

Optimal Voting Rules*

Alex Gershkov, Benny Moldovanu and Xianwen Shi[†]

September 3, 2014

Abstract

We modify the successive voting procedure, predominantly used by European legislatures, by including flexible majority requirements for each of several alternatives. This modification allows us to replicate the outcome of any anonymous, unanimous and dominant strategy incentive compatible mechanism. We use this equivalence to compute the optimal (utilitarian) procedure and its associated majority requirements.

*A previous version of this paper has been circulated under the title “Optimal Mechanism Design without Money”.

[†]We are grateful to Felix Bierbrauer, Andreas Klein, Ilya Segal, Martin Schmalz and Tymon Tatur for several very helpful remarks, and to Tilman Börgers, Deniz Dizdar, Frank Rosar, Gabor Virag and various seminar participants for their insightful comments. Gershkov wishes to thank the Israel Science Foundation, the German-Israeli Foundation, the Google Inter-University Center for Electronic Markets and Auctions, and the Maurice Falk Institute for Economic Research for financial support. Moldovanu wishes to thank the German Science Foundation, the German-Israeli Foundation and the European Research Council for financial support. Shi is grateful to the Social Sciences and Humanities Council of Canada for financial support. Gershkov: Department of Economics, Hebrew University of Jerusalem, Israel and School of Economics, University of Surrey, UK, alexg@huji.ac.il; Moldovanu: Department of Economics, University of Bonn, Germany, mold@uni-bonn.de; Shi: Department of Economics, University of Toronto, Canada, xianwen.shi@utoronto.ca.

1 Introduction

The use of the utilitarian principle as a guide for collective decision making or policy goes back to the birth of Economics. This principle uses cardinal information about preferences (and hence about preference intensities) and evaluates social outcomes in terms of the sum of agents' expected utilities, or, equivalently, in terms of average individual expected utility. When monetary transfers are feasible, the maximization of average utility is also a prerequisite for another classical desideratum, Pareto-efficiency.

In practice, however, many collective decisions are taken through simpler mechanisms that only extract ordinal information about the ranking of alternatives, and that do not allow monetary transfers among agents even if these would be feasible (e.g., various voting schemes within committees and legislatures). The lack of monetary transfers both makes it impossible to extract refined information about preference intensities, and also weakens the implications of Pareto efficiency: the set of Pareto efficient allocations can be very large, and it is not clear how to choose among efficient rules while only using the ordinal information obtained via voting. Nevertheless, although information about preferences intensities is not extracted in each instance, available statistical information about preferences can be used to guide the choice of an appropriate mechanism. Thus, an important issue is the evaluation of voting schemes (or other mechanisms whose rules do not take into account cardinal information) in terms of the ex-ante expected utility they generate, and the identification of optimal procedures for given preferences and their distribution in the population.

In this paper we derive the ex-ante welfare maximizing (i.e., utilitarian) mechanism for settings with an arbitrary number of social alternatives where privately informed agents have single-crossing preferences and where monetary transfers are not feasible. Our analysis takes into account the strategic incentives that agents face in such situations. We also show that the optimal outcome can be implemented in practical applications by a variation of the successive voting scheme, a well-known voting procedure that is predominantly used in most European parliaments, including the parliament of the European Union.

As Rasch [2000] documents in Table 1 below, when several alternatives are involved, European legislatures use successive voting:¹ alternatives are brought to the ballot in a pre-specified order, and at each step an alternative is either adopted (and voting stops), or eliminated from further consideration (and the next alternative is considered).

¹This is to be contrasted with amendment voting, predominantly used in the Anglo-Saxon world. In this procedure voting occurs over pairs of alternatives. Apestegua and Ballester [2013] offer a parallel axiomatic characterization of both procedures.

Successive Procedure (Alternatives voted “one-by-one”)	Amendment (Elimination) Procedure (Alternatives voted “two-by-two”)
Austria, Belgium, Denmark France, Germany, Greece Iceland, Ireland, Spain Italy, Luxembourg, Netherlands Norway, Portugal, Czech Republic Hungary, Poland, Slovakia, Slovenia (European Parliament)	Finland, Sweden, Switzerland United Kingdom (Canada), (USA)

Table 1: Parliamentary Floor Voting Procedures in Europe (Rasch, 2000)

While in most applications adoption occurs if the number of “Yes” votes exceeds a pre-defined and fixed threshold, the needed modification for our purposes involves an adoption threshold that depends on the respective alternative. Flexible thresholds, as required by our optimal mechanism, have been, for example, advocated with a clear utilitarian rationale in mind by Gersbach and Pachtl [2009] for the European Central Bank: the size of the required majority should depend monotonically on the proposed change in interest rate within a period of time. In this way, small shocks affecting only a few countries can be readily accommodated, while radical changes that affect the entire Euro area should only be implemented if they command a broad support.

Non-neutral adoption rules that explicitly treat alternatives differently are already common in practice.² Here are several examples:

1. The Austrian, Belgian, Finnish, Dutch, German and US constitutions can be changed only if two-thirds of the parliaments’ members are in favor, while the status quo is preserved otherwise.³ Three-fifths requirements for constitutional changes are used in France, Greece and Spain.
2. Policy changes in the European Union must garner the support of at least 55% of the member states if proposed by the EU Commission itself, but a 72% threshold is required otherwise.
3. In their well-known studies of the relation between profitability and corporate governance, Gompers et al. [2003] and Bebchuk et al. [2009] find that about a third of the

²Note that all common sequential voting procedures are per-se non neutral. The order of voting influences the result.

³The two-thirds requirement probably goes back to the rules for choosing a new pope, devised by Pope Alexander III in 1179. Although only unanimity was thought to reveal the will of God, Pope Pius II summarized his own election in 1458: “What is done by two thirds of the sacred college, that is surely of the Holy Ghost, which may not be resisted” (in Gragg and Gabel, 1959: 88).

firms in their large data sets use various super-majority requirements when shareholders vote about special issues such as mergers.

In spite of the fact that a qualified (or super-) majority rule still uses only ordinal information about preferences, the selective application of such schemes points to an utilitarian reasoning that weighs potential cardinal gains and losses (from a reform, say) against each other. Our results can be seen as generalizing this intuition and its theoretical and practical implications to several alternatives.

We first look at the successive voting procedure with a decreasing adoption threshold, and identify a very simple and robust (ex-post perfect and Markov) Nash equilibrium where, at each stage, agents use sincere strategies. That is, agents vote “Yes” if the current alternative is their preferred one in the remaining set of alternatives under consideration, and “No” otherwise. Our main results are:

1. We show that, by varying the threshold requirement in successive voting, we can replicate the outcome of **any** anonymous, unanimous and dominant strategy incentive compatible (DIC) mechanism. Conversely, for any successive voting scheme with decreasing thresholds, there is an anonymous, unanimous and DIC mechanism that generates the same equilibrium outcome.
2. We explicitly compute, under some standard assumptions on preferences and their distribution, the adoption thresholds that maximize ex-ante expected welfare. In other words, we derive the incentive compatible optimal mechanism (second-best).

Although the first-best utilitarian rule is not implementable in our setting, the second-best rule obtained here approximates the first-best if the population is large: with large populations, the decision is shifted from the alternative preferred by the median voter in the direction of the alternative preferred by the average voter.

It is remarkable that the implementation of our static dominant strategy mechanisms via a dynamic voting procedure parallels to some extent the well-known dynamic implementation (in settings with monetary transfers) of the static Vickrey auction: Ausubel [2004] constructs an ascending auction procedure where it is an ex-post perfect Nash equilibrium for bidders to use sincere demand revelation strategies at each stage, and where the efficient Vickrey outcome is implemented.

Our technical analysis uses a characterization result (Saporiti [2009]) for DIC mechanisms in frameworks with single-crossing preferences. In turn, Saporiti’s result builds on a classical “converse” to the *Median Voter Theorem* due to Moulin [1980]. Roughly speaking, Moulin’s result says that, on the full domain of single-peaked preferences, all DIC, Pareto efficient and anonymous mechanisms can be described as *generalized median* schemes that choose the median among the n real peaks of actual voters and an additional $n - 1$ fixed “phantom”

voters' peaks.⁴ To understand the logic of our results, and the relation to implementation via successive voting, let $m(k)$ be the number of phantom voters with peaks to the left of, and including alternative k in a generalized median mechanism. Note that, by construction, this function is increasing. The outcome of the median mechanism with such phantom location can be replicated by the sincere equilibrium of the successive voting procedure where the adoption threshold for alternative k is given by $\tau(k) = n - m(k)$.⁵ The optimization task is of combinatorial nature, to determine the appropriate threshold function τ as a function of the agents' preferences and their distribution. Just to give an example, the June 1991 successive voting procedure that determined the new capital of the reunited Germany involved 658 members of parliament and 4 alternatives.⁶ This yields 47 698 420 different anonymous, unanimous and incentive compatible mechanisms among which we look for the optimal one.

The rest of the paper is organized as follows. In the remainder of this Section we review the related literature. In Section 2 we describe the social choice model and the design problem. In Subsection 2.1 we illustrate the model and some implications of incentive compatibility in the simple special case where utilities are linear. In particular, we show that the welfare-maximizing rule (first-best) is not implementable although it is monotone. In Section 3 we first introduce a variation of the successive voting procedure and derive an ex-post Nash equilibrium where voters vote sincerely. Next, we prove that, for any unanimous and anonymous DIC mechanism, there exists a successive voting procedure with decreasing majority requirements that generates the same outcome, and vice versa. In Section 4 we use this equivalence result to derive the precise decreasing sequence of the majority thresholds associated with the DIC mechanism that maximizes the ex ante welfare. In Section 5 we also discuss extensions to other welfare criteria. Section 6 concludes. All omitted proofs are in Appendix A. Appendix B discusses in more detail the regularity conditions used in the characterization of optimal mechanisms.

Related Literature

A very large body of work in the realm of social choice has focused on the implementation of desirable social choice rules in abstract frameworks with purely ordinal preferences, and without monetary transfers. Classical results include the Gibbard-Satterthwaite *Impossibility Theorem* (Gibbard [1973] and Satterthwaite [1975]) and the *Median Voter Theorem* for settings with single-peaked preferences (see Black [1948]). When a Pareto-efficient rule, say, is not implementable in a certain framework, that literature often remains silent about how

⁴Saporiti is able to remove Moulin's assumption that the mechanisms only depend on peaks, while obtaining a result in the same spirit that holds for maximal domains of single-crossing preferences.

⁵Although they do not refer to the successive voting procedure and its equilibria, our representation argument is inspired by Barbera, Gul and Stacchetti's [1993] interpretation of generalized median mechanisms in terms of *coalitional systems*. See also the survey by Barbera [2001].

⁶Besides simple alternatives such as Bonn and Berlin, there were composite ones that involved different locations of parliament and government.

to choose among implementable schemes because preference intensities are not part of the model, and because other goals are not easily formulated within it. For similar reasons, when multiple Pareto-efficient rules are implementable, this literature does not offer tools to meaningfully ranking them.

The idea of comparing voting rules in terms of the ex-ante expected utility they generate goes back to Rae [1969]. That paper and almost the entire following literature focus on settings with two social alternatives (a reform and a status quo, say). Schmitz and Tröger [2012] identify qualified majority rules as ex-ante welfare maximizing in the class of DIC mechanisms—this can be seen as an implication of our main result.⁷ Azrieli and Kim [2014] nicely complement this analysis for two alternatives by showing that any interim Pareto efficient, Bayesian incentive compatible (BIC) choice rule must be a qualified majority rule.⁸ The situation dramatically changes when there are three, or more alternatives: the DIC/BIC constraints and the mechanisms themselves are much more numerous and complex.

Apestequia, Ballester and Ferrer [2011] consider a general social choice model where agents derive cardinal utility from several alternatives, and evaluate mechanisms in terms of the ex-ante expected utility they generate.⁹ Their analysis abstracts from incentives constraints: strategic voting is not considered—this would lead there to impossibility results—and the scoring rules that emerge as optimal in their analysis are known to be subject to strategic manipulation.

Börgers and Postl [2009] study a setting with three alternatives: in their model it is common knowledge that the top alternative for one agent is the bottom for the other, and vice versa. The agents also differ in the relative intensity of their preferences for a middle alternative (the *compromise*) when compared to the top and bottom one, respectively. This intensity is private information. In addition to a characterization of BIC mechanisms in terms of monotonicity and an envelope condition, Börgers and Postl conduct numerical simulations and show that the efficiency loss from second-best rules is often small.¹⁰

In a principal-agent model with quadratic utility functions, hidden information but without transfers, Kovac and Mylovanov [2009] find that the optimal mechanism is deterministic. Rosar [2012] looks at a setting with a continuum of alternatives, quadratic utilities and interdependent preferences, and compares two particular aggregation mechanisms, the median and the average, respectively.

Motivated by computer science applications, Hartline and Roughgarden [2008] study how the system designer can use service degradation (money burning) to align the private users' interests with the social objective. Chakravarty and Kaplan [2013] and Condorelli [2012] analyze optimal allocation problems in private value environments without monetary transfers.

⁷These authors also perform an analysis for Bayesian mechanisms, which is not covered by our study.

⁸Again in a setting with two alternatives, Barbera and Jackson [2006] take the qualified majority rule as given, and derive the optimal weight that maximizes the total expected utilities of all agents.

⁹They also consider other goals such as maxmin, etc.

¹⁰See also McLean and Postlewaite [2002] who study Bayesian incentive compatibility in settings where monetary transfers are limited.

In their models agents can send costly and socially wasteful signals (these may be payments to outsiders).

In contrast to the above papers, Drexl and Klein [2013] allow the redistribution of the collected monetary payments among the agents. They confine attention to settings with two social alternatives and show that a principal who wishes to maximize the agents' welfare (i.e., welfare from the physical allocation minus potential transfers to outsiders) will use a mechanism that does not involve any monetary transfers! Hence, their paper offers a powerful argument for the use of voting schemes. In particular, it must be the case that, for settings with two alternatives, their optimal mechanism coincides with the one derived in this paper, where monetary transfers are a-priori ruled out.

A quite different line of study is pursued by Jackson and Sonnenschein [2007] who consider the linkage of many distinct social problems. Even if no monetary transfers within one problem are possible, the linkage with other decisions creates the possibility of fine-tuning incentives, which acts as having some "pseudo-transfers". Efficiency can be attained then in the limit, where the number of considered problems grows without bound.

As already mentioned above, the seminal paper in the social choice literature closely related to the present research is Moulin [1980]. Several authors have extended Moulin's characterization by discarding the assumption that mechanisms can only depend on peaks.¹¹ Excellent examples are Barbera, Gul and Stacchetti [1993], Barbera and Jackson [1994], Sprumont [1991], Ching [1997], Schummer and Vohra [2007], and Chatterjee and Sen [2011].¹² Moulin's characterization does not necessarily hold on sub-domains of single peaked-preferences. Since the ubiquitous and simplest cardinal informational model used here—with one-dimensional private types—cannot generate the full domain of single peaked preferences, we focus on maximal domains of single crossing preferences where Moulin's characterization was shown to hold by Saporiti [2009].

A very well-known application of a social choice framework with single-crossing preferences is to voting over tax schedules—see for example, Roberts [1977], Romer [1975], and Meltzer and Richard [1981]. Persson and Tabellini [2000] (Chapter 6) survey this literature and use such a framework as the starting point to study voting for redistributive programs such as pensions, unemployment insurance, assistance to the poor, and labor market regulations. They assume that there are only two parties who each suggests policy platforms. Hence, although the taxation application can have infinite number of alternatives, it reduces there to the case of two alternatives. Under simple majority voting, the decisive voter is the one with the median type. Given that the real income distribution typically has a mean above median, the government "size" that results from majority voting in this model is too

¹¹See Sprumont [1995] and Barbera [2001] for excellent surveys. Recently, Ehlers, Peters and Storcken [2002] extend Moulin's characterization to probabilistic strategy-proof rules, and Nehring and Puppe [2007] extend it to a class of generalized single-peaked preference domains based on abstract betweenness relations.

¹²Schummer and Vohra [2002] and Dokow et al. [2012] also establish the equivalence between strategy-proof rules and generalized median voter schemes. In both models, however, agents' preferences are quadratic and thus parameterized solely by their peaks. Hence, they can directly focus on peaks-only mechanisms.

large compared to the social optimal one.

Our framework may be useful for the analysis of multi-party elections. For any number of alternatives, our optimal mechanism tailors the voting rule to correct (to some extent) the difference between the mean and the median.

2 The Social Choice Model

We consider n agents who have to choose one out of K mutually exclusive alternatives. Let $\mathcal{K} = \{1, \dots, K\}$ denote the set of alternatives. Agent $i \in \{1, \dots, n\}$ has (cardinal) utility $u^k(x_i)$, where $k \in \mathcal{K}$ is the chosen alternative and where x_i is a parameter (or type) privately known to agent i only. We assume that for any k , $u^k(x_i)$ is bounded. The types x_1, \dots, x_n are distributed on the interval $[0, 1]^n$ according to a commonly known, joint cumulative distribution function Φ with density ϕ having full support.¹³ This is the one-dimensional, private values specification, the most common one in the vast literature on optimal mechanism design with monetary transfers that followed Myerson's (1981) seminal contribution. But monetary transfers are not feasible in our framework.

Given any two alternatives k and l with $k < l$, let $x^{k,l}$ denote the cutoff type that is indifferent between them:¹⁴

$$u^l(x^{k,l}) = u^k(x^{k,l}). \quad (1)$$

To simplify notation, we denote $x^k \equiv x^{k-1,k}$. We assume that utilities are **single-crossing** with respect to the order of alternatives $1, \dots, K$: for any two alternatives k and l with $k < l$ it holds that

$$\begin{cases} u^k(x_i) > u^l(x_i) & \text{if } x_i < x^{k,l} \\ u^k(x_i) < u^l(x_i) & \text{if } x_i > x^{k,l} \end{cases} . \quad (2)$$

We further assume that each alternative is the top alternative for some type of the agents.¹⁵ That is, for any $k \in \mathcal{K}$, there exists $x_i \in [0, 1]$ such that

$$u^k(x_i) > \max_{l \in \mathcal{K}, l \neq k} u^l(x_i). \quad (3)$$

We shall primarily focus on the case of a utilitarian planner whose objective is to maximize the sum of the agents' expected utilities

$$\max_{k \in \mathcal{K}} E \left[\sum_i u^k(x_i) \right].$$

¹³Here agents' types can be correlated. In Section 4 we shall assume independence between the agents' types.

¹⁴We assume that for any two alternatives k and l , the indifference types $x^{k,l}$ are uniquely defined and different from each other. These assumptions are generic.

¹⁵This assumption ensures that our single-crossing preferences are also single-peaked (see Remark 1 below). However, it rules out the setting of Borgers and Postl [2009] where the third alternative, *compromise*, is not the top alternative of any agent.

Remark 1 *The single-crossing property (2), together with assumption (3), implies that the cutoffs are well-ordered:*

$$0 \equiv x^1 < \dots < x^K < x^{K+1} \equiv 1. \quad (4)$$

To see this, we note that, by definition of x^k and the single-crossing property (2),

$$u^k(x_i) < u^{k-1}(x_i) \text{ for all } x_i < x^k.$$

Similarly, by definition of x^{k+1} and the single-crossing property (2), we have

$$u^k(x_i) < u^{k+1}(x_i) \text{ for all } x_i > x^{k+1}.$$

If $x^k \geq x^{k+1}$, any type x_i satisfies either $x_i \leq x^k$ or $x_i \geq x^{k+1}$, and thus alternative k is (weakly) dominated either by alternative $k-1$ or by alternative $k+1$, which contradicts (3). Therefore, we must have $x^k < x^{k+1}$ for all $k \in \mathcal{K}$, which proves (4). By the definition of x^k and by (4), agents with type x_i have k as their top alternative if and only if $x_i \in [x^k, x^{k+1}]$.

Note also that the agents' preferences are single-peaked. To see this, consider agent i with type $x_i \in (x^k, x^{k+1})$. By definition of x^k , agent i prefers alternative k to any alternative $l < k$, and by definition of x^{k+1} , agent i prefers k over any $l > k$. Consider two alternatives l and m with $l < m < k$. Since $x^l < x^m < x^k$, we have $x_i > x^{l,m}$ and agent i prefers m to l . Similarly, agent i prefers m to l if $k < m < l$. Therefore, agent i 's preferences are single-peaked. On the other hand, our preference domain is a strict subset of the full single-peaked preference domain whenever $K > 2$: not all single-peaked preferences are compatible with our environment (see below an explicit illustration in the linear environment).

In a deterministic direct mechanisms agents report their types, and, for any profile of reports, the mechanism chooses one alternative from \mathcal{K} . Formally, a deterministic direct mechanism is a function $g : [0, 1]^n \rightarrow \mathcal{K} = \{1, \dots, K\}$. A deterministic mechanism is *dominant strategy incentive compatible* (DIC) if for any player i and for any x_i, x'_i and x_{-i} :

$$u^{g(x_i, x_{-i})}(x_i) \geq u^{g(x'_i, x_{-i})}(x_i). \quad (5)$$

It is clear from the above definition that two types that have the same ordinal preferences must be treated in the same way by a DIC mechanism. Thus, an implication of the lack of monetary transfers is that deterministic DIC mechanisms cannot depend on preferences intensities.

2.1 An Illustration: Linear Preferences

Suppose the utilities are linear: $u^k(x_i) = a_k + b_k x_i$. These preferences are necessarily single crossing. The linear specification can be used to model important applications such as voting over linear income taxation schedules.¹⁶

¹⁶An analysis of how majority voting affects linear taxation and the size of the government can be found in Persson and Tabellini [2000] (Chapter 6, page 118-121), which, in turn, follows Romer [1975], Roberts [1977], and Meltzer and Richard [1981].

We also assume that $b_k \geq 0$ for all $k \in \mathcal{K}$ and $b_k \neq b_l$ for all $l \neq k$. Without loss of generality (by renaming alternatives if necessary), we assume that $b_K > b_{K-1} > \dots > b_1 \geq 0$ and that $a_1 > a_2 > \dots > a_K$. The cutoff type who is indifferent between two adjacent alternatives k and $k-1$ is given by

$$x^k \equiv x^{k-1,k} = \frac{a_{k-1} - a_k}{b_k - b_{k-1}}. \quad (6)$$

We impose further restrictions on b_k and a_k so that our previous assumption (3) is satisfied. As a result, cutoffs are ordered according to (4). These restrictions, together with the definition of $x^{l,k}$, imply that $x^{l,k} \in (x^{l+1}, x^k)$ for $k > l+1$, because

$$x^{l,k} = \frac{a_l - a_k}{b_k - b_l} = \frac{(a_l - a_{l+1}) + \dots + (a_{k-1} - a_k)}{(b_{l+1} - b_l) + \dots + (b_k - b_{k-1})}.$$

As a result, the cutoffs $x^{l,k}$ are pairwise different.

Our preference domain is a strict subset of the full single-peaked preference domain. Indeed, consider a setting with 4 different alternatives (1, 2, 3 and 4), and suppose that it holds that $x^{1,4} \in (x^{2,3}, x^{3,4})$, as shown in Figure 1. The feasible single-peaked preferences that have alternative 2 on their top are $2 \succ 1 \succ 3 \succ 4$ and $2 \succ 3 \succ 1 \succ 4$. In particular, the preference $2 \succ 3 \succ 4 \succ 1$ is not compatible with the linear environment. Similarly, if $x^{1,4} \in (x^{1,2}, x^{2,3})$, the feasible single-peaked preferences that have alternative 3 on their top are $3 \succ 2 \succ 4 \succ 1$ and $3 \succ 4 \succ 2 \succ 1$. Here the preference profile $3 \succ 2 \succ 1 \succ 4$ is not compatible with our structure.

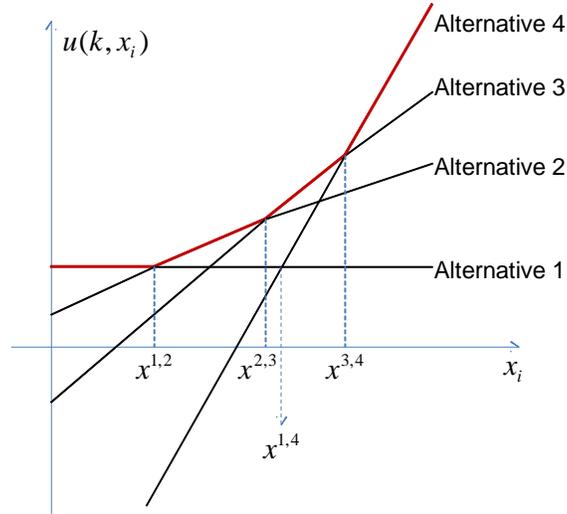


Figure 1: Not all single-peaked preferences are compatible with our linear structure.

Analogously to the classical framework with monetary transfers, a mechanism $g(x_i, x_{-i})$ is DIC if and only if (i) for all x_{-i} and for all i , $g(x_i, x_{-i})$ is increasing in x_i ; and (ii) for any agent i , any $x_i \in [0, 1]$ and $x_{-i} \in [0, 1]^{n-1}$, the following envelope condition holds (see for example Milgrom and Segal [2002]):

$$u^{g(x_i, x_{-i})}(x_i) = u^{g(0, x_{-i})}(0) + \int_0^{x_i} b_{g(z, x_{-i})} dz. \quad (7)$$

When monetary transfers are feasible, **any** monotone decision rule $g(x_i, x_{-i})$ is incentive compatible since it is always possible to augment it with a transfer such that the equality required by (7) holds. Thus, with transfers, only monotonicity really matters for DIC. If monetary transfers were available, the welfare-maximizing allocation would be implementable via the well-known Vickrey-Clarke-Groves mechanisms. But, without monetary transfers, not all monotone decision rules $g(x_i, x_{-i})$ are implementable, and in particular, the welfare maximizing allocation need not be incentive compatible although it is monotone. This phenomenon is illustrated in the next example.

Example 1 (First-best Rule Not Implementable) *Consider the linear environment with two alternatives $\{1, 2\}$ and with two agents $\{i, -i\}$. The designer is indifferent between alternatives 1 and 2 if*

$$2a_1 + b_1(x_i + x_{-i}) = 2a_2 + b_2(x_i + x_{-i}).$$

The first-best rule conditions on the value of the average type, and is given by

$$g(x_i, x_{-i}) = \begin{cases} 1 & \text{if } \frac{1}{2}(x_i + x_{-i}) \in [0, x^2] \\ 2 & \text{if } \frac{1}{2}(x_i + x_{-i}) \in [x^2, 1] \end{cases}$$

where cutoff x^2 is defined in (6): $x^2 \equiv (a_1 - a_2) / (b_2 - b_1)$. The first-best rule is increasing in both x_i and x_{-i} . But, for all $x_{-i} \in [0, 2x^2)$ and $x_i \in [2x^2 - x_{-i}, 1)$, we can rewrite the integral condition (7) as

$$a_2 + b_2 x_i = a_1 + \int_0^{2x^2 - x_{-i}} b_1 dz + \int_{2x^2 - x_{-i}}^{x_i} b_2 dz = a_1 + b_1(2x^2 - x_{-i}) + b_2(x_i - 2x^2 + x_{-i}),$$

which reduces to $x_{-i} = x^2$. Therefore, the integral condition is violated for all $x_{-i} \neq x^2$.

3 Implementation via Successive Voting

The successive voting procedure is the predominant parliamentary voting procedure over multiple alternatives in continental Europe, including the European parliament itself (see Rasch [2000]). In this procedure alternatives are first arranged in some pre-determined voting order, say $1, 2, \dots, K$. The first ballot determines whether there is a majority (often a simple majority) for alternative 1. If so, alternative 1 is adopted and voting ends. If alternative 1 fails to command a majority, this alternative is removed from future consideration, and the parliament proceeds to vote on alternative 2. If a majority supports alternative 2, alternative 2 is adopted; otherwise, the parliament proceeds to vote on alternative 3. Voting continues until one alternative gains majority. If no alternative gains majority in earlier stages, the last two alternatives $K - 1$ and K are paired and the one with majority support is adopted.

In most cases, the required majority for adoption is the same across alternatives, and the voting order is either suggested by the agenda setter or is pre-determined by custom, e.g., more extreme alternatives are voted on first. In general, the voting outcome is sensitive to

the voting order: if voters vote sincerely, later alternatives have better chance to be adopted (Black [1958]), but if voters vote strategically earlier alternatives are more likely to be adopted (Farquharson [1969]).¹⁷

In order to link the successive voting procedure to all unanimous, anonymous and DIC mechanisms, we modify this standard successive voting procedure in two ways. First, we order the alternatives according to the natural order $(1, 2, \dots, K)$ under which the preferences are single-peaked (see Remark 1).¹⁸ This is somewhat consistent with the observation that parliaments vote on extreme alternatives first. The second and more important modification is that the required majority for adoption is no longer constant across alternatives: instead, the required number of votes for choosing alternative k , $\tau(k)$, is a decreasing function. That is, a more stringent majority requirement (which may be more or less than simple majority) is set for earlier alternatives, while a lower majority is required for adopting later alternatives.

The results in this Section are ordinal, and thus do not depend on the particular cardinal specification of utility as long as the single-crossing assumption is satisfied and the domain of preferences is maximal (with respect to single-crossing).

Definition 1 1. A voting strategy for agent i is *sincere* if, at each stage, the agent votes in favor of the respective alternative if and only if it is the best (among the remaining alternatives) given his preferences.

2. A voting strategy for agent i is *monotone* if it consists of a series of “No” in early stages (possibly none), followed by a series of “Yes” in all later stages.

Note that, with single peaked preferences and with our natural voting order, a sincere strategy has a particular structure: the agent votes “No” for all alternatives that appear on the ballot before his most preferred one, and then votes “Yes” for his peak alternative and for all successive ones. Hence, under the successive voting rule with our voting order, monotone voting is a natural generalization of sincere voting.¹⁹

Proposition 1 Consider the successive voting procedure with a decreasing threshold function $\tau(k)$, and assume that all agents except i use monotone voting strategies. Then, the sincere voting strategy is optimal for agent i . In particular, the strategy profile where all agents vote sincerely constitutes an ex-post perfect Nash equilibrium.²⁰

Proof. Assume that all agents other than i use monotone strategies, and let the peak of agent i be on alternative k . Consider first an alternative $k' < k$. The sincere strategy calls

¹⁷See Jung [1990] for extension of the Black-Farquharson analysis of voting order.

¹⁸Alternatively, the successive voting procedure can be modified in an equivalent way if voting takes place in reverse order.

¹⁹Monotone strategies are also *Markov*, i.e., they do not condition on the history of votes before the current one.

²⁰The strategy n -tuple is said to constitute an ex-post perfect equilibrium if at every stage, and for every realization of private information, the n -tuple of continuation strategies constitutes a Nash equilibrium of the game in which the realization of the agents' types is common knowledge.

for i to vote against k' . A deviation from sincere voting matters only if, by changing his strategy from “No” to “Yes” at this stage, alternative k' is chosen whereas it would not be chosen if i voted sincerely. But in this case, the number of “Yes” votes for alternative k' must be $\tau(k') - 1$. Since τ is decreasing, and since all other agents use monotone strategies, voting “No” on alternative k' implies that, in this case, agent i can ensure that the chosen alternative k'' satisfies $k' < k'' \leq k$ (either voting stops before reaching k , or i ensures the choice of k by voting “Yes” on it, and then joining at least $\tau(k') - 1 \geq \tau(k) - 1$ “Yes” votes). All alternatives k'' with $k' < k'' \leq k$ are preferred by i to k' , hence this deviation from sincere voting is not beneficial. Consider now $k' > k$. The sincere strategy calls for i to vote “Yes” at the relevant stage. By deviating to “No”, the chosen alternative must satisfy $k'' \geq k'$. All these alternatives are dominated by k' from i 's point of view, so a deviation is again not beneficial. This completes the proof of optimality of a sincere strategy for agent i . Since the argument applies to all agents, and since voting is monotone, sincere voting constitutes an ex-post perfect Nash equilibrium. ■

We now uncover the connection between the outcome of any dominant strategy mechanisms and the sincere equilibria of the successive voting procedure with decreasing thresholds. We first need to characterize the set of incentive compatible mechanisms. An influential paper by Moulin [1980] shows that, if each agent is restricted to report his top alternative only, then every DIC, Pareto efficient and anonymous voting scheme on the full domain of single-peaked preferences is equivalent to a generalized median voter scheme that is obtained by adding $(n - 1)$ fixed peaks (phantoms) to the n voters' reported peaks and then choosing the median of this larger set of peaks. It turns out that Moulin's characterization also holds in our setting although agents are not restricted to report only their peaks, and although the domain of preferences is a strict subset of the full domain of single-peaked preferences. The relevant result is due to Saporiti [2009]: he provides a characterization of unanimous, anonymous and DIC mechanisms for maximal domains of single-crossing preferences, in a spirit similar to Moulin [1980].

We need several definitions:

- Definition 2**
1. A mechanism g is **unanimous** if $x_i \in (x^k, x^{k+1})$ for all i implies $g(x) = k$.
 2. A mechanism g is **Pareto efficient** if, for any profile of reports $(x_i, x_{-i}) \in [0, 1]^n$, there is no alternative $k \in \mathcal{K}$ such that $u^k(x_i) \geq u^{g(x)}(x_i)$ for all i , with strict inequality for at least one agent.
 3. A mechanism g is **anonymous** if, for any profile of reports $(x_i, x_{-i}) \in [0, 1]^n$, $g(x_1, \dots, x_n) = g(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ where σ denotes any permutation of the set $\{1, \dots, n\}$.

It is clear that a Pareto-efficient mechanism is unanimous. In the presence of dominant strategy incentive compatibility, an anonymous and unanimous mechanism is also Pareto efficient (Saporiti [2009]). We are now ready to state our first main result.

Theorem 1 1. For any unanimous and anonymous DIC mechanism g , there exists a decreasing threshold function $\tau^g(k)$ with $\tau^g(k) \leq n$ for any $k \in \mathcal{K}$ and $\tau^g(K) = 1$ such that, for any realization of types, the outcome of g coincides with the outcome in the sincere equilibrium of successive voting with thresholds $\tau^g(k)$.

2. Conversely, for any decreasing threshold function $\tau(k)$ with $\tau(k) \leq n$ for any $k \in \mathcal{K}$ and $\tau(K) = 1$, there exists an anonymous, unanimous and DIC mechanism g^τ such that, for any realization of types, the outcome of g^τ coincides with the outcome of the sincere equilibrium in successive voting with thresholds $\tau(k)$.

Proof. 1. Our underlying domain of ordinal preferences is **maximal** with respect to single-crossing, i.e., one cannot add to it another ordinal preference profile without violating the single crossing property (see Saporiti [2009] for a formal definition). To see this, note that every two alternatives, say k and l , induce a cutoff $x^{k,l}$, and each cutoff $x^{k,l}$ divides the set of types into two intervals where ordinal preferences differ with respect to the ordering of alternative k and l .²¹ Since each alternative is top for some types, the interval of types is thus partitioned into $K(K-1)/2+1$ parts, each corresponding to a distinct ordinal preference. But this is also the maximum number of ordinal profiles in a maximal domain of single-crossing preferences on K alternatives.

Saporiti [2009] shows that, on a maximal domain of (ordinal) single-crossing preferences any anonymous, unanimous and DIC mechanism in an environment with n voters can be obtained as a generalized median voter mechanism with $n-1$ phantom voters.²² In the dominant strategy equilibrium of such a generalized median voter scheme, all n voters truthfully report their top alternatives, and the outcome is the median of the n real peaks and the $n-1$ fixed phantom peaks.

Let $l_k \geq 0$ denote the number of phantom voters with peak on alternative k in the generalized median voter scheme corresponding to a DIC mechanism g . To construct an equivalent successive voting scheme define the thresholds $\tau^g(k) \equiv n - \sum_{m=1}^k l_m$, and note that $\tau^g(k)$ is decreasing, and that $\tau^g(K) = 1$.

Alternative 1 is the generalized median only if the number of (real) agents who report this alternative as their top alternative exceeds $n - l_1 = \tau(1)$. Alternative 2 is the generalized median if the number of agents who report a peak on alternative 1 is less than $n - l_1$ and if the number of the agents who report either alternative 1 or alternative 2 as their top alternative is at least $n - l_1 - l_2 = \tau(2)$. In general, alternative k is the generalized median if, for any $k' \leq k$, the number of reported peaks on alternatives $1, 2, \dots, k'$ was strictly less than $\tau^g(k')$ and if the number of agents who report their peak on alternative k or lower is at least $n - \sum_{m=1}^k l_m = \tau^g(k)$. Otherwise, alternative K is the generalized median. With this interpretation, it is now clear that the outcome of the sincere equilibrium under successive voting with threshold $\tau^g(k)$ coincides with the outcome of mechanism g .

²¹Recall the (generic) assumption that all $x^{k,l}$ are distinct.

²²See Theorem 3 in the Appendix A for a formal statement of Saporiti's characterization.

2. Conversely, for a given successive voting procedure with decreasing cutoffs $\tau(k)$ such that $\tau(k) \leq n$ for any $k \in \mathcal{K}$ and $\tau(K) = 1$, we can define $l_1 \equiv n - \tau(1)$, and $l_k \equiv \tau(k-1) - \tau(k)$ for $k \geq 2$. Since $\tau(k)$ is decreasing, $\tau(k) \leq n$ for all $k \in \mathcal{K}$ and $\tau(K) = 1$, we have $l_k \geq 0$ for all $k \in \mathcal{K}$ and $\sum_{k=1}^K l_k = n - \tau(K) = n - 1$. The constructed phantom distribution $\{l_k\}$ is part of a generalized median voter scheme which corresponds to some unanimous and anonymous DIC mechanism g^τ . Moreover, it is easy to verify that the outcome of mechanism g^τ is the same as the sincere equilibrium outcome in successive voting with threshold $\tau(k)$. ■

The above theorem simplifies the problem of finding optimal mechanisms. Nevertheless, it should be clear that, with many agents and alternatives, this remains a rather complex discrete optimization problem since the number of feasible decreasing sequence of cutoffs $\tau(k)$, is quite large. The number of DIC, anonymous and unanimous mechanisms for n agents and K alternatives is²³

$$\frac{(n + K - 2)!}{(K - 1)!(n - 1)!}.$$

In the next section we solve the optimization problem under some regularity conditions on the distribution of types and utility functions.

4 The Optimal Mechanism

We now characterize the welfare maximizing allocations that respect the incentive constraints (constrained efficiency, or “second-best”). Following the mechanism design literature, we shall primarily focus on the utilitarian welfare criterion: the social planner wants to maximize the sum of the agents’ expected utilities. Given the equivalence result in Theorem 1, the outcome of any unanimous, anonymous and DIC mechanism can be implemented by a modified successive voting procedure with decreasing thresholds:

$$\tau(k) = n - \sum_{m=1}^k l_m, \tag{8}$$

where l_m is the number of phantom voters with peak on alternative m in the generalized median voter scheme representing the DIC mechanism. Therefore, the task of searching for the optimal mechanism is reduced to that of finding the optimal function $\tau(k)$, or equivalently, the optimal distribution $\{l_k\}_{k=1}^K$ of $(n - 1)$ phantom voters among K alternatives.

We now introduce several assumptions that put some more structure on the optimization problem, allowing us to analytically solve it.

Assumption A *The agents’ signals are distributed identically and independently of each other on the interval $[0, 1]$ according to a cumulative distribution F with density f .*

²³The problem is to partition $(n - 1)$ phantoms into K alternatives, which can be represented by $(K - 1)$ bars placed among $n - 1$ balls. Hence, it is equivalent to choosing $(K - 1)$ out of $(n + K - 2)$ positions to place $(K - 1)$ bars.

This assumption yields the standard symmetric, independent private values model (SIPV) widely used in the literature on trading mechanisms with transfers. We need another assumption that combines requirements on the utility functions and on the distribution of types. Before introducing it, we need some notation. Let us define, for all $k \geq 2$,

$$u_{x < x^k}^k = E \left[u^k(x) \mid x < x^k \right]$$

as the expected utility from alternative k , conditional on the agent's type x being lower than the cutoff x^k . Similarly, we define

$$u_{x > x^k}^k = E \left[u^k(x) \mid x > x^k \right]$$

as the expected utility from alternative k conditional on the agent's type x being higher than the cutoff x^k . Note that, with single crossing, the entire (convex) interval of types below (above) x^k prefer alternative $k - 1$ to k (alternative k to $k - 1$). Finally, let us define the function $\Gamma(k)$ with $k \geq 2$ as

$$\Gamma(k) = \frac{u_{x < x^k}^{k-1} - u_{x < x^k}^k}{u_{x > x^k}^k - u_{x > x^k}^{k-1}}.$$

That is, $\Gamma(k)$ is the ratio of the difference of the “lower” conditional expected utilities for two adjacent alternatives $k - 1$ and k over the difference of the “upper” conditional expected utilities for the same adjacent alternatives. By the definition of cutoff x^k and by the single-crossing property, $u_{x < x^k}^{k-1} > u_{x < x^k}^k$ and $u_{x > x^k}^k > u_{x > x^k}^{k-1}$. Therefore, $\Gamma(k) > 0$ for all $k \geq 2$.

Assumption B *The function Γ is increasing.*

In Appendix B we derive sufficient conditions on the primitives of the social choice model (utility functions and the distribution of types) for the above assumption to hold. For $k \geq 2$, define also

$$\gamma(k) = \frac{\left(u_{x < x^k}^{k-1} - u_{x < x^k}^k \right)}{\left(u_{x < x^k}^{k-1} - u_{x < x^k}^k \right) + \left(u_{x > x^k}^k - u_{x > x^k}^{k-1} \right)} = \frac{1}{1 + 1/\Gamma(k)},$$

and note that, by Assumption B, γ is also increasing.

Consider now an environment with n voters, and let $\tau(k)$ be the threshold of alternative k in successive voting. Our analysis is based on a simple observation: if $\tau(k)$ is part of the optimal voting procedure and $\tau(k - 1) > \tau(k)$, then increasing $\tau(k)$ by 1 or decreasing $\tau(k - 1)$ by 1 should weakly reduce the total expected utility.²⁴ For instance, increasing $\tau(k)$ by 1 (while keeping $\tau(k')$ with $k' \neq k$ unchanged) has an impact only if it changes the

²⁴Increasing $\tau(k)$ by 1 while keeping other cutoffs unchanged corresponds to moving one phantom voter from alternative k to alternative $k + 1$ in the generalized median voter scheme. Similarly, decreasing $\tau(k - 1)$ by 1 while keeping other cutoffs unchanged is equivalent to shifting one phantom from alternative k to alternative $k - 1$. The proposed deviation is feasible only if $\tau(k - 1) > \tau(k)$ since $\tau(k)$ has to be (weakly) decreasing in k . It turns out that the two derived bounds (9) and (10) remain valid even if such deviation is not feasible. See Lemma 1 in Appendix A.

chosen alternative. The proposed deviation will change the chosen alternative, however, only if there are exactly $\tau(k)$ voters with values below x^{k+1} . These kind of arguments generate the following bounds on the threshold function $\tau(k)$:

$$\tau(k-1) \leq n - n\gamma(k) + 1, \text{ for all } k \geq 2, \quad (9)$$

$$\tau(k) \geq n - n\gamma(k+1), \text{ for all } k \leq K-1. \quad (10)$$

Since $\tau(k)$ has to be integer, the above two bounds lead to an essentially unique threshold function.²⁵ Assumption B guarantees that the optimal threshold $\tau^*(k)$ is decreasing, which is required by DIC.

Theorem 2 *Let $\lceil z \rceil$ denote the largest integer that is below z . Under Assumptions A and B, the sincere equilibrium of successive voting with the threshold function*

$$\tau^*(k) = \begin{cases} n - \lceil n\gamma(k+1) \rceil & \text{if } k < K \\ 1 & \text{if } k = K \end{cases}$$

implements the optimal anonymous and unanimous DIC mechanism.

Proof. See Appendix A. ■

The above theorem reveals that adding or eliminating an alternative has only a local effect. That is, adding an alternative k_1 such that an interval $[x^k, x^{k+1}]$ is further divided into $[x^k, x^{k_1}]$ and $[x^{k_1}, x^{k+1}]$ changes only the threshold of alternative k and $k+1$. Similarly, if we eliminate alternative k , it will change the threshold of alternatives $k-1$ and $k+1$ only, without any effect on the other alternatives. This “locality-effect” follows from the single-peaked preferences: the social planner does not want to change the chosen alternative if the peak of the median voter does not change as a result of adding/eliminating the available alternatives.

If there are only two alternatives, Theorem 2 specifies the optimal qualified majority rule (or supermajority). Here are two examples: 1) The choice of $\tau(1) = n$ corresponds to an unanimity rule for adopting alternative 1; 2) For n odd, the choice of $\tau(1) = (n+1)/2$ corresponds to the simple majority rule, and such a rule is optimal in symmetric situations. More generally, each optimal rule is a qualified majority rule, where the bias in favor of one alternative depends on the ratio of expected relative losses suffered in each situation by those whose preferred alternative was not chosen. The following corollary characterizes the optimal voting rule in the case of two alternatives (see Nehring [2004], Barbera and Jackson [2006] and Schmitz and Tröger [2012] for related results). Note that Assumption B is not needed for this result.

Corollary 1 *Suppose there are n agents and only two alternatives, $K = 2$. Under Assumption A, the optimal rule is implemented through a voting game in which alternative 1 is chosen if and only if at least $\tau(1) = n - \lceil n\gamma(2) \rceil$ voters voted in its favour.*

²⁵Several solutions are possible if $n\gamma$ takes integer values.

It is important to note that the single-crossing (and single-peaked) structure allowed us here, as in the standard setting where transfers are allowed, to formulate both a simple cardinal model with one-dimensional private information and a concise regularity assumption (the monotonicity of the function Γ) that yield simple, intuitive formulae for the optimal schemes. Without this structure, say by assuming single-peaked preferences alone, the monotonicity of Γ could not be translated to meaningful conditions on distribution of agents' types and utilities.

Remark 2 *When the number of voters is large, the optimal (second-best) mechanism approximates the welfare maximizing mechanism (first-best) which, as illustrated in Example 1, is not implementable in our setting. In other words, our optimal mechanism corrects for the difference in the alternatives preferred by the (real) median voter and the one yielding the highest average welfare. This result is relatively intuitive since the aggregate uncertainty vanishes in the limit.²⁶ For a simple proof, consider the mechanism that chooses, for any reports, either the unanimous top alternative if such an alternative exists, or the fixed, ex-ante welfare maximizing alternative. It is easy to see that this mechanism is both unanimous and DIC, and hence it must be welfare inferior to the optimal mechanism derived above. It is also clear that the welfare attained by this mechanism converges to the first-best when the population size tends to infinity since the welfare maximizing alternative is chosen with a probability that converges to unity.²⁷*

4.1 The Linear Case

We illustrate our characterization of optimal mechanisms in the linear environment set out in Section 2.1. For this environment, we introduce a simpler assumption to replace Assumption B. Let X be the random variable representing the agents' type. We first define two functions, $C(x)$ and $c(x)$, as follows:

$$C(x) = E[X|X > x] \text{ and } c(x) = E[X|X \leq x].$$

Assumption B' *The functions $x - C(x)$ and $x - c(x)$ are increasing.*

In Appendix B we offer sufficient conditions on the distribution of types for Assumption B' to hold. These are related to ubiquitous conditions on hazard rates, well-known from the theory of optimal mechanism design with quasi-linear utility and monetary transfers. In the linear environment, the function $\gamma(k)$ becomes

$$\gamma(k) = \frac{x^k - c(x^k)}{C(x^k) - c(x^k)}.$$

²⁶A result in the same spirit for settings with only two alternatives has been obtained by Ledyard and Palfrey [2002].

²⁷We wish to thank Tymon Tatur for pointing to us this simple argument.

Under Assumption B', the function

$$\frac{C(x^k) - c(x^k)}{x^k - c(x^k)} = 1 - \frac{x^k - C(x^k)}{x^k - c(x^k)}$$

is decreasing in x^k . It follows that $\gamma(k)$ is increasing since

$$\gamma(k+1) = \frac{x^{k+1} - c(x^{k+1})}{C(x^{k+1}) - c(x^{k+1})} \geq \frac{x^k - c(x^k)}{C(x^k) - c(x^k)} = \gamma(k).$$

Therefore, we obtain the following corollary to Theorem 2.

Corollary 2 *Suppose utilities are linear and Assumptions A and B' hold. The sincere equilibrium of successive voting with the threshold function*

$$\tau^*(k) = \begin{cases} n - \lceil n\gamma(k+1) \rceil & \text{if } k < K \\ 1 & \text{if } k = K \end{cases}$$

implements the optimal anonymous and unanimous DIC mechanism.

This corollary yields immediate and intuitive comparative statics with respect to parameters of the linear utility function $\{a_k, b_k\}_{k=1}^K$. By the definition of the cutoffs x^k , increases in either a_k or b_k decrease x^k and increase x^{k+1} , which in turn leads to a lower threshold for adopting alternative k or higher alternatives. That is, if the attractiveness of any alternative increases, the chances of adopting that or higher alternative increase as well.

Our next proposition shows how the optimal threshold $\tau(k)$ changes with respect to the distribution of types. It uses the following well known stochastic orders (see Shaked and Shanthikumar [2007]).

Definition 3 1. *A random variable Y dominates a random variable X in the hazard rate order (denoted as $X \leq_{hr} Y$) if $[X|X > x] \leq_{st} [Y|Y > x]$ for all x .*

2. *A random variable Y dominates a random variable X in the reverse hazard rate order (denoted as $X \leq_{rh} Y$) if $[X|X < x] \leq_{st} [Y|Y < x]$ for all x .*

3. *A random variable Y dominates a random variable X in the likelihood ratio order (denoted as $X \leq_{lr} Y$) if $[X|a \leq X \leq b] \leq_{st} [Y|a \leq Y \leq b]$ for all $a < b$.*

It is clear from the above definitions that $X \leq_{lr} Y$ implies both $X \leq_{hr} Y$ and $X \leq_{rh} Y$.

Proposition 2 *Consider two type distributions F and \tilde{F} . Let X and \tilde{X} be the random variables representing agent types associated with distribution F and \tilde{F} , respectively. Assume that $X \leq_{hr} \tilde{X}$ and $X \leq_{rh} \tilde{X}$. Let $\tilde{\tau}(k)$ and $\tau(k)$ be the optimal threshold under \tilde{X} and X , respectively. Then, for any $k \in \{1, \dots, K\}$, $\tilde{\tau}(k) \geq \tau(k)$.*

Proof. See Appendix A. ■

This proposition indicates that, if the type distribution shifts up in the likelihood ratio order, the optimal mechanism displays a stronger bias against lower alternatives. Intuitively, as the distribution becomes more skewed towards the right, a utilitarian planner wants to bias in favor of higher alternatives that are likely to be preferred by more agents.

Example 2 Suppose that the distribution of types F is uniform on $[0, 1]$. Then $C(x) = (1 + x)/2$, $c(x) = x/2$, and $\gamma(k) = x^k$. Therefore, the optimal threshold function is given by: $\tau^*(k) = n - \lceil nx^{k+1} \rceil$. Intuitively, the threshold for adopting alternative k is proportional to the share of real types whose top alternative is k or lower.

We can further illustrate the above corollary by describing in more detail the optimal voting rules when there are two agents.

Corollary 3 Suppose there are only two agents. Under Assumptions A and B', the optimal mechanism is characterized by just one integer: the alternative at which the adoption threshold changes from two to one. This alternative is given by

$$k^* \equiv \min \left\{ k \in \mathcal{K} : x^{k+1} \geq \frac{1}{2} \left[C(x^{k+1}) + c(x^{k+1}) \right] \right\}.$$

Proof. See Appendix A. ■

Note that the condition for determining the optimal shifting point can be rewritten as $k^* = k$ if $x^* \in [x^k, x^{k+1}]$, where $x^* = \frac{1}{2} [C(x^*) + c(x^*)]$. The critical value x^* has the same distance to the upper conditional mean $C(x^*)$ as to the lower conditional mean $c(x^*)$. In particular, if the distribution is symmetric, then x^* coincides with the mean of the distribution.

5 Other Objective Functions

Other, non-utilitarian, objective functions can be considered as well, assuming that utilities $u^k(x)$ are increasing in x for all $k \in \{1, \dots, K\}$.²⁸ For example, if the designer's preferences are *maximin*, then the desired allocation is

$$g^{\min}(x_1, \dots, x_n) = k^m$$

where k^m satisfies $x^m \in (x^{k^m}, x^{k^m+1}]$ with $x^m = \min\{x_1, \dots, x_n\}$. That is, k^m is the most preferred alternative of the agent with the lowest signal. In such a case, $\tau(k) = 1$ for all $k \in \mathcal{K}$.

²⁸If $u^k(x)$ is not increasing in x for all k , implementation in maximin or maximax may not be possible. To see this, suppose there are three players, three alternatives $k \in \{1, 2, 3\}$, and types are drawn from the interval $[1, 3]$. Players have quadratic preferences $u^k(x) = -(x - k)^2$. Consider the following realized type profile: $x_1 = x_2 = 1$ and $x_3 = 1.8$. Under truthful reporting, the maximin choice $k^m = 1$. But this is not incentive compatible, because player 3 can report \hat{x}_3 slightly above 2, say $\hat{x}_3 = 2.1$, and induce maximin choice $k^m = 2$ which is player 3's best alternative.

Similarly, if the designer’s preferences are *maximax*, then the designer would like to implement allocation

$$g^{\max}(x_1, \dots, x_n) = k^M$$

where k^M satisfies $x^M \in (x^{k^M}, x^{k^M+1}]$ with $x^M = \max\{x_1, \dots, x_n\}$. That is, k^M is the most preferred alternative of the agent with the highest signal. In such a case, $\tau(k) = n$ for all $k < K$ and $\tau(K) = 1$.

6 Concluding Remarks

We have characterized constrained efficient (i.e., second-best) dominant strategy incentive compatible and deterministic mechanisms in a setting where privately informed agents have single-crossing utility functions, but where monetary transfers are not feasible. Our approach allows a systematic choice among Pareto-efficient mechanisms based on the ex-ante utility they generate. We have also shown that the optimal mechanism can be implemented by a modification of a widely used voting procedure. This modification is an extension to several alternatives of the idea behind qualified majorities (or supermajorities) that are also widely used for binary decisions. In practice, one could use flexible thresholds in a simplified way (e.g., by using only one switching point from a high threshold to a low one) instead of changing the required threshold for each alternative. Such schemes are already welfare superior to those using a fixed threshold.

The implementation of static Pareto efficient and dominant strategy mechanisms via a dynamic voting procedure parallels the dynamic implementation (in settings with monetary transfers) of the static Vickrey auction.

An open question is whether random mechanisms can yield an improvement over the deterministic mechanisms studied in this paper. The answer would be clearly negative if one could show that any probabilistic, DIC and anonymous mechanism is a lottery over deterministic, DIC and anonymous mechanisms. Peters et al. [2014] prove exactly that on single-peaked domains satisfying a minimally richness condition. But, their result is not immediately applicable here, mainly because their incentive compatibility concept is ordinal: a deviation from truth-telling must be disadvantageous for any cardinal utility representation of the ordinal single peaked preferences; thus, their concept is stronger than the incentive compatibility concept for a specific and given cardinal utility function, and it potentially excludes more mechanisms.

Another open question is whether using the more permissible Bayesian incentive compatibility concept can improve the performance of constrained efficient mechanisms.²⁹ It is instructing to note that in the standard setting with independent types, linear utility and **with** monetary transfers, a general welfare equivalence result between dominant strategy

²⁹It is well known that more or less sophisticated Bayesian mechanisms can be used to increase welfare when types are correlated.

incentive compatible and Bayes-Nash incentive compatible mechanisms has been established by Gershkov et al. [2013].

7 Appendix A: Proofs

The formal statement of the main theorem in Saporiti [2009] we use to prove Theorem 1 is the following.

Theorem 3 (Saporiti, 2009) *An unanimous, anonymous mechanism g is DIC if and only if there exists $(n - 1)$ numbers $\alpha_1, \dots, \alpha_{n-1} \in \mathcal{K}$ such that for any type profile $(x_1, \dots, x_n) \in [0, 1]^n$ with $x_i \in (x^{k_i}, x^{k_i+1}]$ for all i , it holds that*

$$g(x_1, \dots, x_n) = M(\alpha_1, \dots, \alpha_{n-1}, k_1, \dots, k_n),$$

where the function $M(\alpha_1, \dots, \alpha_{n-1}, k_1, \dots, k_n)$ returns the median of $(\alpha_1, \dots, \alpha_{n-1}, k_1, \dots, k_n)$.

Proof of Theorem 2. Consider the optimal mechanism and its equivalent representation as a generalized median with phantom voters. Let $k \geq 2$, and suppose that $l_k > 0$ is part of the optimal allocation of $(n - 1)$ phantoms. By optimality, the social planner must prefer this allocation of phantoms over allocating $l_k - 1$ phantoms on alternative k and $l_{k-1} + 1$ phantoms on alternative $k - 1$. This change matters only if it affects the median among $n - 1$ phantom and n real voters. For this to happen, it must be that the total number of voters (“real” and “phantom”) with values below x^k is $(n - 1)$: there are exactly $(n - 1 - \sum_{m=1}^{k-1} l_m)$ “real” voters with values below x^k and $(\sum_{m=1}^{k-1} l_m + 1)$ “real” voters with values above x^k . In this case, by moving a phantom from alternative k to alternative $k - 1$, the planner changes the median from k to $k - 1$. In this case, the total expected utility from alternative k is given by

$$\left(n - 1 - \sum_{m=1}^{k-1} l_m\right) u_{x < x^k}^k + \left(\sum_{m=1}^{k-1} l_m + 1\right) u_{x > x^k}^k.$$

The total expected utility from alternative $k - 1$ is given by

$$\left(n - 1 - \sum_{m=1}^{k-1} l_m\right) u_{x < x^k}^{k-1} + \left(\sum_{m=1}^{k-1} l_m + 1\right) u_{x > x^k}^{k-1}.$$

Since the planner (weakly) prefers k to $k - 1$, the total expected utility from alternative k must be higher than the total expected utility from alternative $k - 1$. This gives us the following “first-order condition” for all $k \geq 2$ with $l_k > 0$:

$$\left(n - 1 - \sum_{m=1}^{k-1} l_m\right) \left(u_{x < x^k}^k - u_{x < x^k}^{k-1}\right) + \left(\sum_{m=1}^{k-1} l_m + 1\right) \left(u_{x > x^k}^k - u_{x > x^k}^{k-1}\right) \geq 0 \quad (11)$$

Similarly, if $l_k > 0$ with $k \leq K - 1$ is part of the optimal allocation of $(n - 1)$ phantoms, then the social planner must prefer this allocation of phantoms to allocating $l_k - 1$ phantoms

on alternative k and $l_{k+1} + 1$ phantoms on alternative $k + 1$. This change matters only if it affects the median among $n - 1$ phantom and n real voters. For this to happen, it must be that the total number of voters (“real” and “phantom”) with values below x^{k+1} is n . In other words, there are exactly $\left(n - \sum_{m=1}^k l_m\right)$ “real” voters with values below x^{k+1} and $\sum_{m=1}^{k-1} l_m$ “real” voters with values above x^{k+1} . In this case, the total expected utility from alternative k is given by

$$\left(n - \sum_{m=1}^k l_m\right) u_{x < x^{k+1}}^k + \left(\sum_{m=1}^k l_m\right) u_{x > x^{k+1}}^k.$$

The total expected utility from alternative $k + 1$ is given by

$$\left(n - \sum_{m=1}^k l_m\right) u_{x < x^{k+1}}^{k+1} + \left(\sum_{m=1}^k l_m\right) u_{x > x^{k+1}}^{k+1}.$$

This yields another “first-order condition” for all $k \leq K - 1$ with $l_k > 0$:

$$\left(n - \sum_{m=1}^k l_m\right) \left(u_{x < x^{k+1}}^k - u_{x < x^{k+1}}^{k+1}\right) + \left(\sum_{m=1}^k l_m\right) \left(u_{x > x^{k+1}}^k - u_{x > x^{k+1}}^{k+1}\right) \geq 0. \quad (12)$$

These two first-order conditions can be rewritten as bounds on phantom distributions for alternatives k with $l_k > 0$. These were inequalities (9) and (10) from the main text, which are equivalent to

$$\sum_{m=1}^{k-1} l_m \geq n\gamma(k) - 1, \text{ for all } k \geq 2, \quad (13)$$

$$\sum_{m=1}^k l_m \leq n\gamma(k+1), \text{ for all } k \leq K - 1. \quad (14)$$

We can use the definition of $\gamma(k)$ to rewrite them as

$$\begin{aligned} \sum_{m=1}^{k-1} l_m &\geq \frac{n \left(u_{x < x^k}^{k-1} - u_{x < x^k}^k\right)}{\left(u_{x < x^k}^{k-1} - u_{x < x^k}^k\right) + \left(u_{x > x^k}^k - u_{x > x^k}^{k-1}\right)} - 1 = n\gamma(k) - 1, \text{ for all } k \geq 2, \\ \sum_{m=1}^k l_m &\leq \frac{n \left(u_{x < x^{k+1}}^k - u_{x < x^{k+1}}^{k+1}\right)}{\left(u_{x < x^{k+1}}^k - u_{x < x^{k+1}}^{k+1}\right) + \left(u_{x > x^{k+1}}^{k+1} - u_{x > x^{k+1}}^k\right)} = n\gamma(k+1), \text{ for all } k \leq K - 1. \end{aligned}$$

Lemma 1 below shows that the above two conditions hold for alternative k with $l_k = 0$.

Therefore, we can construct the (generically unique) candidate distribution of phantom voters’ peaks as follows. We first derive bounds for l_1^* by taking $k = 2$ in (13) and $k = 1$ in (14):

$$n\gamma(2) - 1 \leq l_1^* \leq n\gamma(2).$$

Since the two bounds differ by 1 and l_1^* must be an integer, l_1^* is generically unique and must be equal to $\lceil n\gamma(2) \rceil$, where $\lceil z \rceil$ denotes the largest integer that is below z .

Next note that, for all $2 \leq k \leq K - 1$, conditions (13) and (14) imply that

$$n\gamma(k+1) - 1 \leq \sum_{m=1}^{k-1} l_m^* \leq n\gamma(k+1).$$

Hence, $\sum_{m=1}^k l_m^*$ is also generically unique and must be equal to $\lceil n\gamma(k+1) \rceil$. As a result, we can deduce l_2^* as

$$l_2^* = \sum_{m=1}^2 l_m^* - l_1^* = \lceil n\gamma(3) \rceil - \lceil n\gamma(2) \rceil.$$

Similarly, we can obtain recursively for all l_k^* with $2 \leq k \leq K - 1$:

$$l_k^* = \lceil n\gamma(k+1) \rceil - \lceil n\gamma(k) \rceil.$$

Since by Assumption B, $\gamma(k)$ is increasing in k . Hence, we obtain that $l_k^* \geq 0$.

Finally, since there are $(n - 1)$ phantom voters in total, we have

$$l_K^* = n - 1 - \sum_{m=1}^{K-1} l_m^* = n - 1 - \lceil n\gamma(K) \rceil.$$

It is clear that $\gamma(K) < 1$, so $l_K^* \geq 0$.

To complete the proof, we need to argue that the phantom distribution we constructed above is indeed optimal. Note that we are optimizing a bounded function over a discrete domain, so that the optimal solution always exists. Because the optimal solution has to satisfy the two necessary conditions (13) and (14), and because there is essentially unique distribution that satisfies these two conditions, our candidate distribution $\{l_k^*\}$ must be optimal. The optimal threshold function $\tau^*(k)$ for successive voting can be then easily computed: $\tau^*(k) = n - \sum_{m=1}^k l_m^*$. ■

Lemma 1 *The bounds (13) and (14) hold for all $k \in \mathcal{K}$ with $l_k = 0$.*

Proof. First let us define κ_1 and κ_2 as follows:

$$\begin{aligned} \kappa_1 &= \max \{m \in \mathcal{K} : l_k = 0 \text{ for all } k \leq m\}, \\ \kappa_2 &= \min \{m \in \mathcal{K} : l_k = 0 \text{ for all } k \geq m\}. \end{aligned}$$

We need to consider several cases.

Case 1: Both κ_1 and κ_2 exist. Then we have $l_1 = \dots = l_{\kappa_1} = 0$, and $l_{\kappa_2} = \dots = l_K = 0$. An alternative k with $l_k = 0$ could belong to one of the following three possible scenarios:

(i) $k \leq \kappa_1$. Since $l_1 = \dots = l_{\kappa_1} = 0$, condition (14) holds trivially and we only need to prove condition (13). By definition of κ_1 , $l_{\kappa_1+1} > 0$. Thus, (13) must hold at $\kappa_1 + 1$:

$$\sum_{m=1}^{\kappa_1} l_m \geq n\gamma(\kappa_1 + 1) - 1.$$

Since $l_1 = \dots = l_{\kappa_1} = 0$, we have

$$\sum_{m=1}^{k-1} l_m = \sum_{m=1}^{\kappa_1} l_m \geq n\gamma(\kappa_1 + 1) - 1 \geq n\gamma(k) - 1,$$

where the second inequality follows because γ is increasing and $\kappa_1 + 1 > k$.

(ii) $k \geq \kappa_2$. Since $l_{\kappa_2} = \dots = l_K = 0$, for all $k \geq \kappa_2$, we have

$$\sum_{m=1}^{k-1} l_m = n - 1 - \sum_k^K l_m = n - 1.$$

Hence, condition (13) is trivially satisfied, and we only need to prove condition (14). By definition of κ_2 , $l_{\kappa_2-1} > 0$. So we have (14) hold at $\kappa_2 - 1$:

$$\sum_{m=1}^{\kappa_2-1} l_m \leq n\gamma(\kappa_2).$$

Therefore,

$$\sum_{m=1}^k l_m = n - 1 = \sum_{m=1}^{\kappa_2-1} l_m \leq n\gamma(\kappa_2) \leq n\gamma(k + 1).$$

Again the last inequality follows from the monotonicity of $\gamma(\cdot)$ and the fact that $\kappa_2 < k + 1$.

(iii) $k \in (\kappa_1, \kappa_2)$. Define k_1 and k_2 as follows:

$$\begin{aligned} k_1 &= \max \{m \in \mathcal{K} : m < k \text{ and } l_m > 0\}, \\ k_2 &= \min \{m \in \mathcal{K} : m > k \text{ and } l_m > 0\}. \end{aligned}$$

Both k_1 and k_2 are well defined for all $k \in (\kappa_1, \kappa_2)$. By definition of k_1 and k_2 , we have

$$\sum_{m=1}^k l_m = \sum_{m=1}^{k_1} l_m \text{ and } \sum_{m=1}^{k-1} l_m = \sum_{m=1}^{k_2-1} l_m,$$

and condition (13) holds at $k_2 - 1$ and (14) holds at k_1 :

$$\sum_{m=1}^{k_2-1} l_m \geq n\gamma(k_2) - 1, \text{ and } \sum_{m=1}^{k_1} l_m \leq n\gamma(k_1 + 1).$$

Since $\gamma(\cdot)$ is increasing and $k_1 < k < k_2$, we have

$$\sum_{m=1}^{k-1} l_m \geq n\gamma(k) - 1, \text{ and } \sum_{m=1}^k l_m \leq n\gamma(k + 1).$$

Case 2: Neither κ_1 nor κ_2 exists. Then the argument of Case 1(iii) applies for all k with $l_k = 0$.

Case 3: κ_1 exists but κ_2 does not. Consider alternative k with $l_k = 0$. If $k \leq \kappa_1$, the argument of Case 1(i) applies. If $k > \kappa_1$, the argument of Case 1 (iii) applies.

Case 4: κ_2 exists but κ_1 does not. Consider alternative k with $l_k = 0$. If $k \geq \kappa_2$, the argument of Case 1(ii) applies. If $k < \kappa_2$, the argument of Case 1(iii) applies. ■

Proof of Corollary 3. Recall that the candidate position k^* is defined as

$$k^* \equiv \min \left\{ k \in \mathcal{K} : x^{k+1} \geq (C(x^{k+1}) + c(x^{k+1}))/2 \right\}.$$

By definition of k^*

$$x^{k+1} \geq (c(x^{k+1}) + C(x^{k+1}))/2 \text{ for all } k \geq k^*,$$

and

$$x^{k+1} < (c(x^{k+1}) + C(x^{k+1}))/2 \text{ for all } k < k^*.$$

This implies that

$$\frac{x^{k+1} - c(x^{k+1})}{C(x^{k+1}) - c(x^{k+1})} \geq 1/2, \text{ for all } k \geq k^*, \quad (15)$$

and

$$\frac{x^{k+1} - c(x^{k+1})}{C(x^{k+1}) - c(x^{k+1})} < 1/2 \text{ for all } k < k^*. \quad (16)$$

Moreover we note that for all k

$$\frac{x^{k+1} - c(x^{k+1})}{C(x^{k+1}) - c(x^{k+1})} < 1.$$

Therefore, in order to satisfy both (15) and (16), we must have, in the optimal phantom distribution, $l_{k^*}^* = 1$, and $l_k^* = 0$ for all $k \neq k^*$. ■

Proof of Proposition 2. Observe that

$$\begin{aligned} \frac{x^k - c(x^k)}{C(x^k) - c(x^k)} &= \frac{x^k - E[X|X \leq x^k]}{E[X|X > x^k] - E[X|X \leq x^k]} \\ &= \frac{x^k - E[X|X \leq x^k]}{E[X|X > x^k] - x^k + x^k - E[X|X \leq x^k]} \\ &= \frac{1}{1 + \frac{E[X|X > x^k] - x^k}{x^k - E[X|X \leq x^k]}} \end{aligned}$$

Let $\tilde{G}(k)$ and $G(k)$ be the optimal threshold under \tilde{X} and X , respectively. Then

$$\begin{aligned} G(k) &= \frac{\sum_{m=1}^k l_m^*}{n-1} = \frac{1}{n-1} \left[n \frac{1}{1 + \frac{E[X|X > x^k] - x^k}{x^k - E[X|X \leq x^k]}} \right], \\ \tilde{G}(k) &= \frac{\sum_{m=1}^k \tilde{l}_m^*}{n-1} = \frac{1}{n-1} \left[n \frac{1}{1 + \frac{E[\tilde{X}|\tilde{X} > x^k] - x^k}{x^k - E[\tilde{X}|\tilde{X} \leq x^k]}} \right]. \end{aligned}$$

Since x^k is independent of the distribution, in order to show $\tilde{G}(k) \leq G(k)$, it is sufficient to show that $E[X|X > x^k] \leq E[\tilde{X}|\tilde{X} > x^k]$ and $E[X|X \leq x^k] \leq E[\tilde{X}|\tilde{X} \leq x^k]$. However, for any $x \in [0, 1]$,

$$\begin{aligned} X \leq_{hr} \tilde{X} &\Rightarrow E[X|X > x] \leq E[\tilde{X}|\tilde{X} > x] \\ X \leq_{rh} \tilde{X} &\Rightarrow E[X|X \leq x] \leq E[\tilde{X}|\tilde{X} \leq x] \end{aligned}$$

Taking $x = x^k$ completes the proof of $\tilde{G}(k) \leq G(k)$. Note that

$$\tau(k) = n - \sum_{m=1}^k l_m^* = n - (n-1)G(k).$$

Therefore, we must have $\tilde{\tau}(k) \leq \tau(k)$ for all k . ■

8 Appendix B: Sufficient Conditions for Assumption B and Assumption B'

To derive sufficient conditions for Assumption B, we let $h_k(x)$ denote the utility difference for a type- x agent from two adjacent alternatives k and $k-1$:

$$h_k(x) = u^{k-1}(x) - u^k(x).$$

We claim that if the random variables $\{h_k(x)\}_{k \in \mathcal{K}}$ are ordered in terms of both hazard rate order and reverse hazard rate order, that is, $h_k \leq_{hr} h_{k+1}$ and $h_k \leq_{rh} h_{k+1}$, then Assumption B holds.³⁰ To see this, note that we can write

$$u_{x < x^k}^{k-1} - u_{x < x^k}^k = E[h_k(x) | x < x^k] = E[h_k(x) | h_k(x) > 0]$$

where the second equality follows from the definition of cutoff x^k and the single-crossing property. By rewriting $u_{x > x^k}^k - u_{x > x^k}^{k-1}$ analogously, we obtain

$$\Gamma(k) = -\frac{E[h_k(x) | h_k(x) > 0]}{E[h_k(x) | h_k(x) < 0]}.$$

Note that $h_k \leq_{hr} h_{k+1}$ implies that $E[h_k(x) | h_k(x) > 0]$ is increasing in k , and $h_k \leq_{rh} h_{k+1}$ implies that $E[h_k(x) | h_k(x) < 0]$ is increasing in k . Therefore, $\Gamma(k)$ is increasing in k .

To derive sufficient conditions for Assumption B' to hold in the linear case, we first recall a well-known concept used in the theory of reliability.

³⁰Note that conditions $h_k \leq_{hr} h_{k+1}$ and $h_k \leq_{rh} h_{k+1}$ impose restrictions on the shapes of both the distribution F and the utility function u . Alternatively, if we assume F is uniform, we could explicitly derive the required conditions for Assumption B only on function u . On the other hand, if we assume that the utility function u is linear as in our linear case that will be investigated below, we can derive the required conditions for Assumption B only on the distribution F .

Definition 4 1. The mean residual life (MRL) of a random variable $X \in [0, \bar{\theta}]$ is defined as

$$MRL(x) = \begin{cases} E[X - x | X \geq x] & \text{if } x < \bar{\theta} \\ 0 & \text{if } x = \bar{\theta} \end{cases}$$

2. A random variable X satisfies the decreasing mean residual life (DMRL) property if the function $MRL(x)$ is decreasing in x .

If we let X denote the life-time of a component, then $MRL(x)$ measures the expected remaining life of a component that has survived until time x .³¹ Assuming that $x - C(x)$ is increasing is equivalent to assuming a decreasing mean residual life (DMRL). Assuming that $x - c(x)$ is increasing is equivalent to assuming log-concavity of $\int_0^x F(s) ds$, because

$$x - c(x) = \frac{\int_0^x F(s) ds}{F(x)} \text{ and } \frac{F(x)}{\int_0^x F(s) ds} = \frac{d}{dx} \log \left[\int_0^x F(s) ds \right]$$

A sufficient condition for $\int_0^x F(s) ds$ to be log-concave is that $F(x)$ is log-concave. A sufficient condition for both log-concavity of F and DMRL of F is that the density f is log-concave.³²

References

- [2011] Apestegua, J, Ballester, M., and Ferrer, R. (2011), “On the Justice of Decision Rules,” *Review of Economic Studies* **78**(1), 1-16.
- [2013] Apestegua, J, and Ballester, M. (2013), “Choice by Sequential Procedures,” *Games and Economic Behavior* **77**, 90-99.
- [2004] Ausubel, L. M. (2004), “An Efficient Ascending-Bid Auction for Multiple Objects,” *American Economic Review*, **94**(5), 1452-1475.
- [2014] Azrieli, Y. and Kim, S. (2014), “Pareto Efficiency and Weighted Majority Rules,” *International Economic Review*, forthcoming.
- [2005] Bagnoli, M. and Bergstrom, T. (2005), “Log-Concave Probability and Its Applications,” *Economic Theory* **26**(2), 445-469.

³¹The MRL function is related to the hazard rate (or failure rate) $\lambda(x) = f(x) / [1 - F(x)]$. The “increasing failure rate” (IFR) assumption is commonly made in the economics literature. DMRL is a weaker property, and it is implied by IFR.

³²The log-concavity of density is stronger than (and implies) increasing failure rate (IFR) which is equivalent to log-concavity of the reliability function $(1 - F)$. The family of log-concave densities is large and includes many commonly used distributions such as uniform, normal, exponential, logistic, extreme value etc. The power function distribution ($F(x) = x^k$) has log-concave density if $k \geq 1$, but it does not if $k < 1$. However, one can easily verify the above two conditions hold for $F(x) = x^k$ even with $k < 1$. Therefore, log-concave density is not necessary. See Bagnoli and Bergstrom [2005] for an excellent discussion of log-concave distributions.

- [2001] Barbera, S (2001), “An Introduction to Strategy-Proof Social Choice Functions,” *Social Choice and Welfare* **18**, 619-653.
- [1993] Barbera, S., Gul, F. and Stacchetti, E. (1993): “Generalized Median Voter Schemes and Committees,” *Journal of Economic Theory* **61**(2): 262-289
- [1994] Barbera, S., and Jackson, M. (1994), “A Characterization of Strategy Proof Mechanisms for Economies with Pure Public Goods,” *Social Choice and Welfare* **11**, 241-252.
- [2006] Barbera, S., and Jackson, M. (2006), “On the Weights of Nations: Assigning Voting Weights in a Heterogeneous Union,” *Journal of Political Economy* **114**, 317-339.
- [1990] Barbera, S. and Peleg, B. (1990), “Strategy-proof Voting Schemes with Continuous Preferences,” *Social Choice and Welfare* **7**, 31-38.
- [2009] Bebchuk, L., Cohen, A., and Ferrell, A. (2009), “What Matters in Corporate Governance,” *The Review of Financial Studies* **22**, 783-827.
- [1948] Black, D. (1948), “On the Rationale of Group Decision-Making,” *Journal of Political Economy* **56**, 23-34
- [1958] Black, D. (1958), *The Theory of Committees and Elections*, Cambridge University Press, Cambridge, U.K.
- [2009] Börgers, T., and Postl, P. (2009), “Efficient Compromising,” *Journal of Economic Theory* **144**, 2057-2076.
- [2013] Chakravarty, S. and Kaplan, T. (2013), “Optimal Allocation without Transfer Payments,” *Games and Economic Behavior* **77**(1), 1-20.
- [1997] Ching, S. (1997): “Strategy-Proofness and ‘Median Voters’, ” *International Journal of Game Theory* **26**, 473-490.
- [2011] Chatterje, S. and Sen, A. (2011), “Tops-only Domains,” *Economic Theory* **46**, 255-282.
- [2012] Condorelli, D. (2012), “What Money Can’t Buy: Efficient Mechanism Design with Costly Signals,” *Games and Economic Behavior* **75**(2), 613-624.
- [2012] Dokow, E., Feldman, M., Meir, R. and Nehama, I. (2012), “Mechanism Design on Discrete Lines and Cycles,” *EC’12*, June 4-8, 2012, Valencia, Spain.
- [2013] Drexl, M. and Klein, A. (2013), “Collective Decision Making with Transferable Utilities,” discussion paper, University of Bonn.
- [2002] Ehlers, L., Peters, H., and Storcken, T. (2002), “Strategy-Proof Probabilistic Decision Schemes for One-Dimensional Single-Peaked Preferences,” *Journal of Economic Theory* **105**, 408-434.

- [1969] Farquharson, R. (1969), *The Theory of Voting*, Yale University Press, New Haven, U.S.A.
- [2009] Gersbach, H. and Pacht, B. (2009), “Flexible Majority Rules for Central Banks,” *Journal of Money, Credit and Banking* **41** (2-3), 507-516.
- [2013] Gershkov, A., Goeree J., Kushnir, A., Moldovanu, B. and Shi, X. (2013), “On the Equivalence of Bayesian and Dominant Strategy Implementation,” *Econometrica* **81**(1), 197-220.
- [1973] Gibbard, A. (1973), “Manipulation of Voting Schemes: A General Result,” *Econometrica* **41**(4), 587-601.
- [2003] Gompers, P., Ishi., J. and Metrick, A (2003), “Corporate Governance and Equity Prices,” *Quarterly Journal of Economics* **118**(1), 107-155.
- [2008] Hartline, J. and Roughgarden, T. (2008), “Mechanism Design and Money Burning,” *STOC*, May 17-20, 2008.
- [2007] Jackson, M. and Sonnenschein, H. (2007), “Overcoming Incentive Constraints by Linking Decisions,” *Econometrica* **75**, 241-258.
- [1990] Jung, J. (1990), “Black and Farquharson on Order-of-Voting Effects: An extension,” *Social Choice and Welfare* **7**: 319-329.
- [2009] Kovac, E., and Mylovanov, T. (2009), “Stochastic Mechanisms in Settings without Monetary Transfers,” *Journal of Economic Theory*, **144**(4), 2009, 1373-1395
- [2002] Ledyard, T., and Palfrey, T. (2002), “The Approximation of Efficient Public Good Mechanisms by Simple Voting Schemes,” *Journal of Public Economics*, **83** (2), 151-171.
- [2002] McLean, R., and Postlewaite, A. (2002), “Informational Size and Incentive Compatibility,” *Econometrica* **70**(6), 2421–2453.
- [2002] Milgrom, P., and Segal, I. (2002), “Envelope Theorems for Arbitrary Choice Sets”, *Econometrica* **70**(2), 583–601,
- [1981] Meltzer A., and Richard, S. (1981), “A Rational Theory of the Size of Government,” *Journal of Political Economy* **89**, 914-927.
- [1980] Moulin, H. (1980), “On Strategy-Proofness and Single Peakedness,” *Public Choice* **35**, 437-455.
- [1981] Myerson, R. (1981), “Optimal Auction Design,” *Mathematics of Operations Research* **5**, 58-73.

- [2004] Nehring, K. (2004): “The Veil of Public Ignorance,” *Journal of Economic Theory* **119**, 247-270
- [2007] Nehring, K., and Puppe, C. (2007), “The Structure of Strategy-Proof Social Choice – Part I: General Characterization and Possibility Results on Median Spaces,” *Journal of Economic Theory* **135**, 269-305.
- [2000] Persson, T., and Tabellini, G. (2000), *Political Economics: Explaining Economic Policy*, The MIT Press, Cambridge, Massachusetts.
- [2014] Peters, H., Roy, S., Sen, A., and Storcken, T. (2014): “Probabilistic Strategy-Proof Rules over Single-Peaked Domains,” *Journal of Mathematical Economics* **52**, 123-127.
- [1969] Rae, D. (1969), “Decision Rules and Individual Values in Constitutional Choice,” *American Political Science Review* **63**, 40-56.
- [2000] Rasch, B. E. (2000), “Parliamentary Floor Voting Procedures and Agenda Setting in Europe,” *Legislative Studies Quarterly* **25**(1), 3-23.
- [1977] Roberts, K. (1977), “Voting over Income Tax Schedules,” *Journal of Public Economics* **8**, 329-340.
- [1975] Romer, T. (1975), “Individual Welfare, Majority Voting and the Properties of a Linear Income Tax,” *Journal of Public Economics* **7**, 163-168.
- [2012] Rosar, F. (2012): “Continuous Decisions by a Committee: Median versus Average Mechanisms,” discussion paper, University of Bonn.
- [2009] Saporiti, A. (2009): “Strategy-proofness and Single-crossing,” *Theoretical Economics* **4**, 127-163.
- [1975] Satterthwaite, M. (1975), “Strategy-Proofness and Arrow’s Conditions. Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions,” *Journal of Economic Theory* **10**, 187-217.
- [2002] Schummer, J., and Vohra, R. (2002), “Strategy-proof Location on a Network,” *Journal of Economic Theory* **104**, 405-428.
- [2007] Schummer, J., and Vohra, R. (2007), “Mechanism Design without Money,” in *Algorithmic Mechanism Design*, N. Nisan et al. (eds), Cambridge: Cambridge University Press.
- [2007] Shaked, M., and Shanthikumar, J. (2007), *Stochastic Orders*, Springer, New York.
- [1991] Sprumont, Y. (1991), “The Division Problem with Single-Peaked Preferences: A Characterization of the Uniform Allocation Rule,” *Econometrica* **59**, 509-519.

- [1995] Sprumont, Y. (1995), "Strategy Proof Collective Choice in Economic and Political Environments," *Canadian Journal of Economics* **28**, 68-107.
- [2012] Schmitz, P. and Tröger T. (2012), "The (Sub-) Optimality of the Majority Rule," *Games and Economic Behavior* **74**(2), 651-665.