# SEARCH, ADVERSE SELECTION AND MARKET CLEARING

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Abstract. This paper studies a dynamic matching model with adverse selection to examine whether or not the market almost clears if search friction is small. The economy is populated by two unit mass of infinitesimal (infinitely-lived) sellers, high type and low type sellers of equal size, and a positive unit mass of infinitesimal (infinitely-lived) buyers. In each period, sellers who know the quality of the good and buyers who do not observe the quality are randomly matched in pairs with a long side being rationed. For each pair, a price is randomly drawn. If either party disagrees, then the two agents return to the pool, waiting for another chance to be matched to another agent. If both parties agree, then the trade occurs, and the two agents leave the pool of unmatched agents (but not the economy), generating surplus from trading in each period at the drawn price while the agreement is in place. The long term agreement is dissolved by the decision of either party or by an exogenous shock. Upon dissolution of the long term relationship, both agents return to the respective pools of agents. In any stationary equilibrium with a positive probability of trading, both rates of unemployment and vacancy are uniformly bounded away from 0, even in the limit as search friction vanishes. We identify adverse selection as a fundamental source of the coexistence of unemployment and vacancy in addition to search friction and coordination failure caused by directed search.

Keywords: Matching, Search friction, Adverse selection, Undominated equilibrium, Market clearing, Unemployment, Vacancy

## 1. Introduction

Persistent coexistence of unemployment and vacancy is a major challenge to general equilibrium theory. Search theoretic models were developed to explain this coexistence as a stationary equilibrium outcome in an economy with non-negligible amount of friction. The source of friction can be the time needed to find a suitable match between a worker and a job (e.g., Mortensen and Pissarides (1994)), or the coordination failure among individual agents in the (directed) search process (e.g., Burdett, Shi, and Wright (2001), Lagos (2000) and Matsui and Shimizu (2005)). This paper demonstrates that adverse selection can lead to the persistent coexistence of unemployment and vacancy, even in the limit as the search friction vanishes. In other words, it identifies adverse selection as a

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fundamental source of the coexistence of unemployment and vacancy in addition to search friction and coordination failure caused by directed search.

Following Rubinstein and Wolinsky (1985), a large number of dynamic decentralized trading models demonstrated that if search friction is sufficiently small, then the market "almost" clears to approximate the competitive equilibrium outcome (e.g., Gale (1987), Satterthwaite and Shneyerov (2007) and Cho and Matsui (2012)). If the agent has private information about his own type, then the decentralized trading procedure can aggregate the dispersed private information to achieve the competitive market outcome in the limit, as the friction vanishes. We demonstrate that in the presence of adverse selection, both rates of unemployment and vacancy may be uniformly bounded away from zero, as search friction vanishes. Unemployment and vacancy can persist, however efficient search technology might become through, say, the use of internet. Combined with Cho and Matsui (2012), this paper shows that if agents are sufficiently patient, then the presence of adverse selection is sufficient but also necessary for the persistent unemployment and vacancy in an economy where the interaction among agent is very frequent.[1]

We consider an economy which is populated by two unit mass of infinitesimal (infinitely-lived) sellers, high quality and low quality sellers of equal size, and a positive unit mass of infinitesimal (infinitely-lived) buyers. In each period, sellers who know the quality of the good and buyers who do not observe the quality are randomly matched in pairs with a long side being rationed. For each pair, a price is randomly drawn. If either party disagrees, then the two agents return to the pool, waiting for another chance to be matched to another agent. If both parties agree, then the trade occurs and the two agents leave the pool of unmatched agents (but not the economy), generating surplus from trading in each period while the agreement is in place.[2] The long term agreement is dissolved by the decision of either party or by an exogenous shock.[3] Upon dissolution of the long term relationship, both agents return to the respective pools of agents. The objective function of each agent is the expected discounted average payoff. We examine stationary equilibria in which trading occurs with a positive probability. In order to crystallize the impact of asymmetric information, we examine a sequence of stationary equilibria as the friction, quantified by the time span of each period, vanishes.

We obtain a complete characterization of the equilibrium outcomes in the limit as the friction vanishes. If buyers are on the long side, then equilibrium unemployment and vacancy are *uniformly* bounded away from zero, and low quality sellers grab the entire equilibrium surplus. If buyers are on the short side, the same result holds as long as agents are sufficiently patient; otherwise, vacancy disappears in the limit, and the buyers obtain

[1] If the agents are infinitely impatient, then the model is essentially Akerlof (1970), where the market clears.

[2] We choose the random proposal model mainly for analytic tractability. The main conclusion is carried over to models with different trading protocols, for example, the one in which one party makes a take-it-or-leave-it offer, and the other party responds by accepting or rejecting the offer. See Cho and Matsui (2013a).

[3] If the true quality is revealed with a small but positive probability, then the buyer can decide whether or not to continue the long term relationship, conditioned on the truthfully revealed quality of the good. To simplify notation, however, most of the paper focuses on the case where the quality of the good is not revealed during the long term relationship, as would be true of a market for whole life insurance. In subsection 5.2, we examine the case where the true quality is revealed with a positive probability.

a positive surplus. In particular, if agents are extremely impatient, then the outcome of the model essentially becomes the same as that of the static model of Akerlof (1970).

The adverse selection problem is exacerbated by the dynamic trading process from the viewpoint of the uninformed buyers. In the static model of Akerlof (1970), trade can occur only between low quality sellers and buyers. Thus, if buyers are on the short side, then they can extract positive surplus from trading. However, the static equilibrium outcome cannot be sustained as an equilibrium in a dynamic model. Instead of trading with a low quality seller, a buyer can wait until most low quality sellers are matched away and trade with remaining sellers who are likely to have a high quality good and are willing to agree upon any price above their production cost. It turns out that in an equilibrium of the dynamic model, goods are traded at two different price ranges, one below the reservation value of a high quality seller and the other above it.

Suppose that a buyer and a seller are faced with a price in the low price range, which a buyer knows that only the low quality seller is willing to accept. Note that a low quality seller has an option of waiting for a high price in the future if they cannot reach an agreement today. If a low quality seller is sufficiently patient, she would not accept a price today unless the price is sufficiently high. As a result, the probability of reaching an agreement becomes so small that the buyers are left out in the pool for an extended amount of time.

We choose a model with fixed stock of agents instead of a model with a constant inflow of new agents (e.g., Rubinstein and Wolinsky (1985)) because we are interested in the *rate* of unemployment rather than the *size* of unemployment. We cannot assess the significance of unemployment unless we have a well-defined size of population of workers. One million unemployed people in Singapore would be a national scandal, while the same number of unemployment in China would be a bliss point.[4]

Adverse selection in a search model has recently drawn considerable attention. Guerrieri, Shimer, and Wright (2010) investigates a static matching model with adverse selection. Chang (2012) embeds Guerrieri, Shimer, and Wright (2010) into Mortensen and Pissarides (1994) to investigate the information revelation in a decentralized financial market under adverse selection. As in most models following the framework of Mortensen and Pissarides (1994), these models are built on a matching function, which presumes the coexistence of unemployment and vacancy. In our model, we derive, rather than assume, the coexistence.

By letting the friction vanish, we crystallize the impact of asymmetric information on unemployment and vacancy in an equilibrium, in contrast to existing models on the labor market search in which the amount of friction is a free parameter to be specified (e.g., Mortensen and Pissarides (1994)). Existing models to provide the foundation for the matching function (e.g., Burdett, Shi, and Wright (2001)) focus on the coordination failure of the search processes among individual agents.

Matsui and Shimizu (2005) examined the coordination failure among agents, who are searching a particular post to trade, out of many posts. But, Matsui and Shimizu (2005) proves that the coordination failure can be resilient: the coordination failure may not vanish even in the limit as the friction vanishes. As a result, some post can experience

[4]In a model with a constant inflow of agents, the rate of unemployment converges to zero over time, if the workers are perpetually employed. If they are employed for finite $T$ period, then the rate of unemployment is determined by the arbitrarily chosen duration time $T$.

a positive amount of unemployment, while some other experiences a positive amount of vacancy, which could have been avoided if the agents could have coordinated their search for the trading posts.

This paper differs from the existing papers on information aggregation under adverse selection in a decentralized dynamic procedure, as we sustain positive *rates* of both unemployment and vacancy in an equilibrium *steady* state. For example, Wolinsky (1990) and Moreno and Wooders (2010) investigated the information aggregation and delay in a model with constant inflow of buyers and sellers, where we cannot calculate the equilibrium *rates* of unemployment and vacancy. Blouin and Serrano (2001) studied the same question, in a model with a fixed mass of agents who leave the market permanently as they reach agreement. Because the population size shrinks as the game continues, the equilibrium steady state does not exist in Blouin and Serrano (2001), which makes it impossible to explain the persistence of the coexistence of unemployment and vacancy.[5]

Section 2 describes the model in which the masses of sellers and buyers are exogenously given. Section 3 presents the preliminary results and concepts. Section 4 formally describes the main results. Section 5 explores a couple of extensions. Subsection 5.1 discusses a model with free entry of buyers (i.e., firms). Subsection 5.2 considers a model in which true quality of the good is revealed dring the long term relationship. Section 6 concludes the paper.

## 2. Model

2.1. **Static model.** We consider an economy which is populated by 2 unit mass of infinitesimal (infinitely-lived) sellers, high type and low type sellers of equal size, and $x_b > 0$ unit mass of infinitesimal (infinitely-lived) buyers.[6]

High type sellers produce one unit of high quality good at the cost of $s_h$, while low type sellers produce one unit of low quality good at the cost of $s_l$. Assume $s_h > s_l$. The goods are indivisible. The marginal utility of the high quality good for a buyer is $\phi_h$, while that of the low quality good is $\phi_l$, where $\phi_h > \phi_l$. Each seller produces at most one unit of the good, and each buyer consumes at most one unit of the good.

We make the following three standard assumptions on the parameter values, which are critical for capturing the lemons problem.

*A*1. $\phi_h > s_h > \phi_l > s_l$, which implies that the existence of the gains from trading under each state is common knowledge.

*A*2. $\phi_h - s_h > \phi_l - s_l$ so that it is socially efficient for the high quality sellers to deliver the good to the buyers.

*A*3. $\dfrac{\phi_h + \phi_l}{2} < s_h$ so that the lemons problem is severe in the sense that random transactions lead to a negative payoff either to a buyer or to a high quality seller.

If $p$ is the delivery price of the good, and $y(\in \{h, l\})$ is the quality of the good, seller's profit is $p - s_y$ and buyer's surplus is $\phi_y - p$. Under the assumptions we made, only the

[5]Our model shares many common features with Moreno and Wooders (2010). Yet, we prove that the dynamic trading can make the lemons problem "worse" in the sense that the informed low quality seller can extract the entire gain from surplus, even if the buyer is in the short side of the market.

[6]No main result is qualitatively sensitive to the fact that the masses of high and low quality sellers are the same.

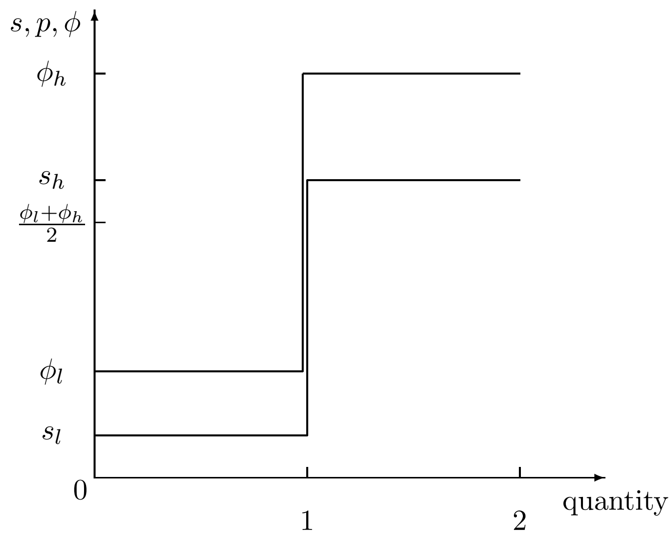


FIGURE 1. Lemons market

low quality good is traded in any competitive equilibrium, and the equilibrium price $p^*$ is given by

$$p^* \in \begin{cases} \{s_l\} & \text{if } x_b < 1, \\ [s_l, \phi_l] & \text{if } x_b = 1, \\ \{\phi_l\} & \text{if } x_b > 1. \end{cases}$$

2.2. **Dynamic model.** Let us embed the above static model into a decentralized dynamic trading model. Time is discrete, and the horizon is infinite. When a buyer and a seller are initially matched at period $t$, conditioned on her type $k \in \{h, l\}$, the seller reports her type as $k'$, possibly in a randomized fashion, to a third party (or mechanism) which draws a price $p$ according to a probability density function $f_{k'}$ over $\mathbb{R}$. We assume that the support of $f_{k'}$ is $[s_l, \phi_h]$.

We assume

$$\forall k' \in \{h, l\},\ \forall p \in [s_l, \phi_h], \quad f_{k'}(p) > 0 \text{ and is continuous.} \tag{2.1}$$

Conditioned on $p$ drawn by the mechanism, each party has to decide whether or not to form a long term relationship. After forming the long term relationship, the buyer can purchase the good at the agreed price, and the seller can sell the good at the same price to the buyer. If the good is delivered at $p$, the seller's surplus is $p - s_k$ and the buyer's surplus is $\phi_k - p$ ($k \in \{h, l\}$).

Then, at the end of the period, either one of two events will occur. The long term relationship breaks down with probability $1 - \delta$, and then, both agents are dumped back to the respective pools. The long term relationship continues with probability $\delta$ without the true quality being revealed.

In each period, the buyer and the seller in a long term relationship can choose to maintain or to terminate it. If one of the agents decides to terminate the long term relationship, both agents return to their respective pools, waiting for the next round of matching. If both agents decide to continue the long term relationship, the long term relationship continues with probability $\delta = e^{-d\Delta}$ where $d > 0$, and with probability $1-\delta$, the long term relationship dissolves, and the two agents are forced to return to the pool.

We assume that the true quality of the good is not revealed to the buyer during the long term relationship, like a life insurance policy, until the long term relationship dissolves. This assumption is only to simplify exposition.[7]

The objective function of each agent is the long run discounted average expected payoff:

$$(1-\beta)\mathsf{E}\sum_{t=1}^{\infty}\beta^{t-1}u_{i,t}$$

where $u_{i,t}$ is the payoff of agent $i$ in period $t$ and $\beta = e^{-b\Delta}$ is the discount factor.

We focus on the undominated stationary equilibrium, which is a stationary equilibrium where no dominated strategy is used, to exclude a "no trading equilibrium" in which every agent refuses to reach an agreement. We simply refer to an undominated stationary equilibrium as an equilibrium, whenever the meaning is clear from the context.

To simplify exposition, we assume for the rest of the paper that $p$ is drawn from $[s_l, \phi_h]$ according to the uniform distribution regardless of the report of the seller. The extension to the case where the price is drawn from a general distribution satisfying (2.1) is cumbersome but straightforward (Cho and Matsui (2013b)).

## 3. Preliminaries

Let $W_s^h(p)$, $W_s^l(p)$, and $W_b(p)$ be the continuation values of a high quality seller, a low quality seller, and a buyer, respectively, after the two agents agree on $p \in [s_l, \phi_h]$. Also, let $W_s^h$, $W_s^l$, and $W_b$ be the continuation values of respective agents after they do not form a long term relationship. Given the equilibrium value functions, let us characterize the optimal decision rule of each agent. In what follows, we write $x \leq O(\Delta)$ if

$$\lim_{\Delta\to 0}\frac{x}{\Delta} < \infty.$$

Let $z_s^l$ and $z_s^h$ be the mass of $s_l$ and $s_h$ sellers in the pool. Similarly, let $z_b$ be the mass of buyers in the pool. Since the mass of paired buyers and the mass of paired sellers are of equal size, we have

$$2 - z_s = x_b - z_b, \tag{3.2}$$

where $z_s = z_s^h + z_s^l$. Let

$$\mu_h = \frac{z_s^h}{z_s}$$

be the proportion of high quality sellers in the pool of sellers, and let $\mu_l = 1-\mu_h$ be the proportion of low quality sellers in the pool of sellers.

[7] In subsection 5.2, we extend the model to the one in which the true quality is revealed to the buyer during the long term relationship with probability $1-\lambda$ per period, and upon the revelation of the true quality, the buyer can decide to continue or terminate the existing long term relationship.

Our main goal is to find conditions under which

$$\lim_{\Delta \to 0} z_b > 0,$$

and

$$\lim_{\Delta \to 0} z_s > 0$$

hold simultaneously. Throughout the paper, $z_s$ is interpreted as unemployment, while $z_b$ as vacancy.

Because the relative size of buyers and sellers in the pool is an important variable, let us define

$$\rho_{bs} = \frac{z_b}{z_s}.$$

Since $\rho_{bs}$ determines the frequency of meeting the other party with a long side rationed, let us define

$$\zeta = \min\{1, \rho_{bs}\}$$

as the probability that a seller meets a buyer, and

$$\xi = \min\left\{1, \frac{1}{\rho_{bs}}\right\} \tag{3.3}$$

as the probability that a buyer meets a seller. Due to (3.2), we have

$$\begin{cases} \zeta = \rho_{bs} < 1 \text{ and } \xi = 1 & \text{if } x_b < 2, \\ \zeta = \rho_{bs} = 1 \text{ and } \xi = 1 & \text{if } x_b = 2, \\ \zeta = \rho_{bs} = 1 \text{ and } \xi < 1 & \text{if } x_b > 2. \end{cases}$$

Let $\Pi_s^h$ be the set of prices that a high quality seller and a buyer agree to accept, and let $\pi_s^h = \mathsf{P}(\Pi_s^h)$. For $p \in \Pi_s^h$, we can write

$$W_s^h(p) = (1-\beta)(p - s_h) + \beta\left(\delta W_s^h(p) + (1-\delta) W_s^h\right).$$

The first term is the payoff in the present period. At the end of the present period, with probability $1-\delta$, the long term relationship dissolves, and the high quality seller's continuation payoff is $W_s^h$. With probability $\delta$, the high quality seller continues the relationship, of which continuation value is given by $W_s^h(p)$.

A simple calculation shows

$$W_s^h(p) = \frac{(1-\beta)(p-s_h) + \beta(1-\delta)W_s^h}{1-\beta\delta}. \tag{3.4}$$

The high quality seller agrees to form a long term relationship with delivery price $p$ if

$$W_s^h(p) > W_s^h$$

which is equivalent to

$$p > s_h + W_s^h. \tag{3.5}$$

On the other hand, $W_s^h$ is given by

$$W_s^h = \beta\zeta\pi_s^h \mathsf{E}[W_s^h(p) | \Pi_s^h] + \beta(1-\zeta\pi_s^h)W_s^h. \tag{3.6}$$

Substituting (3.4) into (3.6), we obtain, after some calculation,

$$W_s^h = \frac{\beta\zeta\pi_s^h}{1-\beta\delta}\mathsf{E}[p - s_h - W_s^h|\Pi_s^h]. \tag{3.7}$$

Similarly, we obtain

$$W_s^l = \frac{\beta\zeta\pi_s^l}{1-\beta\delta}\mathsf{E}[p - s_l - W_s^l|\Pi_s^l], \tag{3.8}$$

where $\Pi_s^l$ is the set of prices that a low quality seller and a buyer agree to accept, and $\pi_s^l = \mathsf{P}(\Pi_s^l)$. In any undominated equilibrium, $s_l$ seller accept $p$ if

$$p > s_l + W_s^l.$$

Imitating the behavior of high quality sellers, a low quality seller can always obtain a higher (or equal) continuation value than a high quality seller.[8] Therefore, we have $W_s^l \geq W_s^h$. Now, we would like to claim that the threshold price for a low quality seller is lower than that for a high quality seller.

**Lemma 3.1.**

$$s_h - s_l > W_s^l - W_s^h.$$

*Proof.* If a high quality seller imitates a low quality seller, then the long run expected payoff from the deviation is

$$W_s^l - (s_h - s_l)\frac{\beta\pi_s^l}{1-\beta\delta+\beta\pi_s^l}.$$

Since the deviation payoff is less than the equilibrium payoff,

$$W_s^l - W_s^h \leq (s_h - s_l)\frac{\beta\pi_s^l}{1-\beta\delta+\beta\pi_s^l} < s_h - s_l$$

as desired. □

Let $\Pi_s^l$ (resp. $\Pi_s^h$) be the set of prices where $L$-type (resp. $H$-type) sellers and buyers trade with a positive probability. Lemma 3.1 says

$$s_l + W_s^l = \inf \Pi_s^l < s_h + W_s^h = \inf \Pi_s^h.$$

Since the decision rule of each seller is a threshold rule, this inequality implies

$$\Pi_s^h \subset \Pi_s^l.$$

Thus, we can partition the set of prices into three regions, $\Pi_s$, $\Pi_p$, and the rest:

$$\begin{aligned} \Pi_s &= \Pi_s^l \setminus \Pi_s^h, \\ \Pi_p &= \Pi_s^l \cap \Pi_s^h, \end{aligned}$$

where $\Pi_s$ is the set of the prices at which trade occurs only with low quality sellers (the subscript stands for separating), $\Pi_p$ is the set of the prices at which trade occurs with

[8]If the true quality is revealed with a positive probability after the good is delivered, then we cannot invoke the same argument to prove the inequality. Yet, the main result is carried over.

both low and high quality sellers (the subscript stands for pooling), and the remaining region is the one in which no trade occurs. Note that we have

$$\Pi_s \subset [s_l + W_s^l, s_h + W_s^h],$$
$$\Pi_p \subset [s_h + W_s^h, \infty).$$

Let $\pi_s = \mathsf{P}(\Pi_s)$ and $\pi_p = \mathsf{P}(\Pi_p)$. Since we focus on an equilibrium in which trading occurs with a positive probability,

$$\pi_s + \pi_p > 0$$

in an equilibrium.

**Definition 3.2.** *If $\pi_p = 0$ in an equilibrium, we call such an equilibrium a separating equilibrium. If $\pi_s = 0$, then the equilibrium is called a pooling equilibrium. If $\pi_s > 0$ and $\pi_p > 0$, then it is called a semi-pooling equilibrium.*

Let us calculate the value function of a buyer. In the private value model in which a buyer knows exactly how valuable the objective is (Cho and Matsui (2012)), the informational content of $p$ is irrelevant for a buyer to deciding whether or not to accept $p$. In contrast, in the present model, the expected quality conditioned on $p$ is a critical factor for a buyer to make a decision.[9] Let $\phi^e(p)$ be the expected quality if $p$ is the price to be agreed upon. If $p \in (s_l + W_s^l, s_h + W_s^h)$, then only low quality sellers agree to accept the price, and therefore, we have $\phi^e(p) = \phi_l$. On the other hand, if $p > s_h + W_s^h$ holds, then both low and high quality sellers agree to do so, and therefore, we have

$$\phi^e(p) = \phi(\mu_l) \equiv \mu_l \phi_l + (1 - \mu_l)\phi_h.$$

If a buyer and a seller agree to form a long term relationship at price $p$, then the expected continuation value of the buyer is given by

$$W_b(p) = (1 - \beta)(\phi^e(p) - p) + \beta \left[\delta W_b(p) + (1 - \delta) W_b\right].$$

Therefore, we have

$$W_b(p) = \frac{(1 - \beta)(\phi^e(p) - p) + \beta(1 - \delta)W_b}{1 - \beta\delta}.$$

Also, the continuation value after no match is given by

$$W_b = \beta\xi\mu_l\pi_s \mathsf{E}\left[W_b(p)|\Pi_s\right] + \beta\xi\pi_p \mathsf{E}\left[W_b(p)|\Pi_p\right] + \beta(1 - \xi\mu_l\pi_s - \xi\pi_p)W_b.$$

After substitutions and tedious calculation, we obtain

$$W_b = \frac{\beta\xi\mu_l\pi_s}{1 - \beta\delta}\mathsf{E}\left[\phi_l - p - W_b|\Pi_s\right] + \frac{\beta\xi\pi_p}{1 - \beta\delta}\mathsf{E}\left[\phi(\mu_l) - p - W_b|\Pi_p\right] \tag{3.9}$$

where $\xi$ is the probability that a buyer is matched to a seller, as defined in(3.3).

A buyer is willing to accept $p$ if

$$W_b(p) > W_b,$$

[9]Even if each individual is infinitesimally small, the informational content of $p$ affects the decision of all buyers. In this sense, each individual is not "informationally small" in the sense of Gul and Postlewaite (1992).

or equivalently,

$$\phi^e(p) - p > W_b.$$

Since $\phi^e(p)$ may change as $p$ changes, the buyer's equilibrium decision rule may not be characterized by a single threshold.

Combining these results and including the endpoints as they are measure zero events, we have

$$\Pi_s = \begin{cases} [s_l + W_s^l, \phi_l - W_b] & \text{if } s_l + W_s^l \le \phi_l - W_b, \\ \emptyset & \text{otherwise}, \end{cases}$$

$$\Pi_p = \begin{cases} [s_h + W_s^h, \phi(\mu_l) - W_b] & \text{if } s_h + W_s^h \le \phi(\mu_l) - W_b, \\ \emptyset & \text{otherwise}. \end{cases}$$

By the assumption that $p$ is uniformly distributed over $(s_l, \phi_h)$, we obtain

$$\mathsf{E}[p - s_h - W_s^h | \Pi_s^h] = \mathsf{E}[p - s_h - W_s^h | \Pi_p] = A\pi_p,$$

where

$$A = \frac{1}{2}(\phi_h - s_l).$$

Therefore, (3.7) can be rewritten as

$$W_s^h = \frac{\beta A (\pi_p)^2 \zeta}{1 - \beta\delta}. \tag{3.10}$$

Similarly, we have

$$\begin{aligned} \mathsf{E}[p - s_l - W_s^l | \Pi_s] &= A\pi_s, \\ \mathsf{E}[\phi(\mu_l) - p - W_b | \Pi_p] &= A\pi_p, \\ \mathsf{E}[\phi_l - p - W_b | \Pi_s] &= A\pi_s. \end{aligned}$$

Thus, $W_s^l$ and $W_b$ can be rewritten as

$$W_s^l = \frac{\beta A (\pi_s)^2 \zeta}{1 - \beta\delta} + \frac{\beta \pi_p \zeta}{1 - \beta\delta} \mathsf{E}[p - s_l - W_s^l | \Pi_p] \tag{3.11}$$

$$W_b = \frac{\beta A (\pi_s)^2 \mu_l \xi}{1 - \beta\delta} + \frac{\beta A (\pi_p)^2 \xi}{1 - \beta\delta}, \tag{3.12}$$

respectively. Following Cho and Matsui (2013b), one can obtain a similar expression for a general distribution of price, given a sufficiently small $\Delta > 0$.

Also, rewrite $\pi_s$ and $\pi_p$ as

$$\pi_s = C[\phi_l - s_l - W_b - W_s^l] \tag{3.13}$$

$$\pi_p = C[\phi(\mu_l) - s_h - W_b - W_s^h] \tag{3.14}$$

where

$$C = \frac{1}{\phi_h - s_l}.$$

The size of population of each type of the agents is determined by the balance equations:

$$1 - z_s^l = \left(\frac{\pi_s \zeta}{1-\delta} + \frac{\pi_p \zeta}{1-\delta}\right) z_s^l \tag{3.15}$$

$$1 - z_s^h = \frac{\pi_p \zeta}{1-\delta} z_s^h \tag{3.16}$$

$$x_b - z_b = \left(\frac{\pi_s \mu_l \xi}{1-\delta} + \frac{\pi_p \xi}{1-\delta}\right) z_b. \tag{3.17}$$

An equilibrium in the baseline model is characterized by $(z_b, z_s^h, z_s^l, W_b, W_s^l, W_s^h)$. We use the following notion of market clearing.

**Definition 3.3.** *A market is cleared if*

$$\lim_{\Delta \to 0} z_b z_s = \lim_{\Delta \to 0} z_b (z_s^h + z_s^l) = 0. \tag{3.18}$$

## 4. Results

Let us state the asymptotic properties of the equilibrium payoffs for the case where $A1 - A3$ hold.

**Theorem 4.1.** *For any sequence of undominated stationary equilibria,*

$$\lim_{\Delta \to 0} W_s^h = 0$$

$$\lim_{\Delta \to 0} W_s^l + W_b = \phi_l - s_l$$

*Proof.* See Appendix A. □

In order to understand how the equilibrium surplus $\phi_l - s_l$ is split between a seller and a buyer, we need to investigate the structure of an equilibrium further. The next lemma is a critical step toward characterizing the condition under which the market fails to clear.

**Lemma 4.2.**

$$\lim_{\Delta \to 0} \zeta W_b = 0.$$

*Proof.* See Appendix B. □

The alternative characterization of Nash bargaining solution by Harsanyi (1956) offers a useful insight toward Lemma 4.2. Let us imagine a bargaining situation between a low quality seller and a buyer, and assume for a moment that $p < s_h$ is on the table.[10] The option value of rejecting the present offer is roughly the future gain multiplied by the probability of reaching an agreement. Alluding to Harsanyi (1956), we may claim that the bargaining power can be represented as a ratio of the option value of rejecting the current offer over the present value of accepting it. The bargaining outcome is determined by equating the bargaining powers of the two parties. Equivalently, the ratio between the equilibrium payoffs of the two parties must be equal to the ratio between the option

[10] The buyer knows only the low quality seller will accept such a price.

values of each party's rejecting the present offer in an equilibrium. The precise ratio of the option values is given by

$$\frac{A\zeta\pi_s^2 + \zeta\pi_p \mathsf{E}[p - s_l - W_s^l | \Pi_p]}{A\left(\mu_l \pi_s^2 + \pi_p^2\right)}.$$

In equilibrium, this must be equal to $W_s^l / W_b$.

Note that the low quality sellers trade either at $p \in \Pi_s$, or at $p \in \Pi_p = [s_h + W_s^h, \phi^e - W_b]$. The net gain from reaching agreement over staying in the pool is $p - s_l - W_s^l > 0$. If $p$ is drawn from $\Pi_p = [s_h + W_s^h, \phi^e - W_b]$, then the net gain from trading at a price $p \in \Pi_p$ does not vanish even if $\Delta \to 0$:

$$p - s_l - W_s^l \geq s_h - s_l - W_s^l = (s_h - \phi_l) + (\phi_l - s_l - W_s^l) \to (s_h - \phi_l) + \lim_{\Delta \to 0} W_b \geq s_h - \phi_l > 0$$

by Theorem 4.1, while the last (strict) inequality follows from A1.

Suppose that $\zeta > 0$ is uniformly bounded away from 0 as $\Delta \to 0$. The low quality seller has an opportunity to make a large profit if she can sell the good at a price drawn from $\Pi_p$. However, this profit can be realized, only if she can be matched to a buyer. On the other hand, the buyer does not have an opportunity to make a large profit as the seller, because he has no private information. If $\zeta > 0$ is bounded away from 0, a low quality seller has a positive chance to realize a large profit from trading at a price drawn from $\Pi_p$. The ratio between the option values of rejecting the present price between the buyer and the low quality seller converges to 0. In an equilibrium, this ratio must be the same as the ratio of the equilibrium payoffs of the buyer and the low quality seller. Therefore, $W_b \to 0$ as $\Delta \to 0$.

Lemma 4.2 reveals the complementary slackness between $z_b$ (or $\zeta$), and $W_b$ in the limit as $\Delta \to 0$. It would be convenient to analyze the baseline model, conditioned on whether or not $\lim_{\Delta \to 0} z_b > 0$ (or equivalently, $\lim_{\Delta \to 0} \zeta > 0$).

**Theorem 4.3.** $\lim_{\Delta \to 0} z_b > 0$ *only if*

$$\frac{\phi_h - \phi_l}{s_h - s_l + \dfrac{b}{d}(\phi_l - s_l)} - (2 - x_b) > 0 \tag{4.19}$$

*holds.*

Theorem 4.3 warrants some discussion to understand its intuition and scope. While the theorem states that (4.19) is a necessary condition for a significant amount of vacancy in the limit, the intuition also indicates that (4.19) is also sufficient, which the ensuing analysis is about to prove.

Observe that given other things, (4.19) will fail if the agents are very impatient so that $b/d$ is large. For example, if $b/d = \infty$ and (4.19) fails, our model is essentially identical with the static model of Akerlof (1970), and the market clears in the sense that $\lim_{\Delta \to 0} z_b = 0$ when the buyers are on short side. The substance of Theorem 4.3 is to show that the intuition of Akerlof (1970) is carried over, as long as the agents are impatient in the sense that $b/d$ is large.

The low quality seller can generate a large profit by agreeing on $p \in \Pi_p$ even if $\Delta > 0$ is small. However, trading at a high price from $\Pi_p$ can be realized after possibly many rounds of matching and bargaining. If $b > 0$ is large so that (4.19) fails, then the seller is too impatient to exploit the future opportunity of trading at a high price, and is content with reaching an agreement quickly, which leads to $\lim_{\Delta\to 0} z_b = 0$, as Theorem 4.3 implies.

By the definition, $z_b \le x_b$ must hold. Thus, one might wonder if

$$\frac{\phi_h - \phi_l}{s_h - s_l + \frac{b}{d}(\phi_l - s_l)} < 2 \tag{4.20}$$

always holds under $A1 - A3$. To verify this inequality, it is sufficient to show that

$$\phi_h - \phi_l \le 2(s_h - s_l).$$

Suppose that

$$\phi_h - \phi_l > 2(s_h - s_l).$$

Then, we have

$$\phi_h - \phi_l > 2(s_h - s_l) > \phi_h + \phi_l - 2s_l$$

where the second inequality follows from $A3$. From the first and the last terms, we conclude that

$$s_l > \phi_l$$

which violates $A1$.

In order to prove the theorem, we need some preliminary results, which reveal the important properties of the equilibrium outcome. From (3.15) and (3.16), we have

$$z_s^h = \frac{1}{1 + \frac{\zeta\pi_p}{1-\delta}}$$

and

$$z_s^l = \frac{1}{1 + \frac{\zeta\pi_p + \zeta\pi_s}{1-\delta}}.$$

**Lemma 4.4.** *Suppose that* $\lim_{\Delta\to 0} z_b > 0$. *Then*

$$\lim_{\Delta\to 0} \frac{\zeta\pi_p}{1-\delta} = Q_p \equiv \frac{b+d}{d}\left[\frac{\phi_l - s_l}{s_h - \phi_l}\right], \tag{4.21}$$

$$\lim_{\Delta\to 0} \frac{\zeta\pi_s}{1-\delta} = Q_s \equiv \left[\frac{2s_h - (\phi_h + \phi_l)}{\phi_h - s_h}\right](1 + Q_p). \tag{4.22}$$

*Proof.* See Appendix C. □

Lemma 4.4 implies

$$\lim_{\Delta\to 0} z_s^h = \frac{1}{1+Q_p} = \frac{s_h - \phi_l}{s_h - s_l + \frac{b}{d}(\phi_l - s_l)}$$

and

$$\lim_{\Delta\to 0} z_s^l = \frac{1}{1+Q_s+Q_p} = \frac{\phi_h - s_h}{s_h - s_l + \frac{b}{d}(\phi_l - s_l)}$$

which are independent of $x_b$. Thus, if $\lim_{\Delta\to 0} z_b > 0$, then

$$0 < \lim_{\Delta\to 0} z_s = \lim_{\Delta\to 0} z_s^h + z_s^l = \frac{\phi_h - \phi_l}{s_h - s_l + \frac{b}{d}(\phi_l - s_l)} < 2 \tag{4.23}$$

where the second inequality follows from (4.20). Note that the right hand side of (4.23) is independent of $x_b$. From (3.17), we know that $z_b$ is a positive linear function of $x_b$. An increase of $x_b$ affects $z_s$ in two ways. As $x_b$ increases, $z_b$ increases, which increases the probability $\zeta$ of a seller meeting a buyer. One may conclude that if $x_b$ increases, then $z_s$ must decrease, since more sellers are matched away. This observation misses the second way of $x_b$ affecting $z_s$. As a seller faces a better chance of meeting a buyer, her long run average payoff increases, and she bargains more aggressively. As a result, $\pi_s$ and $\pi_p$ decrease as a linear function of $\zeta$. In an equilibrium, the two effects of an increase in $x_b$ are perfectly cancelled out so that $z_s$ remains unaffected.

Recall that

$$z_b = x_b - 2 + z_s.$$

Therefore, we have

$$\lim_{\Delta\to 0} z_b = \frac{\phi_h - \phi_l}{s_h - s_l + \frac{b}{d}(\phi_l - s_l)} - (2 - x_b).$$

Thus,

$$\lim_{\Delta\to 0} z_b > 0$$

only if (4.19) holds, which proves Theorem 4.3.

Since the first term of (4.19) is positive, this condition holds automatically if $x_b \geq 2$.[11] If (4.19) holds, the market does not clear as neither unemployment nor vacancy vanishes in the limit as the friction vanishes.

If $\lim_{\Delta\to 0} \zeta = \lim_{\Delta\to 0} z_b/z_s > 0$, then Lemma 4.4 implies that $\pi_s$ and $\pi_p$ vanish at the rate of $\Delta > 0$. Exploiting Lemma 4.4, we can calculate the rate at which $W_b$ vanishes by substituting $\pi_s$ and $\pi_p$. For any small $\Delta > 0$,

$$W_b = \frac{F}{\zeta^2}\Delta + o(\Delta) \tag{4.24}$$

where

$$F = A\frac{d^2}{b+d}\left[\frac{\phi_h - s_h}{\phi_h - \phi_l}Q_s^2 + Q_p^2\right],$$

and

$$\lim_{\Delta\to 0} \frac{o(\Delta)}{\Delta} = 0.$$

We claim that (4.19) is also a sufficient condition for the results we have obtained so far. Note that (4.19) is violated only when $x_b < 2$, i.e., the buyers are on the short side.

**Proposition 4.5.** *Suppose that* $\lim_{\Delta\to 0} z_b = 0$. *Then,*

(1) $\lim_{\Delta\to 0} z_s = 2 - x_b$.
(2) $\frac{\pi_p}{1-\delta} \to \infty$ *and* $\frac{\pi_s}{1-\delta} \to \infty$ *as* $\Delta \to 0$.

[11] It is expected, because the buyer is in a long side if $x_b \geq 2$.

(3) $\lim_{\Delta\to 0} W_b \geq 0$ *and the equality holds only if* (4.19) *is violated with equality.*
(4) (4.19) *is violated.*

*Proof.* See Appendix D. □

A sketch of the proof is as follows. Suppose $b/d$ is small so that (4.19) holds. Indeed, if $b/d$ is sufficiently close to zero, (4.19) always holds under $A1$-$A3$. Then the low type sellers will have an incentive to wait for the future gain from trading at a price in $\Pi_p$ when a price in $\Pi_s$ is drawn.

The more patient the agents are, the larger the gain from trading at a price drawn from $\Pi_p$ becomes. The larger this gain becomes, the more aggressive the low quality seller becomes in bargaining, which leads to a lower probability of reaching an agreement. As $b$ becomes small, $\pi_s$ and $\pi_p$ become so small at some point that they converge to zero at the rate of $\Delta$. From (3.12), we have

$$\lim_{\Delta\to 0} W_b = 0.$$

Since $\lim_{\Delta\to 0} W_b + W_s^l = \phi_l - s_l$ holds due to Theorem 4.1, we have

$$\lim_{\Delta\to 0} W_s^l > 0.$$

One can rewrite (3.11) as

$$W_s^l = \zeta \left[ \frac{\beta A(\pi_s)^2}{1-\beta\delta} + \frac{\beta\pi_p}{1-\beta\delta} \mathsf{E}[p - s_l - W_s^l | \Pi_p] \right]$$

Since $\pi_s$ and $\pi_p$ vanish at the rate of $\Delta$, the term inside of the bracket is uniformly bounded. Since the left hand side is uniformly bounded away from 0, $\lim_{\Delta\to 0} \zeta > 0$ must hold.

## 5. Extensions

5.1. **Free entry.** We have treated $x_b > 0$ as an exogenous parameter. Let us examine the case where buyers enter the market after paying $F^* > 0$ for one period before posting a vacancy. Since a buyer will enter the market only if the long run expected average payoff can recover the fixed cost,

$$-(1-\beta)F^* + \beta W_b = 0 \tag{5.25}$$

must hold in any equilibrium. Re-arranging the terms, (5.25) becomes

$$W_b = \frac{(1-\beta)F^*}{\beta}.$$

We obtain (4.24), and can invoke the same analysis as for the case where (4.19) holds to show

$$\lim_{\Delta\to 0} z_s = \frac{1}{1+Q_p} + \frac{1}{1+Q_p+Q_s} > 0.$$

The limit value of $z_b$ is affected by $x_b$, which is determined by (5.25):

$$\lim_{\Delta\to 0} z_b = \begin{cases} \left[\dfrac{1}{1+Q_p} + \dfrac{1}{1+Q_s+Q_p}\right]\sqrt{\dfrac{F}{F^*}} & \text{if } F < F^*; \\ \left[\dfrac{1}{1+Q_p} + \dfrac{1}{1+Q_s+Q_p}\right]\left[\dfrac{F}{F^*}\right] & \text{if } F > F^*. \end{cases}$$

5.2. **Revelation of quality.** To simplify notation, we have assumed so far that the true quality of the good is not revealed until the existing long term relationship is dissolved. In order to understand how the information revelation affects the equilibrium outcome, suppose that a buyer and a seller are in the long term relationship, who have agreed to deliver one unit of the good from the seller to the buyer at price $p$. After the good is delivered to the buyer, the true quality is revealed with probability $1 - \lambda = 1 - e^{-\Delta\theta}$ ($\theta > 0$). Based upon the available information about the good, if any, the buyer and the seller decide whether to continue the long term relationship or not. If both agents decide to continue the long term relationship, then the two agents remain in the same relationship with probability $\delta = e^{-\Delta d}$. Even if both agents choose to stay in the long term relationship, with probability $1 - \delta$, the relationship is dissolved immediately, and the two agents return to the respective pools. If either agent decides to terminate the long term relationship, then the relationship is dissolved immediately and the two agents return to the respective pools. The rest of the rules of the game remain the same.

An important implication of the new information is that the buyer has an option to terminate the long term relationship, if he discovers the quality is low, and to continue the relationship, if the quality is high. While the new information allows the buyer to get rid of low quality goods, the ensuing analysis reveals that as long as the lemon's problem is severe, the results in the previous section are carried over.

Since the new information arrives in each period with a positive probability, however, we need to modify assumption $A3$ accordingly:

$A3'$. The lemons problem is severe in the sense that

$$\frac{\phi_h + \phi_l}{2} + \frac{1}{2}\frac{\theta}{b+d}\phi_h < s_h.$$

The first term of the left hand side is the average quality of the good when the good is purchased. After the good is purchased, the true quality is revealed with probability $1 - e^{-\Delta\theta}$, while the agent discounts the future payoff at the rate of $e^{-\Delta b}$, and the long term relationship lasts with a probability of $e^{-\Delta d}$. After the true quality is revealed, only the high quality good will be kept, which make up one half of the goods purchased by the buyer. The second term is the expected average discounted quality, conditioned on the event that the quality is revealed, and only the high quality good is kept.

Purchasing a good has an option value of observing the true quality, in addition to consuming the average quality. A tedious calculation shows that if price $p$ is sufficiently high so that both high and low quality sellers agree to sell the good, the buyer accepts $p$ when

$$\tilde{\phi}^e - p \geq W_b$$

where

$$\tilde{\phi}^e = \frac{\mu_l \phi_l + (1-\mu_l)\phi_h + \dfrac{\beta(1-\lambda)(1-\mu_l)}{1-\beta\delta}\phi_h}{1 + \dfrac{\beta(1-\lambda)(1-\mu_l)}{1-\beta\delta}}.$$

Define

$$\Pi_p = [s_h + W_s^h, \tilde{\phi}^e - W_b] \qquad \text{and} \qquad \Pi_s = [s_l + W_s^l, \phi_l - W_b].$$

Then, we can calculate the value of each type of the agent conditioned on the event that he is in the pool:

$$W_s^h = \frac{\zeta\beta\pi_p \mathsf{E}(p - s_h - W_s^h \mid \Pi_p)}{1-\beta+\beta\lambda(1-\delta)},$$

$$W_s^l = \frac{\zeta\beta\pi_s \mathsf{E}(p - s_l - W_s^l \mid \Pi_s)}{1-\beta\lambda\delta} + \frac{\xi\beta\pi_p \mathsf{E}(p - s_l - W_s^l \mid \Pi_p)}{1-\beta\lambda\delta},$$

and

$$W_b = \frac{\xi\beta\mu_l\pi_s}{1-\beta\delta}\mathsf{E}(\phi_l - p - W_b \mid \Pi_s) + \frac{\zeta\beta\pi_p}{1-\beta\lambda\delta}\left(1 + \frac{\beta(1-\lambda)(1-\delta)}{1-\beta\delta}\right)\mathsf{E}(\tilde{\phi}^e - p - W_b \mid \Pi_p).$$

These values can be rewritten in a form more convenient for the analysis if the price is drawn from a uniform distribution:

$$W_s^h = A\frac{\zeta\beta\pi_p^2}{1-\beta+\beta\lambda(1-\delta)},$$

$$W_s^l = A\frac{\zeta\beta\pi_s^2}{1-\beta\lambda\delta} + \frac{\zeta\beta\pi_p \mathsf{E}(p - s_l - W_s^l \mid \Pi_p)}{1-\beta\lambda\delta},$$

and

$$W_b = A\frac{\xi\beta\mu_l\pi_s^2}{1-\beta\delta} + A\frac{\xi\beta\pi_p^2}{1-\beta\lambda\delta}\left(1 + \frac{\beta(1-\lambda)(1-\delta)}{1-\beta\delta}\right).$$

Along with the balance equations, we can solve for the equilibrium outcome $(z_b, z_s^h, z_s^l; W_b, W_s^h, W_s^l)$. We are interested in the case where

$$\lim_{\Delta\to 0} z_b > 0.$$

If

$$\lim_{\Delta\to 0} z_b > 0,$$

then $\lim_{\Delta\to 0} W_b = 0$ and $\lim_{\Delta\to 0} W_s^h = 0$ imply

$$\lim_{\Delta\to 0} \tilde{\phi}^e - s_h = 0.$$

Thus, we have

$$\lim_{\Delta\to 0} \mu_l = \frac{\left(1 + \frac{\theta}{b+d}\right)\phi_h - s_h}{\left(1 + \frac{\theta}{b+d}\right)\phi_h - \phi_l}.$$

We need to modify (4.19) accordingly:

$$\frac{\left(1+\frac{\theta}{b+d}\right)\phi_h-\phi_l}{s_h-s_l+\dfrac{b}{d}(\phi_l-s_l)}-(2-x_b)>0$$

which is a sufficient and necessary condition for

$$\lim_{\Delta\to 0} W_b = 0.$$

## 6. Concluding Remarks

This paper examines a dynamic matching model with adverse selection (Akerlof (1970) and Burdett and Wright (1998)) to see whether or not the market almost clears if search friction is small. We identify adverse selection as a fundamental source of the coexistence of unemployment and vacancy other than search friction and coordination failure caused by directed search.

Vacancy and unemployment are important objects of investigation in the labor market search models. A typical labor market search model (e.g., Mortensen and Pissarides (1994)) assumes a matching function $m(u, v)$ which specifies the rate at which unemployed workers ($u$) are matched to vacant positions ($v$). Indeed, Blanchard and Diamond (1989) pointed out that the matching function itself presumes the coexistence of a positive amount of unemployment ($u$) and a positive amount of vacancy ($v$). We have demonstrated that if the labor market is subject to adverse selection, then the equilibrium outcome can entail the coexistence of vacancy and unemployment, even in the limit as search friction vanishes.

We chose the random proposal model as a bargaining protocol mainly for the analytic convenience. The preliminary investigation reveals that the main conclusion of this paper is robust against the details of the bargaining protocols. In Cho and Matsui (2013a), for example, we demonstrate that the result is carried over to the model with a bargaining protocol in which the buyer makes the ultimatum offer in each period to the seller.

## Appendix A. Proof of Theorem 4.1

Define $O(\Delta)$ as a function that vanishes at the rate of $\Delta$:

$$\lim_{\Delta\to 0} \frac{O(\Delta)}{\Delta} < \infty.$$

**Lemma A.1.** $\lim_{\Delta\to 0}(\pi_p)^2 \leq O(\Delta)$

*Proof.* The second term of the buyer's value function and $W_b < \infty$ imply the statement. □

**Lemma A.2.** $\lim_{\Delta\to 0} \frac{\zeta\pi_p}{1-\beta\delta} < \infty$.

*Proof.* Suppose $\lim_{\Delta\to 0} \frac{\zeta\pi_p}{1-\beta\delta} = \infty$. Since $\lim_{\Delta\to 0} W_s^l < \infty$,

$$\frac{\zeta\pi_p}{1-\beta\delta}\mathsf{E}(p - W_s^l - s_l \mid \Pi_p) < \infty.$$

Under the hypothesis of the proof,

$$\lim_{\Delta\to 0} \mathsf{E}(p - W_s^l - s_l \mid \Pi_p) = 0.$$

Since $\pi_p > 0$ and $\lim_{\Delta\to 0} \pi_p = 0$,

$$0 < \phi^e - W_b - s_l - W_s^l \to 0.$$

Recall

$$\phi_l < s_h.$$

Thus,

$$\phi_l - W_b < s_h + W_s^h$$

and the gap between the left and the right hand sides does not vanish as $\Delta \to 0$. Since $\pi_s > 0$,

$$s_l + W_s^l < \phi_l - W_b < s_h + W_s^h < \phi^e - W_b$$

while

$$\phi^e - W_b - s_l - W_s^l \to 0.$$

This is a contradiction. □

Based upon these two observations, we conclude that the high quality seller's equilibrium payoff vanishes as $\Delta \to 0$, which proves the first part of Theorem 4.1.

**Lemma A.3.** $\lim_{\Delta\to 0} W_s^h = 0$.

*Proof.* Apply Lemmata A.1 and A.2 to $W_s^h$. □

Since $\pi_s > 0$, an $s_l$ seller and a buyer trades with a positive probability, which imposes an upper bound on $W_s^l + W_b$.

**Lemma A.4.** $W_s^l + W_b < \phi_l - s_l$.

*Proof.* A direct implication of $\pi_s > 0$. □

The next lemma shows that the low quality seller cannot be completely sorted out in a semi-pooling equilibrium, even in the limit as $\Delta \to 0$. As the pool contains a non-negligible portion of low quality sellers, the buyer needs to sort out the sellers, which is costly for the buyer and for the society as a whole, even if the friction vanishes. On the other hand, the low quality seller has an option to imitate the high quality seller, which provides significant bargaining power to a low quality seller when she is matched to a buyer.

**Lemma A.5.** $\lim_{\Delta\to 0} \mu_l > 0$.

*Proof.* Suppose $\lim_{\Delta\to 0}\mu_l = 0$. Then $\lim_{\Delta\to 0}\phi(\mu_l) = \phi_h$ holds. Thus, from (3.14), Lemmata A.3 and A.4 together with $W_s^l \geq 0$, we have

$$\lim_{\Delta\to 0}\pi_p = \lim_{\Delta\to 0} C[\phi_h - s_h - W_b - W_s^h] \geq C[(\phi_h - s_h) - (\phi_l - s_l)] > 0,$$

which contradicts with Lemma A.1. □

As in Lemma A.2, we can compute the rate at which $\zeta\pi_s$ vanishes.

**Lemma A.6.** $\lim_{\Delta\to 0}\frac{\zeta\pi_s}{1-\beta\delta} < \infty$.

*Proof.* Suppose $\lim_{\Delta\to 0}\frac{\zeta\pi_s}{1-\beta\delta} = \infty$. Then from Lemma A.2 and the balance equations of the sellers, $\lim_{\Delta\to 0}\mu_l = 0$ holds, which contradicts to Lemma A.5. □

The next lemma is the seller's counterpart of Lemma A.1.

**Lemma A.7.** $\lim_{\Delta\to 0}\pi_s \leq O(\Delta)$.

*Proof.* This statement is directly implied by Lemma A.5 and (3.12). □

A corollary of Lemma A.7 is that the sum of the long run average payoffs of a buyer and $s_l$ seller converges to $\phi_l - s_l$, which proves the second part of Theorem 4.1.

**Lemma A.8.** $\lim_{\Delta\to 0} W_s^l + W_b = \phi_l - s_l$.

*Proof.* From Lemma A.7 together with (3.13), we have

$$\lim_{\Delta\to 0}\pi_s = \lim_{\Delta\to 0} C[(\phi_l - s_l) - (W_b + W_s^l)] = 0.$$

□

## Appendix B. Proof of Lemma 4.2

From (3.15), (3.16) and (3.17), we know that in order to investigate the asymptotic properties of $z_b$ and $z_s$, we need to understand the asymptotic properties of $\zeta\pi_p/(1-\delta)$ and $\zeta\pi_s/(1-\delta)$.

**Lemma B.1.** $\lim_{\Delta\to 0}\frac{\zeta\pi_s}{1-\beta\delta} > 0$

*Proof.* Suppose that $\lim_{\Delta\to 0}\frac{\zeta\pi_s}{1-\beta\delta} = 0$. From the balance equations of the sellers, we have

$$\frac{\mu_l}{1-\mu_l} = \frac{\frac{\pi_p\zeta}{1-\delta}+1}{\frac{\pi_s\zeta}{1-\delta}+\frac{\pi_p\zeta}{1-\delta}+1} \to 1$$

which implies that

$$\mu_l \to \frac{1}{2}.$$

Since the lemons problem is severe (assumption $A3$),

$$\phi(\mu_l) - s_h \to \frac{\phi_h+\phi_l}{2} - s_h < 0.$$

Recall that $W_s^h \to 0$. Since any equilibrium must be semi-pooling, $\pi_p > 0$. For a sufficiently small $\Delta > 0$, however,

$$0 < \phi(\mu_l) - W_b - W_s^h - s_h \leq \phi(\mu_l) - s_h \to \frac{\phi_h+\phi_l}{2} - s_h < 0$$

which is impossible. □

**Lemma B.2.**

$$0 < \lim_{\Delta\to 0}\frac{\pi_s}{\pi_p} < \infty.$$

*Proof.* Since we have

$$0 < \lim_{\Delta\to 0} \frac{\pi_s \zeta}{1-\delta} < \infty,$$

by way of Lemmata A.6 and B.1 , and

$$\lim_{\Delta\to 0} \frac{\pi_p \zeta}{1-\delta} < \infty,$$

by way of Lemma A.2,

$$\lim_{\Delta\to 0} \frac{\pi_p}{\pi_s} < \infty.$$

holds. To prove

$$\lim_{\Delta\to 0} \frac{\pi_p}{\pi_s} > 0$$

by way of contradiction, suppose that

$$\lim_{\Delta\to 0} \frac{\pi_p}{\pi_s} = 0.$$

Since

$$0 < \lim_{\Delta\to 0} \frac{\pi_s \zeta}{1-\delta} < \infty,$$

$$\lim_{\Delta\to 0} \frac{\pi_p}{\pi_s} = 0$$

implies

$$\lim_{\Delta\to 0} \frac{\pi_p \zeta}{1-\delta} = 0.$$

We claim that $\zeta \to 0$ as $\Delta \to 0$ under the hypothesis of the proof. If

$$\lim_{\Delta\to 0} \zeta > 0,$$

then $\pi_s = O(\Delta)$ and $\pi_p = O(\Delta)$. As a result,

$$\lim_{\Delta\to 0} W_s^l = \lim_{\Delta\to 0} W_b = 0,$$

which is impossible since

$$W_b + W_s^l \to \phi_l - s_l.$$

□

**Lemma B.3.** $\lim_{\Delta\to 0} \frac{\zeta\pi_p}{1-\beta\delta} > 0$

*Proof.* Note

$$\lim_{\Delta\to 0} \frac{\zeta\pi_p}{1-\beta\delta} = \lim_{\Delta\to 0} \frac{\zeta\pi_s}{1-\beta\delta}\frac{\pi_p}{\pi_s}.$$

The desired conclusion follows from Lemmata B.1 and B.2. □

**Lemma B.4.** $\lim_{\Delta\to 0} \mathsf{E}[p|\Pi_p] = s_h$.

*Proof.* Since $\lim_{\Delta\to 0} \pi_p = 0$, $\Pi_p = [s_h + W_s^h, \phi^e(p) - W_b]$ shrinks to a single point. Since $\lim_{\Delta\to 0} W_s^h = 0$, all points in $\Pi_p$ converge to $s_h$, from which the conclusion follows. □

**Lemma B.5.** $\lim_{\Delta\to 0} W_s^l > 0$

*Proof.* Recall the equilibrium value function of $W_s^l$, and observe that the second term of the value function is strictly positive, even in the limit as $\Delta \to 0$. □

We are ready to prove Lemma 4.2. Note

$$\frac{W_s^l}{W_b} = \frac{A\zeta\pi_s^2 + \zeta\pi_p \mathsf{E}[p - s_l - W_s^l|\Pi_p]}{A\left(\mu_l\pi_s^2 + \pi_p^2\right)}.$$

Thus,

$$\frac{\mu_l W_s^l}{\zeta W_b} \propto \frac{\mu_l\zeta\pi_s^2 + \mu_l\zeta\pi_p \mathsf{E}(p - s_l - W_s^l|\Pi_p)}{\mu_l\zeta\pi_s^2 + \zeta\pi_p^2} = \frac{\mu_l\pi_s\frac{\pi_s}{\pi_p} + \mu_l\mathsf{E}(p - s_l - W_s^l|\Pi_p)}{\mu_l\pi_s\frac{\pi_s}{\pi_p} + \pi_p}. \tag{B.26}$$

The denominator converges to zero by way of Lemmata A.1, A.7, and B.2, while the numerator converges to a value greater than or equal to $\mu_l(s_h - \phi_l) > 0$ due to Lemma B.4 and $\lim_{\Delta\to 0} W_s^l \leq \phi_l - s_l$. Therefore, since $\lim_{\Delta\to 0} \mu_l W_s^l > 0$, $\zeta W_b \to 0$.

## Appendix C. Proof of Lemma 4.4

Suppose $\lim_{\Delta\to 0} z_b > 0$. Then Lemma 4.2 implies $\lim_{\Delta\to 0} W_b = 0$, which in turn implies $\lim_{\Delta\to 0} W_s = \phi_l - s_l$ due to Theorem 4.1. We derive (4.21) from $W_s^l$ by using the fact that the first term converges to zero, and Lemma B.4. As for (4.22), note that $\mu_l = z_s^l/z_s$. Taking the limit of this expression and equating it with $\lim_{\Delta\to 0} \mu_l = \frac{\phi_h - s_h}{\phi_h - \phi_l}$, we derive (4.22).

## Appendix D. Proof of Proposition 4.5

Suppose $\lim_{\Delta\to 0} z_b = 0$.

(1) follows from the fact that $2 - z_s = x_b - z_b$.

(2) Note that $\zeta \to 0$ if and only if $z_b \to 0$. Lemma B.1 and Lemma B.3 imply that $\frac{\pi_p}{1-\delta} \to \infty$ and $\frac{\pi_s}{1-\delta} \to \infty$ as $\Delta \to 0$.

(3) To simplify notation, let us write

$$\begin{aligned} \bar{\mu} &= \lim_{\Delta\to 0} \mu_l = \frac{\phi_h - s_h}{\phi_h - \phi_l} \\ \bar{Q}_s &= \lim_{\Delta\to 0} \frac{\zeta\pi_s}{1-\delta} \\ \bar{Q}_p &= \lim_{\Delta\to 0} \frac{\zeta\pi_p}{1-\delta}. \end{aligned}$$

Under the assumption that $\zeta \to 0$, one can derive from the balance equations that

$$\frac{x_b}{2 - x_b} = \bar{\mu}\bar{Q}_s + \bar{Q}_p$$

and

$$\frac{\bar{\mu}}{1 - \bar{\mu}} = \frac{\bar{Q}_p + 1}{\bar{Q}_s + \bar{Q}_p + 1}.$$

From the value function of $s_l$ seller, one can show that

$$\lim_{\Delta\to 0} W_s^l = \frac{\frac{d}{b+d}\bar{Q}_p(s_h - s_l)}{1 + \frac{d}{b+d}\bar{Q}_p}.$$

Since

$$W_s^l + W_b \to \phi_l - s_l,$$

$\lim_{\Delta\to 0} W_b > 0$ if and only if

$$\frac{\frac{d}{b+d}\bar{Q}_p(s_h - s_l)}{1 + \frac{d}{b+d}\bar{Q}_p} < \phi_l - s_l.$$

We know that if $\bar{Q}_p = Q_p$, then

$$\frac{\frac{d}{b+d}\bar{Q}_p(s_h - s_l)}{1 + \frac{d}{b+d}\bar{Q}_p} = \phi_l - s_l.$$

Thus, $\lim_{\Delta\to 0} W_b > 0$ if and only if $\bar{Q}_p < Q_p$. One can show that $\bar{Q}_p$ solves

$$\bar{Q}_p + 1 = \left(1 + \frac{d}{d+b}\bar{Q}_p\right)\left(\frac{\phi_h - \phi_l}{s_h - s_l}\frac{1}{2 - x_b}\right),$$

where we use the balance equations, $\lim_{\Delta\to 0} W_s^l + W_b = \phi_l - s_l$, and

$$\bar{\mu}\phi_l + (1-\bar{\mu})\phi_h = s_h + \lim_{\Delta\to 0} W_b.$$

Note that $\bar{Q}_p \leq Q_p$ if and only if (4.19) is violated, and the equality holds only if (4.19) is violated with an equality.

(4) follows from the last part of the proof of (3).

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