum_s p_\alpha(s) V(u, w - \mu, q^s) > V(u, w, p) = u(w)$$

and *rejects* it otherwise.

Denote by $\mathcal{A}$ the class of information transactions described above. Additionally, denote by $\mathcal{A}_\epsilon$ the sub-class of these information transactions such that $\|p - q^s\|_\infty < \epsilon$ for all s. An *index of appeal* of information transactions is a function from the class of information transactions to the positive reals $Q : \mathcal{A} \to \mathbb{R}_+$. The index of appeal A suggested by Cabrales et al. [2014] is defined by

$$A(a) = -\frac{1}{\mu} \log \left(\sum_s p_\alpha(s) \exp\left(-d\left(p||q^s\right)\right) \right),$$

where

$$d\left(p||q\right) = \sum_k p_k \log \frac{p_k}{q_k}$$

is the *Kulback-Leibler divergence* [Kullback and Leibler, 1951].

Cabrales et al. [2013] suggest the *entropy reduction* as a measure of informativeness of an information structure for investors. It is defined by

$$I_e(\alpha) = H(p) - \sum_s p_\alpha(s) \cdot H(q^s),$$

where,

$$H(q) = -\sum_{k \in K} q_k \log(q_k).$$

In the current context, consider the index J_e, the *cost adjusted entropy reduction* defined by

$$J_e(\mu, \alpha) = \frac{I_e(\alpha)}{\mu}.$$

To apply the techniques from the previous sections, some more definitions are required. Given an index of informativeness Q, a utility function u, a wealth level w and $\epsilon > 0$:

Definition 16. $R^\epsilon_Q(u, w) := \inf\left\{Q(a) |\, a \in \mathcal{A}_\epsilon \text{ and } a \text{ is accepted by } u \text{ at } w\right\}$

Definition 17. $S^\epsilon_Q(u, w) := \sup\left\{Q(a) |\, a \in \mathcal{A}_\epsilon \text{ and } a \text{ is rejected by } u \text{ at } w\right\}$

$R^\epsilon_Q(u, w)$ is the Q-informativeness of the *least informative* accepted transaction according to Q, which is in $\mathcal{A}_\epsilon$. $S^\epsilon_Q(u, w)$ is the Q-informativeness of the *most informative* rejected transaction according to Q, again restricting the support of the transactions to $\mathcal{A}_\epsilon$.

Definition. u at w has *at least as much taste for Q-informativeness* as v at w' if for every $\delta > 0$ there exists $\epsilon > 0$ such that $S^\epsilon_Q(u, w) \le R^\epsilon_Q(v, w') + \delta$.

The interpretation of u at w having at least as much taste for Q-informativeness as v at w' is that, at least for small transactions, if v at w' accepts *any* small transactions with a certain level of Q-informativeness, u at w accepts *all* small transactions which are significantly (by at least δ) more Q-informative. The following definitions will also prove useful:

Definition 18. $R_Q(u,w) := \lim_{\epsilon \to 0^+} R_Q^\epsilon(u,w)$

Definition 19. $S_Q(u,w) := \lim_{\epsilon \to 0^+} S_Q^\epsilon(u,w)$.

$S_Q(u,w)$ is the Q-appeal of the most Q-appealing transaction that is rejected, and never provides a lot of information, in the sense that the posterior and the prior are close.[53] Finally, define the *coefficient of local taste for Q-informativeness* of an agent u with wealth w as the inverse of $S_Q(u,w)$.

9.2 The Index

Theorem 17 is the analogue of Theorem 1 in the current context. It shows that the coefficient of local taste for Q-informativeness coincides with the inverse of ρ for the two indices of informativeness discussed above.[54]

Theorem 17. *(i) For every u and w, $R_A(u,w) = S_A(u,w) = \rho_u(w)$. (ii) For every u and w, $R_{J_e}(u,w) = S_{J_e}(u,w) = \rho_u(w)$.*

Proof. See appendix. □

Corollary 3. *For $Q \in \{A, J_e\}$ u at w has* at least as much taste for Q-informativeness *as v at w' iff $\rho_u(w) \leq \rho_v(w')$.*

The following two theorems are the analogues of Theorems 2 and 3.

Axiom. *Homogeneity. There exists $k > 0$ such that for every information transaction $a = (\mu, \alpha)$ and every $\lambda > 0$, $Q(\lambda \cdot \mu, \alpha) = \frac{1}{\lambda^k} \cdot Q(a)$.*

The homogeneity axiom states that Q is homogeneous of degree $-k$ in transaction prices. This axiom entails the cardinal content of the index. It is particularly interesting if $k = 1$. In this case, the units of the index could be interpreted as *information per dollar*.

Axiom. *Local consistency. $\forall u\ \forall w\ \exists \lambda > 0\ \forall \delta > 0\ \exists \epsilon > 0\ R_Q^\epsilon(u,w) + \delta > \lambda > S_Q^\epsilon(u,w) - \delta$.*

Definition. *Reflexivity.* The relation has at least as much taste for Q-informativeness is *reflexive* if for all u and w, u at w has at least as much taste for Q-informativeness as u at w.

Proposition. *If Q satisfies local consistency, then the relation has at least as much taste for Q-informativeness is reflexive.*

[53] Note that in this setting the index is not independent of the prior p, even when the dependence is not made explicit by the notation I use.

[54] The relations between risk aversion and the taste for information have been discussed extensively in the literature [e.g. Willinger, 1989].

Theorem 18. *Fix $k > 0$. If Q satisfies local consistency and homogeneity of degree $-k$ in prices, then the coefficient of local taste for Q-informativeness is ordinally equivalent to ρ^{-1}, and the relation has at least as much taste for Q-informativeness induces the same order as ρ^{-1}.*

Proof. See appendix. □

Remark 5. Both axioms in Theorem 18 are essential: omitting either admits indices to which the local taste is not ordinally equivalent to ρ^{-1}.

Proof. Follows from following examples. □

Example 14. $Q \equiv 5$ satisfies local consistency, but it does not satisfy homogeneity of degree $k < 0$. The coefficient of local taste for this index induces the trivial order.

Example 15. $Q := \frac{1}{\mu}$ satisfies homogeneity, but violates local consistency. The coefficient of local taste for this index induces the trivial order.

Theorem 19. *(i) Given $k > 0$, there exists a continuum of locally consistent homogeneous of degree $-k$ indices of appeal for which the coefficient of local taste equals to the inverse of ρ. (ii) Moreover, some of these indices are not monotone with respect to Blackwell dominance.*[55]

Proof. See appendix. □

Definition. *Q-informativeness globally more attractive.* For an index Q, say that Q-informativeness *is globally at least as attractive* for u as it is for v (written $v \precsim_Q u$) if for all w, w', u at w has at least as much taste for Q-informativeness as v at w' . Q-informativeness *is globally more attractive* for u than to v (written $v \prec_Q u$) if $v \precsim_Q u$ and not $u \precsim_Q v$.

Axiom. *Global consistency. For any w, any u, v, and any $a, b \in \mathcal{A}$, if $v \prec_Q u$, $A(a) < A(b)$ and v accepts a at w, then u accepts b at w.*

Theorem 20. *For a given $k > 0$, $A^k(\cdot)$ is the unique index that satisfies local consistency, global consistency and homogeneity of degree $-k$ in prices, up to a multiplication by a positive number.*

Proof. Let Q' satisfy the conditions and consider $Q = (Q')^{1/k}$. It is homogeneous of degree -1 and still locally consistent, so by Theorem 18 the relation has at least as much taste for Q-informativeness induces the same order as ρ^{-1}. This, in turn, implies that if $v \underset{Q}{\prec} u$ then v is uniformly more risk averse than u. Combined with this fact, global consistency and homogeneity of degree -1 in prices imply the two axioms that are uniquely satisfied by positive multiples of A, according to Theorem 4 in Cabrales et al. [2012]. That A satisfies local consistency follows from Theorem 17. This implies that A^k also satisfies local consistency. That other axioms are satisfied follows from Cabrales et al. [2012] using Theorem 17. The same holds for positive multiples of A^k. □

[55] Q is monotone with respect to Blackwell dominance if for any cost $\mu > 0$ and all information structures α, β, if α Blackwell dominates β then $Q(\mu, \alpha) > Q(\mu, \beta)$.

Corollary 4. *J_e, the cost adjusted entropy reduction index, does not satisfy global consistency.*

Example 16. (Based on Example 2 of Cabrales et al. [2012]). Let $K = \{1, 2, 3\}$ and fix a uniform prior. Consider the information structures

$$\alpha_1 = \begin{bmatrix} 1-\epsilon_1 & \epsilon_1 \\ 1-\epsilon_1 & \epsilon_1 \\ \epsilon_1 & 1-\epsilon_1 \end{bmatrix}, \ \alpha_2 = \begin{bmatrix} 1-\epsilon_2 & \epsilon_2 \\ 0.1 & 0.9 \\ \epsilon_2 & 1-\epsilon_2 \end{bmatrix},$$

and the information transactions $a_1 = (1, \alpha_1)$ and $a_2 = (1, \alpha_2)$. It can be shown that

$$A(a_1) \approx -\log\left(\frac{2}{3}\epsilon_1^{1/3} + \frac{1}{3}\epsilon_1^{2/3}\right),$$

and

$$A(a_2) \approx -\log\left(\epsilon_2^{1/3}\right).$$

This means that the ordering of the two transactions according to A depends on the choices of $\epsilon_1, \epsilon_2 > 0$. Even when they are both small, their relative magnitude matters.

In contrast, the cost adjusted entropy reduction index, J_e, ranks a_2 higher than a_1 for small $\epsilon_1, \epsilon_2 > 0$. To see this, note that

$$J_e\left(a_1\right) \approx \ln 3 - 0.462,$$

and

$$J_e\left(a_2\right) \approx \ln 3 - 0.550.$$

This means that there exists a choice of small enough ϵ_1, ϵ_2 such that $A(a_1) < A(a_2)$ and $J_e(a_1) > J_e(a_2)$. Hence, there exists two CARA functions with different ARA coefficients (between $A(a_1)$ and $A(a_2)$), which both accept a_2 but reject a_1, demonstrating that J_e violates global consistency.

As discussed previously, the homogeneity axiom has some cardinal content. In what follows, it will be removed and replaced with less demanding conditions: monotonicity in prices, and continuity with respect to prices. Example 17 will show that these conditions do not suffice to ensure that the local taste for Q-informativeness does not induce the trivial order or even that the index is monotonic with respect to Blackwell's order. As in previous sections, with a stronger version of global consistency, these conditions will suffice to pin down a unique index of informativeness (up to a monotonic transformation), and this index will have all of the desirable properties mentioned above.

Definition. *Continuity.* An index of informativeness is *continuous* (in price) if for every α, $Q(\cdot, \alpha)$ is a continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$.

Example 17. $Q\left(\mu, \alpha\right) := 1 - \exp\left\{-\left(1 + \frac{1}{\mu}\right)\right\}$ is positive and continuous. It satisfies local consistency, but the relation has at least as much taste for Q-informativeness applies to any two utilitiy-wealth pairs. Hence, for any u and v, Q-informativeness is not more attractive for u than it is for v, and so global consistency is satisfied. The coefficient of local taste for Q-informativeness is equal to 1 for all agents at all wealth levels. Since Q is independent of the signal structure, it is clearly not monotonic with respect to Blackwell's order.

Axiom. *Strong global consistency. For any w, any u, v, and any $a, b \in \mathcal{A}$, if $v \precsim_Q u$, $A(a) < A(b)$ and v accepts a at w, then u accepts b at w.*

Strong global consistency is clearly violated by the index from Example 17, as any two utilities u, v satisfy $v \precsim_Q u$.

Theorem 21. *If Q is a continuous index of the appeal of information transactions that satisfies monotonicity in price and strong global consistency, and the relation has at least as much taste for Q-informativeness is reflexive, then Q is ordinally equivalent to A.*

Proof. See appendix. □

Corollary. *If Q is a continuous index of the appeal of information transactions that satisfies monotonicity in price and strong global consistency, and the relation has at least as much taste for Q-informativeness is reflexive, then Q satisfies monotonicity with respect to Blackwell dominance and local consistency, and the coefficient of local taste for Q-informativeness is ordinally equivalent to ρ^{-1}.*

9.3 The Demand for Information Transactions

Proposition 11. *An information transaction a with $A(a) = b$ is rejected by u only if there exist some w such that local transactions with A of b are rejected. An information transaction a with $A(a) = b$ is accepted by u only if there exist some w such that local transactions with A of b are accepted.*

Proof. Omitted. □

Corollary. *[Cabrales et al., 2014, Theorem 2] If $A(a) > \sup_w \rho_u(w)$ then u rejects a at any wealth level. If $A(a) < \inf_w \rho_u(w)$ then u accepts a at any wealth level.*

Remark. Cabrales et al. [2014] derive a result on *sequential transactions*,[56] which could be generalized to a result in the spirit of compound gamble property. Since this result requires some notation, I do not provide it here.

Axiom. *Generalized Samuelson property. $\forall u, w'$ $S_Q^{\infty}(u, w') \leq \sup_w S_Q(u, w)$ and $R_Q^{\infty}(u, w') \geq \inf_w R_Q(u, w)$.*

Theorem 22. *If Q satisfies the generalized Samuelson property, reflexivity, monotonicity with respect to first order time dominance and continuity then Q is ordinally equivalent to A.*

Proof. Omitted. □

[56]Section 7.3 of Cabrales et al. [2014].

9.4 Properties of the Index A

The setting of information transactions is somewhat different than other settings that are discussed in this paper, in that the index depends on the prior, and is therefore not completely objective. Example 18 below shows that the order induced by A is different for different priors. Thus, the prior is a relevant part of the specification of the decision making problem that the index is derived from. The fact that in the setting presented here the prior and the prices (which are more likely to be observable) coincide is comforting in this regard.[57]

An important property of the index A is that it is monotonic with respect to *Blackwell's [1953] partial ordering* of information structures [Cabrales et al., 2014]. According to Blackwell's order, one information structure is more informative than another if the latter is a garbling of the prior. Blackwell [1953] proved that one information structure is more informative than another according to this partial ordering if and only if every decision maker prefers it to the other. Cabrales et al. [2014] show that if α is more informative than β in the sense of Blackwell, then $A(\mu,\alpha) > A(\mu,\beta)$ for every $\mu > 0$ and every prior.[58] As Blackwell's ordering is the parallel of stochastic dominance and time dominance, this property is analogous to the properties of the indices presented in previous sections. It is important to note that monotonicity with respect to Blackwell dominance was not one of the requirements in Theorem 21. Other desirable properties of the index include monotonicity in prices and being jointly continuous in p, μ, and q^s. For an extensive discussion of the properties of this index see Cabrales et al. [2014].

Finally, the cardinal interpretation of the index A is relatively more compelling, as the homogeneity (of degree -1) axiom may be interpreted as stating that the index measures information per dollar payed. If this interpretation is taken seriously, then the index may be used in practice for comparing different information providers, charging a fixed fee.

9.5 Prior-Free Implications

In this section I make the dependence of A on the prior, p, explicit and write $A(\cdot, p)$. First, I note that the order induced by the index of the appeal of information transactions depends on the prior in the strict sense. This can be seen easily in the following example:

Example 18. Let $K = \{1,2,3\}$ and let $p_1 = (.5-\epsilon, .5-\epsilon, 2\epsilon)$ and $p_2 = (2\epsilon, .5-\epsilon, .5-\epsilon)$. Consider the information structures

$$\alpha_1 = \begin{bmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \\ .5 & .5 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} .5 & .5 \\ 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{bmatrix}$$

for some small ϵ, and the information transactions $a_1 = (1, \alpha_1)$ and $a_2 = (1, \alpha_2)$. It is easy to verify that $A(a_1, p_1) > A(a_2, p_1)$, but $A(a_1, p_2) < A(a_2, p_2)$. Informally, this is true since, given p_i, α_i

[57]See also the next subsection which discusses the prior-free implications of A.

[58]Recall that A depends on the prior p, even though this fact is not reflected in the notation I use.

reveals almost all of the information that an investor could hope for, but α_{-i} could be improved upon significantly.

The upshot of the example is that without knowledge of the prior, an analyst cannot deduce the "correct" complete order which was derived previously. But some comparisons could still be made, even in the absence of knowledge about the prior. For example, since A is monotonic with respect to Blackwell dominance for all p, whenever one structure, α, Blackwell dominated another, β, it is the case that $A\left(\left(\mu,\alpha\right),p\right)\geq A\left(\left(\mu,\beta\right),p\right)$ for all prices, μ, and all prior beliefs, p. The same holds for comparisons of structures that differ only in price.

Definition. An information transaction a is *at least as appealing as* b *independently of the prior* if $A(a,p)\geq A(b,p)$ for all prior beliefs, p.

As explained above, the order *prior-independent at least as appealing* is strictly partial, but it includes all the comparisons that could be made by Blackwell's partial order and monotonicity in prices. I now turn to show that it could compare strictly more pairs of information transactions. I base my proof on an example used in Peretz and Shorrer [2014] to show that, even though the index I_e of Cabrales et al. [2013] depends on the prior, it can compare strictly more pairs of information structures than Blackwell's order.

Theorem 23. *There exists information transactions* $a=(1,\alpha)$ and $b=(1,\beta)$, *such that* α *does not dominate* β *in the Blackwell sense, yet* a *is at least as appealing as* b *independently of the prior, and* b *is not at least as appealing as* a *independently of the prior.*[59]

This result suggests that, even though the prior-independent order is partial, it still improves upon the more general Blackwell ordering. Thus, restricting attention to the particular decision making problem of investment, allows to derive a more complete order than Blackwell's, even without specifying a prior. This result, therefore, contributes to the literature which attempts to extend partial order of Blackwell by restricting the class of decision problems and agents under consideration [e.g. Persico, 2000, Athey and Levin, 2001, Jewitt, 2007].

Proof. Follows from the example. □

Example 19. Let $K=\{1,2\}$ and and consider the information structures

$$\alpha_1=\begin{bmatrix}.3 & .7\\ .7 & .3\end{bmatrix},\ \alpha_2=\begin{bmatrix}.3 & .7\\ .1 & .9\end{bmatrix},$$

and the transactions $a=(1,\alpha_1)$, $b=(1,\alpha_2)$.

I claim that $A(a,p)\geq A(b,p)$ for all p. Identify p with the probability of state 1, which lies in $[0,1]$. Fixing the two information structures, define a function $\phi_{a,b}:[0,1]\longrightarrow\mathbb{R}$ as follows:

$$\phi_{a,b}\left(\cdot\right):=\exp\left\{-A(b,\cdot)\right\}-\exp\left\{-A(a,\cdot)\right\}.$$

[59] Cabrales et al. [2014] disentangle the roles of p, and propose an index that depends on both security prices and the prior. The theorem will continue to hold in this setting, even if independence of both the prior and prices is required.

For $p \in \{0,1\}$, $A(\cdot,p) \equiv 0$, hence $\phi_{a,b}(p)$ also equals zero. $\phi_{a,b}(\cdot)$ is also a continuous function (this follows from the properties of A) and twice continuously differentiable in $(0,1)$ with a strictly positive second derivatives. This implies that $\phi_{a,b}(\cdot)$ is a convex and continuous function with $\phi_{a,b}(0) = \phi_{a,b}(1) = 0$. But this means that $\phi_{a,b}(p) \leq 0$ for all $p \in [0,1]$ which means that $A(b,p) \leq A(a,p)$ for all $p \in [0,1]$, hence a is at least as appealing as b independently of the prior. It is not hard to verify that b is not at least as appealing as a (by example, or using the strict convexity of $\phi_{a,b}$).

Finally, it remains to check that the comparison is not due to monotonicity in price or Blackwell's order. The first is obvious, as a and b involve the same price. It is not very hard to verify that α_1 does not dominate α_2 in the Blackwell sense. To do this, note that the set of all 2×2 information structures which are dominated by α_1 is

$$\text{Conv}\left\{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} .3 & .7 \\ .7 & .3 \end{pmatrix}, \begin{pmatrix} .7 & .3 \\ .3 & .7 \end{pmatrix}\right\},$$

where Conv denotes the convex hull of the four matrices. α_2 is not included in this set, as Figure 1 illustrates.

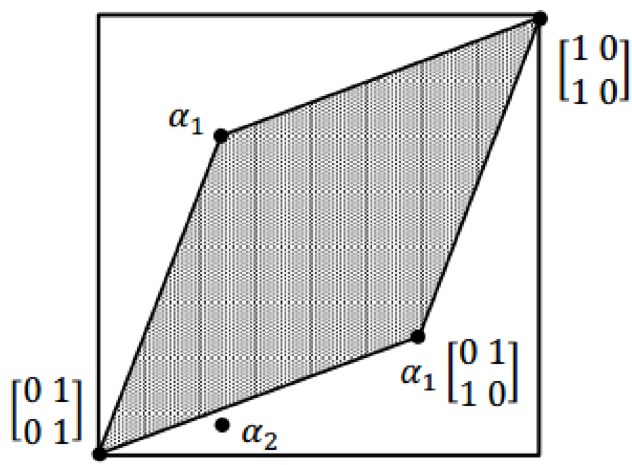

Figure 1: The figure depicts the two dimensional space of 2×2 information structures. These matrices could be written as $\begin{bmatrix} x & 1-x \\ y & 1-y \end{bmatrix}$, where both x and y are in $[0,1]$. In the figure, x is represented by the horizontal axis and y is represented by the vertical axis. The shaded area are the matrices which represent information structures which are dominated by α_1 in the Blackwell sense. The point α_2 is outside the shaded area.

10 Discussion

This paper presented an axiomatic approach for deriving an objective index which could serve as a guide for decision making for different decision makers. The approach was shown to be quite general; it pins down a unique index with desirable properties in five decision making settings. Future research should focus on characterizing the class of decision making problems to which the approach is applicable.

References

Vikas Agarwal and Narayan Y. Naik. Multi-period performance persistence analysis of hedge funds. *Journal of Financial and Quantitative Analysis*, 35(03):327–342, 2000.

Pol Antràs, Davin Chor, Thibault Fally, and Russell Hillberry. Measuring the upstreamness of production and trade flows. Technical report, National Bureau of Economic Research, 2012.

Kenneth J. Arrow. *Aspects of the Theory of Risk-Bearing*. Helsinki: Yrjö Jahnssonin Säätio, 1965.

Kenneth J. Arrow. *Essays in the Theory of Risk Bearing*. Chicago: Markham, 1971.

Kenneth J. Arrow. The value of and demand for information. *Decision and organisation. Londres: North-Holland*, 1972.

Kenneth J. Arrow and David Levhari. Uniqueness of the internal rate of return with variable life of investment. *The Economic Journal*, 79(315):pp. 560–566, 1969.

Philippe Artzner. Application of coherent risk measures to capital requirements in insurance. *North American Actuarial Journal*, 3(2):11–25, 1999.

Philippe Artzner, Freddy Delbaen, Jean-Marc Eber, and David Heath. Coherent measures of risk. *Mathematical finance*, 9(3):203–228, 1999.

Susan Athey and Jonathan Levin. The value of information in monotone decision problems. 2001.

Anthony B. Atkinson. On the measurement of inequality. *Journal of economic theory*, 2(3):244–263, 1970.

Anthony B. Atkinson. *The economics of inequality*. Clarendon Press Oxford, 1983.

Anthony B. Atkinson. On the measurement of poverty. *Econometrica*, 55(4):pp. 749–764, 1987.

Robert J. Aumann and Roberto Serrano. An economic index of riskiness. *Journal of Political Economy*, 116(5):pp. 810–836, 2008.

John F. Banzhaf III. Weighted voting doesn't work: A mathematical analysis. *Rutgers L. Rev.*, 19:317, 1964.

Robert J. Barro. Rare disasters and asset markets in the twentieth century. *The Quarterly Journal of Economics*, 121(3):823–866, 2006.

Robert J. Barro. Rare disasters, asset prices, and welfare costs. Technical report, National Bureau of Economic Research, 2007.

David Blackwell. Equivalent comparisons of experiments. *The Annals of Mathematical Statistics*, 24(2):pp. 265–272, 1953.

Øyvind Bøhren and Terje Hansen. Capital budgeting with unspecified discount rates. *The Scandinavian Journal of Economics*, 82(1):pp. 45–58, 1980.

Phillip Bonacich. Power and centrality: A family of measures. *American journal of sociology*, pages 1170–1182, 1987.

Antonio Cabrales, Olivier Gossner, and Roberto Serrano. The appeal of information transactions. *Working paper*, 2012.

Antonio Cabrales, Olivier Gossner, and Roberto Serrano. Entropy and the value of information for investors. *The American Economic Review*, 103(1):360–377, 2013.

Antonio Cabrales, Oliver Gossner, and Roberto Serrano. The inverse demand for information and the appeal of information transactions. *Working paper*, 2014.

Avi J. Cohen and Geoffrey C. Harcourt. Retrospectives: Whatever happened to the cambridge capital theory controversies? *Journal of Economic Perspectives*, pages 199–214, 2003.

Gerard Debreu. *Theory of value: An axiomatic analysis of economic equilibrium*, volume 17. Yale University Press, 1972.

W. Erwin Diewert. Index number issues in the consumer price index. *The Journal of Economic Perspectives*, pages 47–58, 1998.

Thomas Dohmen, Armin Falk, David Huffman, Uwe Sunde, Jürgen Schupp, and Gert G. Wagner. Individual risk attitudes: Measurement, determinants, and behavioral consequences. *Journal of the European Economic Association*, 9(3):522–550, 2011.

Federico Echenique and Roland G. Fryer Jr. A measure of segregation based on social interactions. *The Quarterly Journal of Economics*, pages 441–485, 2007.

Steinar Ekern. Time dominance efficiency analysis. *The Journal of Finance*, 36(5):1023–1033, 1981.

Eugene F. Fama. The behavior of stock-market prices. *Journal of business*, pages 34–105, 1965.

Irving Fisher. The theory of interest. *New York: Kelley, Reprint of the 1930 Edition*, 1930.

Dean P. Foster and Sergiu Hart. An operational measure of riskiness. *Journal of Political Economy*, 117(5):pp. 785–814, 2009.

Dean P. Foster and Sergiu Hart. A wealth-requirement axiomatization of riskiness. *Theoretical Economics*, 8(2):591–620, 2013.

James E. Foster and Tapan Mitra. Ranking investment projects. *Economic Theory*, 22:469–494, 2003.

Xavier Gabaix. Variable rare disasters: A tractable theory of ten puzzles in macro-finance. *The American Economic Review*, pages 64–67, 2008.

Josef Hadar and William R. Russell. Rules for ordering uncertain prospects. *The American Economic Review*, 59(1):pp. 25–34, 1969.

Giora Hanoch and Haim Levy. The efficiency analysis of choices involving risk. *The Review of Economic Studies*, 36(3):pp. 335–346, 1969.

Sergiu Hart. Comparing risks by acceptance and rejection. *Journal of Political Economy*, 119(4): pp. 617–638, 2011.

Campbell R. Harvey and Akhtar Siddique. Conditional skewness in asset pricing tests. *The Journal of Finance*, 55(3):1263–1295, 2000.

Orris C. Herfindahl. *Concentration in the steel industry*. PhD thesis, Columbia University., 1950.

Ulrich Homm and Christian Pigorsch. An operational interpretation and existence of the Aumann-Serrano index of riskiness. *Economics Letters*, 114(3):265 – 267, 2012a.

Ulrich Homm and Christian Pigorsch. Beyond the sharpe ratio: An application of the aumann–serrano index to performance measurement. *Journal of Banking & Finance*, 36(8):2274–2284, 2012b.

Ian Jewitt. Information order in decision and agency problems. *Nuffield College*, 2007.

Ohad Kadan and Fang Liu. Performance evaluation with high moments and disaster risk. *Journal of Financial Economics*, 113(1):131–155, 2014.

Alex Kane. Skewness preference and portfolio choice. *Journal of Financial and Quantitative Analysis*, 17(01):15–25, 1982.

Harry M. Kat and Chris Brooks. The statistical properties of hedge fund index returns and their implications for investors. *Cass Business School Research Paper*, 2001.

Alan Kraus and Robert H. Litzenberger. Skewness preference and the valuation of risk assets. *The Journal of Finance*, 31(4):1085–1100, 1976.

S. Kullback and R.A. Leibler. On information and sufficiency. *The Annals of Mathematical Statistics*, 22(1):79–86, 1951.

Johann G. Lambsdorff. The methodology of the corruption perceptions index 2007, 2007.

Irving H. LaValle. On cash equivalents and information evaluation in decisions under uncertainty: Part i: Basic theory. *Journal of the American Statistical Association*, 63(321):pp. 252–276, 1968.

Moshe Leshno and Yishay Spector. An elementary proof of blackwell's theorem. *Mathematical Social Sciences*, 25(1):95 – 98, 1992.

David Levhari and Paul A. Samuelson. The nonswitching theorem is false. *The Quarterly Journal of Economics*, 80(4):518–519, 1966.

Haim Levy and Yoram Kroll. Ordering uncertain options with borrowing and lending. *Journal of Finance*, pages 553–574, 1978.

Michael Luca. Reviews, reputation, and revenue: The case of yelp. com. Technical report, Harvard Business School, 2011.

Mark Machina and Michael Rothschild. Risk. in the new palgrave dictionary of economics, edited by steven n. durlauf and lawrence e. blume, 2008.

Jack Meyer. Two-moment decision models and expected utility maximization. *The American Economic Review*, pages 421–430, 1987.

Carl J. Norstrøm. A sufficient condition for a unique nonnegative internal rate of return. *Journal of Financial and Quantitative Analysis*, 7(03):1835–1839, 1972.

Nathan Nunn. Relationship-specificity, incomplete contracts, and the pattern of trade. *The Quarterly Journal of Economics*, 122(2):pp. 569–600, 2007.

Ignacio Palacios-Huerta and Oscar Volij. The measurement of intellectual influence. *Econometrica*, 72(3):963–977, 2004.

Ron Peretz and Ran Shorrer. Entropy and the value of information for investors: the prior free implications. *mimeo*, 2014.

Motty Perry and Philip J. Reny. How to count citations if you must. Technical report, Working paper, 2013.

Nicola Persico. Information acquisition in auctions. *Econometrica*, 68(1):135–148, 2000.

John W. Pratt. Risk aversion in the small and in the large. *Econometrica*, 32(1/2):pp. 122–136, 1964.

John W. Pratt and John S. Hammond. Evaluating and comparing projects: Simple detection of false alarms. *The Journal of Finance*, 34(5):1231–1242, 1979.

Michael Rothschild and Joseph E. Stiglitz. Increasing risk: I. a definition. *Journal of Economic Theory*, 2(3):225–243, September 1970.

Paul A. Samuelson. The st. petersburg paradox as a divergent double limit. *International Economic Review*, 1(1):31–37, 1960.

Paul A. Samuelson. Risk and uncertainty-a fallacy of large numbers. *Scientia*, 98(612):108, 1963.

Amnon Schreiber. Economic indices of absolute and relative riskiness. *Economic Theory*, pages 1–23, 2013.

Lloyd S. Shapley and Martin Shubik. A method for evaluating the distribution of power in a committee system. *American Political Science Review*, 48(03):787–792, 1954.

William F. Sharpe. Mutual fund performance. *The Journal of Business*, 39(1):pp. 119–138, 1966.

Ralph Turvey. Present value versus internal rate of return-an essay in the theory of the third best. *The Economic Journal*, 73(289):93–98, 1963.

Ivo Welch. *Corporate finance: an introduction.* Prentice Hall, 2008.

Marc Willinger. Risk aversion and the value of information. *Journal of Risk and Insurance*, pages 104–112, 1989.

Menahem E. Yaari. Some remarks on measures of risk aversion and on their uses. *Journal of Economic Theory*, 1(3):315–329, 1969.

Shlomo Yitzhaki. On an extension of the gini inequality index. *International Economic Review*, pages 617–628, 1983.

11 Appendix - Proofs

11.1 Fact 2 and Lemmata 5 and 6

Definition. *Full image.* An index of riskiness Q satisfies *full image* if for every $\epsilon > 0$, $\operatorname{Im} Q(\mathcal{G}_\epsilon) = \mathbb{R}_+$.

Full image says that even when the support of the gambles is limited to an ϵ-ball, the image of Q is all of $\mathbb{R}_+$. Both Q^{AS} and Q^{FH} satisfy full image. This is simply demonstrated by considering gambles of the form $g = [\epsilon, \frac{e^{c\epsilon}}{1+e^{c\epsilon}}; -\epsilon, \frac{1}{1+e^{c\epsilon}}]$ and $g' = [\epsilon, \frac{1}{2}; -\frac{\epsilon}{1+\epsilon \cdot c}, \frac{1}{2}]$, as $Q^{AS}(g) = \frac{1}{c}$ and $Q^{FH}(g') = \frac{1}{c}$.

Fact 2. *If Q satisfies full image then $R_Q(u,w) \geq S_Q(u,w)$ for every u and w.*

Proof. By the properties of the supremum, since

$$\{Q(g)|\, g \in \mathcal{G}_\epsilon \text{ and } g \text{ is accepted by } u \text{ at } w\} \cup \{Q(g)|\, g \in \mathcal{G}_\epsilon \text{ and } g \text{ is rejected by } u \text{ at } w\} = \mathbb{R}_+.$$

If the supremum of the first set is less than the infimum of the second, then intermediate points do not belong to either in violation of full-image. □

Lemma 5. *If Q satisfies homogeneity and $0 < S_Q(u,w) < \infty$ for all u and w, then Q satisfies full image.*

Proof. For some u and w, $S_Q(u,w) = c$, $0 < c < \infty$. Hence for some small positive ϵ', for every $0 < \epsilon < \epsilon'$ there exists gambles in $\mathcal{G}_\epsilon$ with Q-riskiness greater than $\frac{c}{2}$. Since multiplying by $0 < \lambda < 1$ keeps the gambles in $\mathcal{G}_\epsilon$, there are gambles with any level of Q-riskiness lower than $\frac{c}{2}$ in $\mathcal{G}_\epsilon$. Since for $\lambda > 1$, $\epsilon < \epsilon'$ implies that $\frac{\epsilon}{\lambda} < \epsilon'$, the same applies to $\mathcal{G}_{\frac{\epsilon}{\lambda}}$. But, using homogeneity, this means that $\mathcal{G}_\epsilon$ includes gambles with any level of Q-riskiness lower than $\lambda \cdot \frac{c}{2}$. Since $\lambda > 1$ was arbitrary, the proof is complete. □

Lemma 6. *If Q satisfies homogeneity and local consistency, then $0 < S_Q(u,w) = R_Q(u,w) < \infty$ for all u and w.*

Proof. Local consistency states that

$$\forall u\ \forall w\ \exists \lambda > 0\ \forall \delta > 0\ \exists \epsilon > 0\ R_Q^\epsilon(u,w) - \delta < \lambda < S_Q^\epsilon(u,w) + \delta,$$

which implies that

$$\forall u\ \forall w\ \exists \lambda > 0\ R_Q(u,w) \leq \lambda \leq S_Q(u,w).$$

Since for any u, w, and $\epsilon > 0$ the set $\{g|\ g \in \mathcal{G}_\epsilon,\ g \,\text{is rejected by}\, u \,\text{at}\, w\}$ is non empty, there exists a sequence of gambles $\{g_n\}$ such that for each n g_n is rejected, $g_n \in \mathcal{G}_{\frac{1}{n}}$ and $Q(g_n) < (1 + \frac{1}{n}) \cdot S_Q^{1/n}(u,w)$. For small $\delta > 0$, let $h_n := (1-\delta)g_n$ for each n. For n large enough, h_n are all accepted since $Q(h_n) = (1-\delta)^k Q(g_n) < S_Q^{1/n}(u,w)$ and h_n is in $\mathcal{G}_{\frac{1}{n}}$. But this implies that $R_Q(u,w) > (1-\delta)^k S_Q(u,w)$ since h_n are almost always accepted and $\lim_{n\to\infty} Q(h_n) = (1-\delta)^k \lim_{n\to\infty} Q(g_n) = (1-\delta)^k S_Q(u,w)$. Since δ was arbitrarily small, this implies $R_Q(u,w) \geq S_Q(u,w)$. So, putting the results together, gives

$$\forall u\ \forall w\ \exists \lambda > 0\ \lambda \leq S_Q(u,w) \leq R_Q(u,w) \leq \lambda,$$

which completes the proof. □

11.2 Theorem 2

Proof. (i) I first show that for every $a > 0$ any combination of the form $Q_a(g) := Q^{FH}(g) + a \cdot \left|Q^{FH}(g) - Q^{AS}(g)\right|$ is an index of riskiness for which the coefficient of local aversion equals the coefficient of local aversion to Q^{FH}. The reason is that for small supports, the second element in the definition is vanishingly small by Inequality 3.0.3, and so Q_a and Q^{FH} should be close.

Fix $a > 0$. First, note that

$$\forall g \in \mathcal{G}\ 0 < Q^{FH}(g) \leq Q^{FH}(g) + a \cdot \left|Q^{FH}(g) - Q^{AS}(g)\right|,$$

so $Q_a(g) \in \mathbb{R}_+$. Additionally, for every $\delta > 0$ there exists $\epsilon > 0$ small enough such that for every $g \in \mathcal{G}_\epsilon$,

$$Q^{FH}(g) \leq Q^{FH}(g) + a \cdot \left|Q^{FH}(g) - Q^{AS}(g)\right| \leq Q^{FH}(g) + \delta. \tag{11.2.1}$$

Inequality 11.2.1 stems from the small support combined with Inequality 3.0.3. It tells us that the coefficient of local aversion to Q_a-riskiness cannot be different from $A_{Q^{\text{FH}}}$ which equals $A_{Q^{\text{AS}}}$ according to Theorem 1. That local consistency is satisfied follows from the same reasoning. The proof of (i) is completed by recalling that $Q^{FH} \neq Q^{AS}$, that both indices are locally consistent (immediate from Theorem 1) and homogeneous.[60]

(ii) Follows from Example 2. □

11.3 Theorem 3

Proof. I start with the first part. In one direction, $\rho_u(w) > \rho_v(w')$ implies that $(u, w) \gtrdot (v, w')$ [Yaari, 1969], so Lemma 1 implies that $A_Q(u, w) \geq A_Q(v, w')$.

To see that $A_Q(u, w) \neq A_Q(v, w')$, define $c := \left(\frac{\rho_u(w)+\rho_v(w')}{2}\right)^{-1}$. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of gambles such that $g_n \in \mathcal{G}_{\frac{1}{n}}$ and $Q^{AS}(g_n) = c$. For a small $\delta > 0$ let $h_n = (1 + \delta)g_n$. By Theorem 1, for large values of n, g_n and h_n will be rejected by u at w and accepted by v at w', so

$$S_Q(v, w') \geq R_Q(v, w') \geq (1 + \delta)^k \cdot S_Q(u, w) > S_Q(u, w) \geq R_Q(u, w),$$

where the strict inequality follows from the fact that $\infty > S_Q(u, w) > 0$ by Lemma 6, the first and the last inequality follow from the local consistency axiom, and the second inequality follows from the definitions of R_Q and S_Q and homogeneity, by the properties of g_n and h_n. This proves that $A_Q(u, w) > A_Q(v, w')$.

In the other direction, if $A_Q(u, w) > A_Q(v, w')$ then, from homogeneity and the fact that $\infty > R_Q(v, w') > R_Q(u, w) > 0$, there exists a sequence of gambles $\{k_n\}_{n=1}^{\infty}$ such that $k_n \in \mathcal{G}_{\frac{1}{n}}$ and $Q(k_n) = c'$, where $c' := \left(\frac{A_Q(u,w)+A_Q(v,w')}{2}\right)^{-1}$. For a small $\delta > 0$ let $l_n = (1 + \delta)g_n$. A similar argument shows that

$$S_{Q^{AS}}(v, w') = R_{Q^{AS}}(v, w') \geq (1 + \delta) \cdot S_{Q^{AS}}(u, w) > S_{Q^{AS}}(u, w) = R_{Q^{AS}}(u, w),$$

where the strict inequality follows from the fact that $S_{Q^{AS}}(u, w) > 0$ by Lemma 2, the equalities follow from the same lemma, and the weak inequality follows from the definitions of $R_{Q^{AS}}$ and $S_{Q^{AS}}$ and the homogeneity of Q^{AS}, by the properties of g_n and l_n. Using Lemma 2 once again, this implies that $\rho_u(w) > \rho_v(w')$.

For the second part, recall that u at w is at least as averse *to Q-riskiness* as v at w' if for every $\delta > 0$ there exists $\epsilon > 0$ such that $S_Q^\epsilon(v, w') \geq R_Q^\epsilon(u, w) - \delta$. This implies that $S_Q(v, w') \geq R_Q(u, w)$, which from Lemma 6 implies that $R_Q(v, w') \geq R_Q(u, w)$.

[60] An alternative proof could use indices of the form: $(Q^{FH})^\alpha (Q^{AS})^{1-\alpha}$, $\alpha \in (0, 1)$. This form may prove to be useful in empirical work, since it enables some flexibility in the estimation. In addition, it allows us to put some weight on the FH measure that "punishes" heavily for rare disasters [Barro, 2006].

In the other direction, if $R_Q(v, w') \geq R_Q(u, w)$, then by Lemma 6 $\infty > S_Q(v, w') \geq R_Q(u, w) > 0$. This means that for every $\delta > 0$ there exists $\epsilon > 0$ such that $S^\epsilon_Q(v, w') \geq R^\epsilon_Q(u, w) - \delta$, as S_Q is the limit of S^ϵ_Q and R_Q is the limit of R^ϵ_Q. □

11.4 Theorem 5

Proof. First, observe that for any CARA utility function u it must be the case that u is globally at least as averse to Q-riskiness as u, by reflexivity and the lack of wealth effects in CARA functions. Now consider two gambles g and g' with $Q^{AS}(g) > Q^{AS}(g')$. Consider u CARA with $\rho_u \equiv \frac{2}{Q^{AS}(g)+Q^{AS}(g')}$. u accepts g' and rejects g, implying that $Q(g) \geq Q(g')$, since otherwise strong global consistency will be violated (the violation would be the fact that u is globally no less averse to Q-riskiness than itself, u accepts g' with $Q(g') > Q(g)$, but rejects g).

Next, I claim that if $Q^{AS}(g) > Q^{AS}(g')$, but $Q(g) = Q(g')$, then there exists a gamble g_ϵ such that $Q^{AS}(g_\epsilon) > Q^{AS}(g')$, but $Q(g_\epsilon) < Q(g')$ in contradiction to the above result. To see this note that from monotonicity of Q, for any small $\epsilon > 0$ a gamble $g_\epsilon = g + \epsilon$ has $Q(g_\epsilon) < Q(g)$, and from continuity of Q^{AS}, for small enough ϵ, $Q^{AS}(g_\epsilon) > Q^{AS}(g')$.

Finally, I claim that if $Q^{AS}(g) = Q^{AS}(g')$, but $Q(g) > Q(g')$, then there exists a gamble g_ϵ such that $Q^{AS}(g_\epsilon) < Q^{AS}(g')$, but $Q(g_\epsilon) > Q(g')$. To see this, apply the same argument from the previous paragraph, only this time use the continuity of Q and the monotonicity of Q^{AS}.

The upshot of the above discussion is that $Q^{AS}(g) > Q^{AS}(g') \iff Q(g) > Q(g')$ as required.

□

11.5 Proposition 2

Definition 20. *Wealth-independent compound gamble [Foster and Hart, 2013].* An index Q has the *wealth-independent compound gamble* property if for every compound gamble of the form $f = g + \mathbf{1}_A h$, where $Q(g) = Q(h)$, $\mathbf{1}$ an indicator, A is an event such that g is constant on A ($g|_A \equiv x$ for some x) and h is independent of A, $Q(f) = Q(g)$.

Proof. Foster and Hart [2013] show that Q^{AS} satisfies wealth-independent compound gamble. If $Q^{AS}(g) \neq Q^{AS}(h)$, take the one with higher (lower) level of AS riskiness, and increase (decrease) all its values be ϵ large enough to equate the level of riskiness of the two gambles. Use wealth independent compound gamble and monotonicity with respect to stochastic dominance to deduce the required conclusion. □

11.6 Theorem 7

Lemma 7. *If Q is a continuous index of performance that satisfies global consistency, reflexivity, translation invariance, monotonicity and homogeneity, and u and v are two CARA utilities with $\rho_u \leq \rho_v$, then u is globally inclined to invest in Q-performers at least as v.*

Proof. From reflexivity and the fact that there are no wealth effects for CARA functions it follows that u is globally inclined to invest in Q-performers at least as itself. The conclusion follows, as for any w, w', v accepts less transactions at w' than u at w in the sense of set inclusion, so for all $\bar{q} > q > 0$ and $\delta > 0$, there exists $\epsilon > 0$ such that

$$0 \leq \sup_{(q,r)\in\mathcal{T}_\epsilon} \{Q(r) \mid (q,r) \textit{ is rejected by } u \textit{ at } w\} \leq \inf_{(q,r)\in\mathcal{T}_\epsilon} \left\{Q(r) \mid (q,r) \textit{ is accepted by } u \textit{ at } w'\right\} + \delta$$

$$\leq \inf_{(q,r)\in\mathcal{T}_\epsilon} \left\{Q(r) \mid (q,r) \textit{ is accepted by } v \textit{ at } w'\right\} + \delta,$$

where $\bar{q}$ is the value that is used for reflexivity at (u, w). □

Lemma 8. *The following are equivalent:*

(i) u at w is locally inclined to invest in P^{AS}-performers at least as v at w'

(ii)$\rho_u(w) \leq \rho_v(w')$

Proof. $\neg(ii) \implies \neg(i)$: By Theorem 1 if $\rho_u(w) > \rho_v(w')$, then for small enough $\epsilon > 0$ v at w' accepts any local transaction such that $Q^{AS}(q{\cdot}r) = \frac{3}{2\rho_u(w)+\rho_v(w')}$ or $Q^{AS}(q{\cdot}r) = \frac{3}{\rho_u(w)+2\rho_v(w')}$, and such transactions are rejected by u at w. Such transactions have $P^{AS}(r) = q \cdot \frac{2\rho_u(w)+\rho_v(w')}{3}$ and $P^{AS}(r) = q \cdot \frac{\rho_u(w)+2\rho_v(w')}{3}$ respectively. This implies that $\sup_{(q,r)\in\mathcal{T}_\epsilon} \left\{P^{AS}(r) \mid (q,r) \text{ is rejected by } u \text{ at } w\right\} \geq q \cdot \frac{2\rho_u(w)+\rho_v(w')}{3} > q \cdot \frac{\rho_u(w)+2\rho_v(w')}{3} \geq \inf_{(q,r)\in\mathcal{T}_\epsilon} \{Q(r) \mid (q,r) \text{ is accepted by } v \text{ at } w'\}$, for all $\epsilon > 0$, so (i) does not hold (use $\delta = \frac{\bar{q}}{3} \cdot \left|\frac{2\rho_u(w)+\rho_v(w')}{3} - \frac{\rho_u(w)+2\rho_v(w')}{3}\right|$ to get a contradiction).

$(ii) \implies (i)$: By Theorem 1 and an argument as above, P^{AS} satisfies reflexivity. Thus, for some $\bar{q}_1$, for all $\bar{q}_1 > q > 0$ and all $\delta > 0$ there exists $\epsilon > 0$ with

$$0 \leq \sup_{(q,r)\in\mathcal{T}_\epsilon} \left\{P^{AS}(r) \mid (q,r) \text{ is rejected by } u \text{ at } w\right\} \leq \inf_{(q,r)\in\mathcal{T}_\epsilon} \left\{P^{AS}(r) \mid (q,r) \text{ is accepted by } u \text{ at } w\right\} + \delta.$$

By the same theorem, there exists $\bar{q}_2$ such that for all $\bar{q}_2 > q > 0$ and δ there exists $\epsilon' > 0$ with

$$\inf_{(q,r)\in\mathcal{T}_{\epsilon'}} \{Q(r) \mid (q,r) \text{ is accepted by } u \text{ at } w\} \leq \inf_{(q,r)\in\mathcal{T}_{\epsilon'}} \left\{Q(r) \mid (q,r) \text{ is accepted by } v \text{ at } w'\right\} + \delta.$$

Thus, for all $\min\{\bar{q}_1, \bar{q}_2\} > q > 0$ and all $\delta'(= 2\delta) > 0$, there exists $\min\{\epsilon, \epsilon'\} > \bar{\epsilon} > 0$ such that

$$0 \leq \sup_{(q,r)\in\mathcal{T}_{\bar{\epsilon}}} \{Q(r) \mid (q,r) \text{ is rejected by } u \text{ at } w\} \leq \inf_{(q,r)\in\mathcal{T}_{\bar{\epsilon}}} \left\{Q(r) \mid (q,r) \text{ is accepted by } v \text{ at } w'\right\} + \delta'.$$

□

Lemma 9. *P^{AS} is a continuous index of performance that satisfies reflexivity, global consistency, translation invariance, monotonicity and homogeneity.*

Proof. Translation invariance is immediate as the index could be expressed as a function of $r - r_f$. From now on, assume without loss of generality that $r_f = 0$. For homogeneity, note that both the expectation operator and Q^{AS} are homogeneous of degree 1, and so their ratio is homogeneous of degree 0. Continuity follows from the continuity of Q^{AS} and the fact that if r^n are bounded and converge to r, $E[r_n]$ converges to $E[r]$ from the bounded convergence theorem.

For any r if $E[r] = c > 0$ then for all $\lambda > 0$ $E[(r + \lambda)] = c + \lambda \equiv (1 + \epsilon) E[r]$ for some $\epsilon > 0$. From homogeneity of degree 1 and monotonicity with respect to first order stochastic dominance of Q^{AS} one has

$$Q^{AS}\left(\frac{r}{E[r]}\right) = Q^{AS}\left(\frac{(1+\epsilon) r}{E[(1+\epsilon) r]}\right) = Q^{AS}\left(\frac{(1+\epsilon) r}{c+\lambda}\right) > Q^{AS}\left(\frac{r}{c+\lambda}\right) > Q^{AS}\left(\frac{(r+\lambda)}{c+\lambda}\right),$$

where the inequalities follows from monotonicity of Q^{AS} with respect to first order stochastic dominance, and from the homogeneity of degree 1 of Q^{AS}. The previous inequality implies that $P^{AS}_{r_f}(r + \lambda) > P^{AS}_{r_f}(r)$.

Reflexivity was proved in Lemma 8. Global consistency is implied by the global consistency of Q^{AS}, by Lemma 8. □

Lemma 10. *If P is a continuous index of performance that satisfies reflexivity, global consistency, translation invariance, monotonicity and homogeneity of degree 0, then it is ordinally equivalent to P^{AS}.*

Proof. Assume, by way of contradiction that P satisfies the conditions but is not ordinally equivalent to P^{AS}. There are three ways such violation happen:

1. There exist $r, r' \in \mathcal{R}_1$ with $P^{AS}(r) > P^{AS}(r')$ and $P(r) < P(r')$
2. There exist $r, r' \in \mathcal{R}_1$ with $P^{AS}(r) > P^{AS}(r')$ and $P(r) = P(r')$
3. There exist $r, r' \in \mathcal{R}_1$ with $P^{AS}(r) = P^{AS}(r')$ and $P(r) < P(r')$

There is no loss of generality in treating only the first case. The reason is that using monotonicity and continuity, we could slightly shift r and r' to break the equalities in the right direction while not effecting the inequalities.

Given a violation of type 1, consider an agent with CARA utility function, u such that $\rho_u \equiv .6P^{AS}(r) + .4P^{AS}(r')$. Note that u accepts $(1, r)$ but rejects $(1, r')$, and that u is globally inclined to invest in P-performers at least as u by Lemma 7. But this means that global consistency is violated by P. □

Proof. (Of the theorem) Follows from the lemmata. □

11.7 Lemma 4

Lemma 11. $g \in \mathcal{H} \iff \log(1 + g) \in \mathcal{G}$.

Proof. In one direction, $g \in \mathcal{H} \Rightarrow g \in \mathcal{G}$ and $Q^{FH}(g) < 1$. Since $Q^{FH}(g) \geq L(g)$ it follows that $\log(1 + g)$ is well-defined. As $g \in \mathcal{G}$, it assumes a negative value with positive probability and therefore so does $\log(1 + g)$. Finally, $Q^{FH}(g) < 1$ implies that $E[\log(1 + g)] > 0$. Hence, $\log(1 + g) \in \mathcal{G}$.

In the other direction, if $\log(1+g) \in \mathcal{G}$ we have that $\log(1+g)$ assumes a negative value with positive probability and therefore so does g. In addition, we have $\sum p_i \log(1+g_i) > 0$. Hence, by Fact 1, $g \in \mathcal{H}$. □

Proof. (of Lemma 4) Note that for every $g \in \mathcal{H}$ and $S > 0$, we have $E\left[(1+g)^{-\frac{1}{S}}\right] = E\left[e^{-\frac{\log(1+g)}{S}}\right]$. Consequentially, Lemma 11 and Theorem A in AS imply that the unique positive solution for the equation is $S(g) = Q^{AS}(\log(1+g))$. □

11.8 Theorem 9

Lemma 12. *For all $g \in \mathcal{H}$, If $u \in \mathcal{U}$ has a constant RRA then $\varrho_u(w) - 1 < \frac{1}{S(g)}$ if and only if $E[u(w+wg)] > u(w)\ \forall w > 0$.*

Proof. As positive affine transformations of the utility function do not change acceptance and rejection, it is enough to treat functions of the form $u(w) = -w^{1-\alpha}$. Now observe that:

$$E[u(w+wg)] > u(w) \iff E\left[-w^{1-\alpha}(1+g)^{1-\alpha}\right] > -w^{1-\alpha} \iff E\left[(1+g)^{1-\alpha}\right] < 1 \iff$$

$$\iff E\left[e^{(1-\alpha)\cdot\log(1+g)}\right] < 1 \iff Q^{AS}(\log(1+g)) < \frac{1}{\alpha-1} \iff \alpha - 1 < \frac{1}{S(g)}.$$

□

Lemma 13. *For every $u, v \in \mathcal{U}$, if $\inf_x \varrho_u(x) \geq \sup_{x'} \varrho_v(x')$ then for every w, if u accepts g at w so does v.*

Proof. Without loss of generality, assume that $v(w) = u(w) = 0$ and that $v'(w) = u'(w) = 1$. For every $t > 1$

$$\log v'(tw) = \log v'(tw) - \log v'(w) = \int_1^t \frac{\partial \log v'(sw)}{\partial s} ds = \int_1^t w\frac{v''(sw)}{v'(sw)} ds =$$

$$= \int_1^t \frac{1}{s}\cdot\left(sw\frac{v''(sw)}{v'(sw)}\right) ds \geq \int_1^t \frac{1}{s}\cdot\left(sw\frac{u''(sw)}{u'(sw)}\right) ds = \log u'(tw)$$

$$\log v'(\frac{w}{t}) = \log v'(\frac{w}{t}) - \log v'(w) = \int_1^t \frac{\partial \log v'(\frac{w}{s})}{\partial s} ds = \int_1^t -\frac{w}{s^2}\frac{v''(\frac{w}{s})}{v'(\frac{w}{s})} ds =$$

$$= \int_1^t \frac{1}{s}\cdot\left(-\frac{w}{s}\frac{v''(\frac{w}{s})}{v'(\frac{w}{s})}\right) ds \leq \int_1^t \frac{1}{s}\cdot\left(-\frac{w}{s}\frac{u''(\frac{w}{s})}{u'(\frac{w}{s})}\right) ds = \log u'(\frac{w}{t})$$

This means that for every $t > 0$:

$$v(tw) = v(tw) - v(w) = \int_1^t wv'(sw)ds \geq \int_1^t wu'(sw)ds = u(tw)$$

And so, if $E[u(w+wg)] > u(w) = 0$ then necessarily $E[v(w+wg)] > v(w) = 0$ as $E[v(w+wg)] \geq E[u(w+wg)]$. □

Lemma 14. For every $u \in \mathcal{U}$ and every $w > 0$, $R_S(u,w) = S_S(u,w)$ and $A_S(u,w) = \varrho_u(w) - 1$.

The proof of Lemma 14 is analogous to the proof of Lemma 2 and is therefore omitted. Recalling that the CRRA utility function with parameter α is often expressed as

$$-w^{1-\alpha} = -w^{-(\alpha-1)},$$

this transformation of $\varrho_u(\cdot)$ seems particularly natural.

Proof. (Of the theorem, sketch). First observe that for every $\alpha > 0$ $S((1+g)^\alpha - 1) = Q^{AS}(\log(1+g)^\alpha) = Q^{AS}(\alpha \cdot \log(1+g)) = \alpha \cdot Q^{AS}(\log(1+g)) = \alpha \cdot S(g)$, so S satisfies Scaling. By Lemma 14, $\infty > R_S(u,w) = S_S(u,w) = \frac{1}{\varrho_u(w)-1} > 0$ (which implies that S satisfies local consistency).

To see that S satisfies global consistency, observe that the fact that A_S is ordinally equivalent to ϱ implies that if $v \succ u$ then there exist $\lambda \geq 1$ with $\inf_w \varrho_v(w) \geq \lambda \geq \sup_{w'} \varrho_u(w')$. Therefore, by Lemma 13 if v accepts g at w so does an agent with a CRRA utility function with RRA equals λ. Furthermore, by Lemma 12, if $S(h) < S(g)$ this agent will accept h at any wealth level. Applying Lemma 13 again implies that u accepts h at w.

For uniqueness, assume that $\hat{Q}$ satisfies the requirements. By Lemma 11 $\hat{P}(g) := \hat{Q}(e^g - 1)$ is an index of riskiness $\hat{P} : \mathcal{G} \to \mathbb{R}_+$. For every $\alpha > 0$, we have $\hat{P}(\alpha g) = \hat{Q}(e^{\alpha g} - 1) = \hat{Q}((1 + e^g - 1)^\alpha - 1) = \alpha \cdot \hat{Q}(e^g - 1) = \alpha \cdot \hat{P}(g)$, so $\hat{P}$ satisfies homogeneity. I next claim that $\hat{Q}(g) > \hat{Q}(h)$ if and only if $S(g) > S(h)$. To see this, note that from Theorem 8 $A_{\hat{Q}}$ is ordinally equivalent to ϱ and that from local consistency and scaling $0 < S_Q(u,w) = R_Q(u,w) < \infty$ (see Lemma 6 for a proof of the analogous case). From these facts it follows that S and $\hat{Q}$ order lotteries in the same manner (as before, using CRRA functions). Hence, $\hat{P}$ and Q^{AS} also agree on the order of lotteries. Since both $\hat{P}$ and Q^{AS} are homogeneous, we have that $\hat{P} = C \cdot Q^{AS}$ for some $C > 0$. This in turn, implies that $\hat{Q} = C \cdot S$, for some $C > 0$. □

11.9 Theorem 10

Proof. First, observe that for any CRRA utility function u it must be the case that u is globally at least as averse to Q-riskiness as u, by reflexivity and the lack of wealth effects in CRRA functions. Now consider two gambles g and g' with $S(g) > S(g')$. Consider u CRRA with $\varrho_u \equiv 1 + \frac{2}{S(g)+S(g')}$. u accepts g' and rejects g, implying that $Q(g) \geq Q(g')$, since otherwise strong global consistency will be violated (the violation would be the fact that u is globally no less averse to Q-riskiness than itself, u accepts g' with $Q(g') > Q(g)$, but rejects g).

Next, I claim that if $S(g) > S(g')$, but $Q(g) = Q(g')$, then there exists a gamble g_ϵ such that $S(g_\epsilon) > S(g')$, but $Q(g_\epsilon) < Q(g')$ in contradiction to the above result. To see this note that from monotonicity of Q, for any small $\epsilon > 0$ a gamble $g_\epsilon = g + \epsilon$ has $Q(g_\epsilon) < Q(g)$, and from continuity of S, for small enough ϵ, $S(g_\epsilon) > S(g')$.

Finally, I claim that if $S(g) = S(g')$, but $Q(g) > Q(g')$, then there exists a gamble g_ϵ such that $S(g_\epsilon) < S(g')$, but $Q(g_\epsilon) > Q(g')$. To see this, apply the same argument from the previous paragraph, only this time use the continuity of Q and the monotonicity of S.

The upshot of the above discussion is that $S(g) > S(g) \iff Q(g) > Q(g)$ as required. □

11.10 Theorem 12

Lemma 15. *Let $c = (x_n, t_n)_{n=1}^N$ be an investment cashflow. If $r_k(s) < r_j(s)$ for all $s \in [t_1, t_N]$ then, for all t, $\sum_n e^{-\int_t^{t_n} r_k(s)ds} x_n \leq 0$ implies that $\sum_n e^{-\int_t^{t_n} r_j(s)ds} x_n < 0$.*

Proof. Denote by n^* the highest index with $x_n < 0$. Then

$$\sum_n e^{-\int_t^{t_n} r_k(s)ds} x_n = \sum_{n \leq n^*} e^{-\int_t^{t_n} r_k(s)ds} x_n + \sum_{n > n^*} e^{-\int_t^{t_n} r_k(s)ds} x_n = -\sum_{n \leq n^*} e^{-\int_t^{t_n} r_k(s)ds} |x_n| + \sum_{n > n^*} e^{-\int_t^{t_n} r_k(s)ds} |x_n|, \tag{11.10.1}$$

and

$$-\sum_{n \leq n^*} e^{-\int_t^{t_n} r_k(s)ds} |x_n| + \sum_{n > n^*} e^{-\int_t^{t_n} r_k(s)ds} |x_n| \leq 0 \iff e^{\int_t^{t_{n^*}} r_k(s)ds} \cdot \left(-\sum_{n \leq n^*} e^{-\int_t^{t_n} r_k(s)ds} |x_n| + \sum_{n > n^*} e^{-\int_t^{t_n} r_k(s)ds} |x_n| \right) \leq 0, \tag{11.10.2}$$

and similar statements hold when r_k is replaced with r_j. But,

$$e^{\int_t^{t_{n^*}} r_k(s)ds} \cdot \left(-\sum_{n \leq n^*} e^{-\int_t^{t_n} r_k(s)ds} |x_n| + \sum_{n > n^*} e^{-\int_t^{t_n} r_k(s)ds} |x_n| \right) = -\sum_{n \leq n^*} e^{-\int_{t_{n*}}^{t_n} r_k(s)ds} |x_n| + \sum_{n > n^*} e^{-\int_{t_{n*}}^{t_n} r_k(s)ds} |x_n| >$$
$$-\sum_{n \leq n^*} e^{-\int_{t_{n*}}^{t_n} r_j(s)ds} |x_n| + \sum_{n > n^*} e^{-\int_{t_{n*}}^{t_n} r_j(s)ds} |x_n|$$

as positives are only multiplied by smaller numbers and negatives are multiplied by greater (positive) numbers. □

Lemma 16. *If $c = (x_n, t_n)_{n=1}^N$ is an investment cashflow then there exists a unique positive number r such that $\sum_n e^{-rt_n} x_n = 0$. Furthermore, if $\tilde{r}(t) > r > \hat{r}(t)$ for all $t \in [t_1, t_N]$, then the NPV of c is negative using $\tilde{r}$, and is positive using $\hat{r}$.*

For general cashflows, multiple solutions to the equation defining the internal rate of return may exit. Interestingly, both Arrow and Pratt took interest in finding simple conditions that would rule out this possibility [Arrow and Levhari, 1969, Pratt and Hammond, 1979]. Lemma 16 generalizes the result of Norstrøm [1972] who had shown that investment cashflows have a unique positive IRR in the discrete setting.

Proof. Define the function $f(\alpha) := \sum_n e^{-\alpha t_n} x_n$. Observe that $f(\cdot)$ is continuous, and satisfies $f(0) > 0$ and $f(\alpha) < 0$ for large values of α. Hence, continuity implies the existence of a solution. Lemma 15 implies its uniqueness, and the second part of the claim. □

Lemma 17. *If T satisfies translation invariance, homogeneity and local consistency, then for all u, w, $0 < S_T(i,t) = R_T(i,t) < \infty$.*

Proof. Local consistency requires that

$$\forall i \ \ \forall t \ \ \exists \lambda > 0 \ \ \forall \delta > 0 \ \ \exists \epsilon > 0 \ \ R_T^{\epsilon}(i,t) - \delta < \lambda < S_T^{\epsilon}(i,t) + \delta,$$

which implies that

$$\forall i \ \ \forall t \ \ \exists \lambda > 0 \quad R_T(i,t) \leq \lambda \leq S_T(i,t).$$

Since for any i, t, and $\epsilon > 0$ the set $\{c | \ c \in \mathcal{C}_{t,\epsilon}, \ c \, \text{is rejected by} \, i\}$ is non empty, there exists a sequence of cashflows $\{c_n\}$ such that for each n, $c_n := (x_i^n, t_i^n)$ is rejected, $c_n \in \mathcal{C}_{t,\frac{1}{n}}$ and $T(c_n) < (1 + \frac{1}{n}) \cdot S_T^{1/n}(i,t)$. For small $\delta > 0$, let $c_n' := (x_i^n, (t_i - t_1)(1-\delta))$ for each n. For n large enough, c_n' are all accepted since $T(c_n') = (1-\delta)^k T(c_n) < S_T^{1/n}(i,t)$ and c_n' is in $\mathcal{C}_{t,\frac{1}{n}}$. But this implies that $R_T(i,t) > (1-\delta)^k S_T(i,t)$ since c_n' are almost always accepted and $\lim_{n\to\infty} T(c_n') = (1-\delta)^k \lim_{n\to\infty} T(c_n) = (1-\delta)^k S_Q(i,t)$. Since δ was arbitrarily small, this implies $R_T(i,t) \geq S_T(i,t)$. So, putting the results together, gives

$$\forall i \ \ \forall t \ \ \exists \lambda > 0 \ \ \lambda \leq S_T(i,t) \leq R_T(i,t) \leq \lambda,$$

which completes the proof. □

Proof. (of the theorem) For the first part, in one direction, if $r_i(t) > r_j(t')$ then there exists a small $\epsilon' > 0$ such that for all $x, y \in (-\epsilon', \epsilon')$ $r_i(t+x) > r_j(t+y)$.[61] For a sequence of cashflows with small support and IRR of $\frac{r_i(t)+r_j(t')}{2}$ their translations which start at t' are almost always accepted, and the translations which starts at t are almost always rejected. The same applies to these translated cashflows with times t_i^n replaced by $(1-\delta)(t_i^n - t)$. By Lemma 17, homogeneity and translation invariance this implies that $R_T(i,t) < R_T(j,t')$.

In the other direction, assume $R_T(i,t) < R_T(j,t')$. From Lemma 17 $0 < R_T(i,t) < R_T(j,t') < \infty$. Consider a sequence of cashflows $\{c_n\}$ with $t_N^n < \frac{1}{n}$, $t_1^n = 0$ and $T(c_n) = \frac{2R_T(i,t)+R_T(j,t')}{3}$. For small δ, let $\{c_n'\}$ be a sequence of cashflows such that $t_i'^n = t_i^n \cdot (1-\delta)$. The translations of both $\{c_n\}$

[61]The proof follows closely the proof of Theorem 3, which provides more details.

and $\{c'_n\}$ which start at t' are almost always accepted by j and both the translations that start at t are almost always rejected by i. This, in turn, implies that $r_i(t) > r_j(t')$ using the previous Lemmata.

The second part follows from the first part and from Lemma 17. □

11.11 Propositions 9 and 10

Proof. (Proposition 9) Note that $\forall i, t \;\; A_D(i,t) = r_i(t)$. The conclusion follows from Lemma 16. □

Proof. (Proposition 10) Follows from Lemma 15. □

11.12 Theorem 13

Proof. To prove (i) I first identify one such index. The construction draws upon the findings of previous sections. First, denote by $\mathcal{C}^1$ the class of investment cashflows with $|t_N - t_1| = 1$. Restricting attention to this class of cashflows, I define a function from $\mathcal{C}^1$ to $\mathcal{G}$, the class of gambles, $\mathcal{T} : \mathcal{C}^1 \to \mathcal{G}$,

$$\mathcal{T}(c) = \left[1, \frac{e^{\frac{1}{D(c)}}}{1+e^{\frac{1}{D(c)}}}; -1, \frac{1}{1+e^{\frac{1}{D(c)}}}\right].$$

Observe that $Q^{AS}(\mathcal{T}(\cdot)) \equiv D(\cdot)$. Now, given a cashflow $c = (x_n, t_n)_{n=1}^N$, let $\alpha_c := |t_N - t_1|$. Given t, define $\hat{c}_t := \left(x_n, t + \frac{1}{\alpha_c}(t_n - t)\right)_{n=1}^N$. By construction, $\hat{c}_t$ is a member of $\mathcal{C}^1$. This allows defining a new index $Z : \mathcal{C} \to \mathbb{R}_+$ in the following way:

$$Z(c) := Q^{FH}(\alpha_c \cdot \mathcal{T}(\hat{c}_t)).$$

Z is homogeneous and translation invariant since Q^{FH} is homogeneous, and $\mathcal{T}$ was constructed to assure these properties.

Noting that for $c \in \mathcal{C}_{t,\epsilon}$

$$|D(c) - Z(c)| = \left|Q^{AS}(\alpha_c \cdot \mathcal{T}(\hat{c}_t)) - Q^{FH}(\alpha_c \cdot \mathcal{T}(\hat{c}_t))\right| \leq 2\alpha_c \leq 2\epsilon,$$

one observes that $R_Z(\cdot,\cdot) = R_D(\cdot,\cdot)$ and $S_Z(\cdot,\cdot) = S_D(\cdot,\cdot)$, so if D is locally consistent so is Z.

D satisfies all the requirements of the theorem (proved later on) and the coefficient of local aversion to D equals to r. Since the relation at least as averse to D-delay induces the same order as r, the same applies to Z-delay, as $0 < A_D < \infty$. This implies that for $a > 0$ combinations of the form $W_a(\cdot) = Z(\cdot) + a\,|D(\cdot) - Z(\cdot)|$ also satisfy the requirements of (i). To see that $D \neq Z$, it is enough to consider a cashflow c with $\alpha_c = 1$ and $D(c) = 1$. For this cashflow $Z(c) \approx 1.26$. Together with the fact that Z and D are uniformly close on small domains, the fact that the coefficient of local aversion to Z equals to r (which is positive and finite) implies that the same holds for W_a, which completes the proof of this part.

(ii) Follows from example 20. □

Example 20. Consider $W_1(\cdot)$ and a cashflow c with $\alpha_c = 1$ for which $D(c) = 1$. This implies that $Z(c) \approx 1.26$, hence $W_1(c) < 1.6$. Now consider another cashflow, c', with $\alpha_{c'} = 1$, which first order time dominates c and has $D(c') = \epsilon$ for a small ϵ.[62] Since $Z(c) \geq 1$ from the properties of Q^{FH} and $\mathcal{T}$, $W_1(c') > 1.6$. Therefore, while c' first order time dominates c, $W_1(c) < W_1(c')$.

11.13 Theorem 14

Proof. I provide the proof for the case $k = 1$, but the generalization is simple. First, I check that D satisfies the axioms. Homogeneity is clearly satisfied as

$$\sum_n \mathrm{e}^{-rt_n} x_n = 0 \iff \mathrm{e}^{rt} \sum_n \mathrm{e}^{-rt_n} x_n = 0 \iff \sum_n \mathrm{e}^{-r(t_n - t)} x_n = 0 \iff \sum_n \mathrm{e}^{-\frac{r}{\lambda} \cdot \lambda (t_n - t)} x_n = 0 \ \ (\forall t \ \forall \lambda > 0) .$$

Translation invariance is also satisfied as

$$\sum_n \mathrm{e}^{-rt_n} x_n = 0 \iff \mathrm{e}^{rt} \sum_n \mathrm{e}^{-rt_n} x_n = 0 \ \ (\forall t) .$$

For local consistency, I use the smoothness of $r_i(\cdot)$ to deduce that for every t and small $\epsilon > 0$ there exists $\delta > 0$ such that if $s \in (t - \delta, t + \delta)$ then $r_i(t) - \epsilon < r_i(s) < r_i(t) + \epsilon$. This fact, together with Lemmata 15 and 16, implies that $0 < S_D(i,t) = R_D(i,t) < \infty$ and that $A_D(i,t) = r_i(t)$, hence the axiom is satisfied.

To see that global consistency is satisfied, first note that i is at least as averse to D-delay as j if and only if $\sup_t r_j(t) \leq \inf_t r_i(t)$. Consider an agent that discounts at the constant rate ν, with $\sup r_j(t) \leq \nu \leq \inf r_i(t)$. Label this agent ν. Lemma 15 implies that ν accepts any cashflow accepted by i, Lemma 16 implies that he also accepts cashflows with higher IRR, and another application of Lemma 15 implies that j accepts these cashflows.

I now turn to show that the only indices that satisfy the five axioms are positive multiples of D. This is done in two steps. In the first step, I show that indices that satisfy the axioms agree with the order induced by D. Then, I show that they are also multiples of this index.

For the first step, assume by way of contradiction that there exists another index, Q, that satisfies the axioms but does not agree with D on the ordering of two cashflows at some given time points. There are three possibilities:

1. $Q(c) > Q(c')$ and $D(c) < D(c')$ for cashflows c and c'.
2. $Q(c) > Q(c')$ and $D(c) = D(c')$ for cashflows c and c'.
3. $Q(c) = Q(c')$ and $D(c) < D(c')$ for cashflows c and c'.

There is no loss of generality in treating just the first case. To see this, note that the second and third cases imply the existence of an example of the first type. Such example in obtained by breaking the tie in the correct direction, using translation invariance and homogeneity, while preserving the strict inequality.

[62] This could be achieved by increasing x_N.

To obtain a contradiction, choose r_1 and r_2 such that

$$D(c) < \frac{1}{r_2} < \frac{1}{r_1} < D(c'),$$

and consider two agents that discount with the constant rates r_1 and r_2, and are labeled accordingly r_1 and r_2 (with a slight abuse of notation). Using Lemma 16 both r_1 and r_2 accept c and rejects c'. Theorem 12 and Lemma 17 imply that $r_1 \underset{Q}{\prec} r_2$. But this means that Q violates global consistency, as r_2, the impatient agent, accepts c, the Q-delayed cashflow, but r_1 does not accept c' which is less Q-delayed. Thus, Q and D must agree on the ordering of any two cashflows at any given time point.

For the second step, choose an arbitrary cashflow $c_0 = (x_n, t_n)_{n=1}^N$ and an index that satisfies the axioms, T. For any cashflow c, there exists a positive number $\lambda > 0$ such that $T\left((x_n, t_1 + \lambda \cdot (t_n - t_1))_{n=1}^N\right) = T(c)$. The first step implies that $D\left((x_n, t_1 + \lambda \cdot (t_n - t_1))_{n=1}^N\right) = D(c)$. But $D\left((x_n, t_1 + \lambda \cdot (t_n - t_1))_{n=1}^N\right) = \lambda \cdot D(c_0)$, and also $T\left((x_n, t_1 + \lambda \cdot (t_n - t_1))_{n=1}^N\right) = \lambda \cdot T(c_0)$. Altogether this means that $T(c) = \frac{T(c_0)}{D(c_0)} D(c)$ for every c. □

11.14 Theorem 15

Proof. First, observe that for any agent with constant discount rate, it must be the case that the agent is globally at least as averse to T-delay as himself, by reflexivity and the invariance of the sign of the NPV of translations of a cashflow when the discount rate is constant. Now consider two cashflows c and c' with $D(c) > D(c')$. Consider i with $r_i \equiv \frac{2}{D(c)+D(c')}$. i accepts c' and rejects c, implying that $T(c) \geq T(c')$, since otherwise strong global consistency will be violated (the violation would be the fact that i is globally at least as averse to T-delay as itself, i accepts c' with $T(c') > T(c)$, but rejects c).

Next, I claim that if $D(c) > D(c')$, but $T(c) = T(c')$, then there exists a cashflow c_ϵ such that $D(c_\epsilon) > D(c')$, but $T(c_\epsilon) < T(c')$ in contradiction to the above result. To see this note that from monotonicity of T, for any small $\epsilon > 0$, given $c = (x_i, t_i)_{i=1}^N$, a cashflow $c_\epsilon = (x_i + \epsilon, t_i)_{i=1}^N$ has $T(c_\epsilon) < T(c)$, and from continuity of D, for small enough ϵ, $D(c_\epsilon) > D(c')$.

Finally, I claim that if $D(c) = D(c')$, but $T(c) > T(c')$, then there exists a cashflow c_ϵ such that $D(c_\epsilon) < D(c')$, but $T(c_\epsilon) > T(c')$. To see this, apply the same argument from the previous paragraph, only this time use the continuity of T and the monotonicity of D.

The upshot of the above discussion is that $D(c) > D(c') \iff T(c) > T(c')$ as required. □

11.15 Theorem 17

Proof. (i) The proof is similar to the proof of Theorem 1. First, note that if $\left\{a_n = (\mu_n, \alpha_n) \in \mathcal{A}_{\frac{1}{n}}\right\}_{n=1}^{\infty}$ are accepted it must be the case that $\mu_n \underset{n\to\infty}{\longrightarrow} 0$. To see this, assume by way of contradiction that there is a sub-sequence of such transactions where the price does not converge to 0, without loss of generality $a_n = (\mu_n, \alpha_n)$, and $\lim_{n\to\infty} \mu_n = \hat{\mu} \in (0, \infty]$. Let $\mu := \min\{\hat{\mu}, 1\}$. Then, there exits N

such that for all $n > N$ $l_n := \left(\frac{\mu}{2}, \alpha_n\right)$ is accepted. Lemma 2 of Cabrales et al. [2014] proves that as $\frac{1}{n}$ approaches 0, so does the scale of the optimal investment $\|b^n\|$. Therefore, for $\frac{1}{n}$ small enough, $w - \frac{\mu}{2} + b^n_k$ is in a small environment of $w - \frac{\mu}{2} < w$ for all k, a contradiction.

For the second step, from the discussion above it follows that for $\frac{1}{n}$ small enough, $w - \mu_n + b^n_k$ is in a δ-environment of w for all k, if $a = (\mu, \alpha) \in \mathcal{A}_\epsilon$ is accepted. $\rho_u(w)$ is continuous, and so for every $\gamma > 0$ there exists a $\delta > 0$ small enough such that $x \in (w - \delta, w + \delta)$ implies $|\rho_u(x) - \rho_u(w)| < \gamma$.

For the final step, choose a small positive number η, and consider the CARA agents with absolute risk aversion coefficients $\rho_u(w) + \eta$ and $\rho_u(w) - \eta > 0$. For a small enough environment of w, I,

$$\rho_u(w) - \eta \leq \inf_{x \in I} \rho_u(x) \leq \sup_{x \in I} \rho_u(x) \leq \rho_u(w) + \eta.$$

This, in turn, implies, using Theorem 3 of Cabrales et al. [2014] and a slightly modified version of their Theorem 2, that the coefficient of local taste for A-informativeness of u with wealth w is equal to $\rho_u^{-1}(w)$, and that $R_A(u, w) = S_A(u, w)$.

(ii) Cabrales et al. [2013] showed that $a = (\mu, \alpha)$ is accepted by an agent with log utility function if and only if $I_e(\alpha) > \log\left(\frac{w}{w-\mu}\right)$. Using a Taylor approximation yields

$$\log\left(\frac{w}{w-\mu}\right) = \log(w) - \log(w - \mu) \approx \frac{1}{w}\mu + \frac{\mu^2}{2w^2}.$$

As shown above, if $a_n = (\mu_n, \alpha_n) \in \mathcal{A}_{\frac{1}{n}}$ are accepted it must be the case that $\mu_n \underset{n\to\infty}{\longrightarrow} 0$. It is therefore the case that for n large enough (when posteriors are close to the prior), a_n is accepted by agents with log utility function if

$$J_e(a_n) = \frac{I_e(\alpha_n)}{\mu_n} > \frac{1}{w} + O(\mu_n) \underset{n\to\infty}{\longrightarrow} \frac{1}{w} = \rho_{\log}(w),$$

and rejected if

$$J_e(a_n) = \frac{I_e(\alpha_n)}{\mu_n} < \frac{1}{w} + O(\mu_n) \underset{n\to\infty}{\longrightarrow} \frac{1}{w} = \rho_{\log}(w).$$

For any $x \in \mathbb{R}_+$, $\frac{1}{x} \equiv w \in \mathbb{R}_+$ satisfies $\rho_{\log}(w) = x$, and so by properly translating the log utility function (and changing all but an environment of the baseline wealth level of the agent), one can use a "sandwich" argument of the form used above to complete the proof. □

11.16 Theorem 18

Proof. The proof uses the same techniques used above. If $\rho_u(w) > \rho_v(w')$ then there exists some $\gamma > 0$ such that $\rho_u(w) > (1 + \gamma) \cdot \rho_v(w')$. Following the arguments used before, for $\epsilon > 0$ small enough, if u accepts $a = (\mu, \alpha) \in \mathcal{A}_\epsilon$ then v accepts $((1 + \frac{\gamma}{2}) \cdot \mu, \alpha)$. Together with local consistency and homogeneity this implies that the coefficient of local taste for Q-informativeness of u at w is smaller than the coefficient of local taste for Q-informativeness of v at w', and that v at w has at least as much taste for Q-informativeness as u at w'. [63]

[63] For details, see Theorem 3.

In the other direction, assume $\rho_u(w) = \rho_v(w')$, and by way of contradiction assume that the coefficient of local taste for Q-informativeness of u at w is not equal to the coefficient of local taste for Q-informativeness of v at w'. Without loss of generality, assume that the coefficient of local taste for Q-informativeness of u at w is greater than the coefficient of local taste for Q-informativeness of v at w' . This means that there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of information transactions, such that for every n, $a_n = (\mu_n, \alpha_n)$ satisfies (a) $a_n \in \mathcal{A}_{\frac{1}{n}}$, (b) For some small $\gamma > 0$, $((1+\gamma)\cdot\mu_n, \alpha_n)$ is accepted by u at w, and (c) a_n is rejected by v at w'. But this implies that A violates local consistency, a contradiction, and so the coefficient of local taste for Q-informativeness of u at w is equal to the coefficient of local taste for Q-informativeness of v at w'. This, in turn, implies that u at w is has at least as much taste for Q-informativeness as v at w' (and vice versa). ☐

11.17 Theorem 19

Proof. For (i), let $\delta := \frac{1}{2}\min_i \{\min\{p_i, 1-p_i\}\}$. Define

$$B(a) = \begin{cases} A(a) & \|p - q^s\| < \delta \ \forall s \\ \frac{1}{\mu^k}\cdot f(\alpha) & else \end{cases}$$

for some positive f. Then B satisfies the required properties since for local transactions (ones with posteriors close to the prior) it is equal to A, and since both A and $\frac{1}{\mu}f(\alpha)$ are homogeneous and changes in the price do not change the distance of the posteriors from the prior (and hence the rule that governs B). Choosing $f \equiv 1$ (or many other choices) completes the proof of (ii). ☐

11.18 Theorem 21

Proof. First, observe that for any CARA utility function u it must be the case that Q-informativeness is globally at least as attractive for u as it is for u, by reflexivity and the lack of wealth effects in CARA functions. Now consider two information transactions, a and a', with $A(a) > A(a')$. Consider u CARA with $\rho_u \equiv \frac{A(a)+A(a')}{2}$. u accepts a and rejects a', implying that $Q(a) \geq Q(a)$, since otherwise strong global consistency will be violated (the violation would be the fact that Q-informativeness is globally at least as attractive for u as is for itself, u accepts a with $Q(a') > Q(a)$, but rejects a').

Next, I claim that if $A(a) > A(a')$, but $Q(a) = Q(a')$, then there exists a transaction a_ϵ such that $A(a_\epsilon) > A(a')$, but $Q(a_\epsilon) < Q(a')$ in contradiction to the above result. To see this denote $a_\epsilon := (\mu+\epsilon, \alpha)$, where $a = (\mu, \alpha)$, and note that from monotonicity of Q, for any small $\epsilon > 0$, $Q(a_\epsilon) < Q(a')$, and from continuity of A, for small enough ϵ, $A(a_\epsilon) > A(a')$.

Finally, I claim that if $A(a) = A(a')$, but $Q(a) > Q(a')$, then there exists a transaction a_ϵ such that $A(a_\epsilon) < A(a')$, but $Q(a_\epsilon) > Q(a')$. To see this, apply the same argument from the previous paragraph, only this time use the continuity of Q and the monotonicity of A.

The upshot of the above discussion is that $A(a) > A(a') \iff Q(a) > Q(a')$ as required. ☐