Network Games

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Abstract

In a variety of contexts – ranging from public goods provision to information collection – a player's well-being depends own action as well as on the actions taken by his or her neighbors. We provide a framework to analyze such strategic interactions when neighborhood structure, modeled in terms of an underlying network of connections, affects payoffs. We provide results characterizing how the network structure, an individual's position within the network, the nature of games (strategic substitutes versus complements and positive versus negative externalities), and the level of information, shape individual behavior and payoffs.

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1 Introduction

In a diverse range of social and economic interactions - including public goods provision, job search, political alliances, trade, friendships, and information collection – an agent's well being depends on her own actions as well as on the actions taken by other agents in close proximity; i.e., her neighbors.[1] For example, the decision of an agent of whether or not to buy a new product, or to attend a meeting, is often influenced by the choices of his or her friends and acquaintances. The pattern of neighborhoods affecting each agent is formalized in terms of a network of relations – the social network.

In this paper we examine how individual behavior varies with position within a network as well as how changes in the network structure - increasing the number of connections or redistributing connections - affect individual behavior and welfare. The paper develops a general framework to address these questions, which has some important features.

First, we allow for a variety of manners in which a player's payoff can depend on the play of his or her neighbors. We focus on the two canonical types of interplay that social interaction might take: that of strategic complements and that of strategic substitutes. For instance, the benefit that an individual obtains from buying a product or undertaking a given behavior is often greater as more of his or her friends do the same, providing for strategic complementarities. This might be due to direct effects of having similar or compatible products, or peer pressures, obtaining information from neighbors, and so forth. The set of applications where such effects are present are wide and include many choices that individuals make. Similarly, there are also many choices an individual makes where they can "free-ride" on friends, for instance not becoming informed directly but instead relying on the expertise of friends for a variety of tasks. These two types of strategic interaction allow for a wide class of games, and include as special cases most of the applications in the literature.

Second, we incorporate the complexity of social networks and the knowledge that players have about their network in a tractable and powerful manner. Since social networks are large and complex, although individuals might have information about local features of their

[1]The literature is much to vast to survey here, but for just a few representative examples of empirical work on network effects see Coleman (1966), Conley and Udry (2005), Granovetter (1994), Topa (2001), and Glaeser, Sacerdote, Scheinkman (1996), and Beaman (2007).

network, they are likely to possess only aggregate statistics about the network at large. This idea guides our approach: we focus on the number of connections that individuals have, i.e., their degree. We measure levels of knowledge in terms of the radius of local knowledge that a player has – whether she knows her own degree or her own degree and the degree of her neighbors, and so on. By conceptualizing a player's local network information as his or her type, we can analyze social interaction as a Bayesian game. We capture correlations in the degrees of neighbors through definitions of positive and negative association, embodying the idea that groups of variables are positively or negatively related. We discuss different associations exhibited by a vareity of random models of network formation in Appendix 8.1.

At this point, it is most useful just to provide a brief summary of our main contributions, as a careful description requires a number of definitions that appear in the text.

- We develop a new model of games on networks that allows for incomplete information on the part of individuals about their neighbors' connectivity that admits almost all random network models as special cases.

- We formalize notions of correlation of the connectedness of groups of players by adapting notions of statistical association.

- By examining games of strategic complements and substitutes we characterize how behavior varies by how connected a player is. In particular we show that:

  - If the game is of strategic complements, so that higher actions by neighbors lead to incentives for higher actions by a given player, and a positive association property holds, then there are equilibria where more connected players take higher actions.

  - If the game is of strategic substitutes, so that higher actions by neighbors lead to incentives for lower actions by a given player, and a negative association property holds, then there are equilibria where more connected players take lower actions.

- We compare how changes in the network structure, modeled through changes in the distribution of degrees and neighboring degrees, affects equilibrium play. In the case of strategic complements, moving to a denser network structure leads to increased behavior in the sense of higher actions by players of all types. In the case of strategic substitutes, a denser network structure leads to decreased behavior in the sense of lower actions by players of each type. The welfare effects can be ordered in some cases, but we show that this depends on additional information about the change in structure of the networks.

- We examine how equilibrium play is affected by the information structure showing that having more extensive information about network structure can lead to changes in equilibrium.

Before proceeding to present the model, let us comment on its relation to the literature.

The first model that explicitly examines games played on a network is the model of "graphical games" as introduced by Kearns, Littman, and Singh, 2001, also analyzed by Kakade, Kearns, Langford, and Ortiz, 2003, among others.[2] Although the basic idea of examining games on networks is the same, the line of study here has no overlap with that in the graphical games literature. The focus there has been on finding efficient algorithms to compute Nash equilibria in two-action games played on networks. Here, we allow for more general games and information structures (and hence the change in names), and our focus is on the structure of equilibria and how that interacts with the network, rather than how to compute equilibria.

There are also a number of papers that have looked at specific games played on a network. For instance, decisions to undertake criminal activity (Ballester, Calvó-Armengol, and Zenou (2005)), public goods provision (Bramoullé and Kranton (2005)), and decisions of buying a product (Galeotti (2005), Goyal and Moraga-Gonzalez (2001)) have been studied in specific formations and under complete information.[3] The role of incomplete information is worth best illustrated via a comparison with the results of Bramoulle and Kranton (2005). Bramoullé and Kranton (2005) consider a game where players search for valuable information and information is freely shared among neighbors. Players' utilities depend on a sum of

[2]There are also models of equilibria in social interactions, where players care about the play of certain other groups of players. See Glaeser and Scheinkman (2003) for an overview.

[3]In particular, regular networks (in which all players have the same degree) and core-periphery structures (the star network is a special case of such structures) have been extensively explored in the literature.

their own efforts and efforts of neighbors. They assume that efforts of players are strategic substitutes and that each player has complete network information. They find that there is multiplicity of equilibria and that the comparative statics within and across networks are ambiguous. The implications of incomplete information are brought out strongly in a setting where the degrees of neighbors are independent. Propositions 4 and 6 in our paper show that if players know only their own degree and aggregate network statistics then equilibria are monotone in actions and payoffs. Moreover, in section 4 we show that the effects of adding links (in a binary version of their game) are clear cut: every degree player chooses actions with greater probability, receives lower expected externalities from their neighbors (and therefore earns lower payoffs). Thus, the ambiguity found in the setting of Bramoullé and Kranton (2005) disappears when we examine incomplete information settings, and this holds across a general class of games. Moreover, we also present some results that show that we can provide some partial ordering conclusions even in the context of complete information.

Two recent papers Galeotti and Vega-Redondo (2005) and Sundararajan (2005) study games with incomplete network knowledge in specific contexts. Sundararajan (2005) studies games with strategic complements and focuses on the case where players know only their own degree. By contrast, the present paper allows for games with complements as well as substitutes, studies the effect of network changes (e.g., as it becomes more connected or more heterogenous), and allows for different levels of information. Galeotti and Vega-Redondo (2005) consider a game with a specific functional form in which payoffs are given by the product of neighbors' actions. Their payoff violates our restriction on payoffs, Assumption A. The methods of analysis and the results are quite different from the present paper.[4]

The use of stochastic dominance techniques for making comparisons across network structures was developed by Jackson and Rogers (2007a) in the context of a diffusion process. The contributions here are to develop techniques using a much more general association definition that admits many more network formation models, and to develop a general analysis of the game theoretic structure of behavior, whereas the Jackson and Rogers analysis works with mechanical contagion more in line with the spread of diseases or information than behav-

[4]Lopez-Pintado and Watts (2005) study social influence games and their general motivation is similar to ours. However, they abstract from network structure altogether.

ior.[5] Recent work by Jackson and Yariv (2007) follows up on the type of analysis conducted here and examines the multiplicity of equilibria of games on networks with incomplete information, but with a binary action model and a different formulation of payoffs, and very complementary results.

The rest of the paper is organized as follows. Section 2 presents the model and also motivates the approach in terms of models of network formation. Section 3 presents results on the equilibrium behavior of agents as a function of their local conditions. Sections 4 and 5 examine the effects on equilibrium that result from changing networks in a number of different ways. Section 6 studies the effects of greater local knowledge, while section 7 concludes. Proofs are presented in appendices at the end of the paper.

2 The Model

We now present the main elements of our theoretical framework.

Networks: The connections between a finite set of players $N = \{1, \dots n\}$ are described by a network.The network is represented by a matrix $g \in \{0,1\}^{n \times n}$, with $g_{ij} = 1$ denoting that i's payoff is affected by j's behavior. We follow the convention of setting $g_{ii} = 0$ for all i.

Note that the network could be asymmetric in that i might be affected by j's behavior, even if j does not care about i's behavior. A special case is where g is a symmetric matrix.

Let $N_i(g) = \{j | g_{ij} = 1\}$ represent the set of direct neighbors of i. For any integer $k \geq 1$, $N_i^k(g)$ denotes the *k-neighborhood* of i in g; that is, all the players that can be reached from i by directed paths of length no more than k. So, inductively $N_i^1 = N_i$ and $N^k = N_i^{k-1} \cup (\cup_{j \in N_i^{k-1}} N_j)$. The *degree*, $k_i(g)$, of player i is the number of i's direct connections:

$$k_i(g) = |N_i(g)|.$$

Strategies and Payoff Functions: Each player i takes an action x_i in X, where X is a compact subset of $[0, 1]$. Without loss of generality, we assume throughout that $0, 1 \in X$. We

[5] See Jackson and Yariv (2005) for an analysis of diffusion of behavior rather than diseases.

consider both discrete and connected action sets X. The payoff of player i when the profile of actions is $x = (x_1, ..., x_n)$ is given by:

$$v_{k_i(g)}(x_i, x_{N_i(g)})$$

where $x_{N_i(g)}$ is the vector of actions taken by the neighbors of i. Thus the payoff of a player depends on his or her own action and the actions that his or her direct neighbors take.

Note that the payoff function depends on the player's degree but not on his or her identity. Therefore, any two players who have the same degree have the same payoff function. We shall also assume that v_k depends on the vector $x_{N_i(g)}$ in an anonymous way, so that if x' is a permutation of x (both k-dimensional vectors) then $v_k(x_i, x) = v_k(x_i, x')$ for any x_i. If X is a connected action set then v_k is taken to be continuous in all arguments and concave in own action.

Strategic Complements and Substitutes: We examine two different conditions regarding how players' strategies interact. These two variations cover most applications of interest.

A game exhibits *strategic complements* if it satisfies increasing differences: for all k, $x_i > x'_i$, and $x \geq x'$: $v_k(x_i, x) - v_k(x'_i, x) \geq v_k(x_i, x') - v_k(x'_i, x')$.

The game exhibits *strategic substitutes* if it satisfies decreasing differences: for all k, $x_i > x'_i$, and $x \geq x'$: $v_k(x_i, x) - v_k(x'_i, x) \leq v_k(x_i, x') - v_k(x'_i, x')$. These notions are said to apply strictly if the payoff inequalities are strict whenever $x \neq x'$.

Externalities:

In addition to the strategic interaction of players' strategies, we also keep track of the nature of externalities. These need not go in the same direction as the strategic conditions. To understand this, consider an example of "doping" in sports. An athlete is more likely to want to cheat if his or her competitors are cheating. Thus, this is a game of strategic complements. However, the externalities are negative. Higher levels of cheating by competitors lowers an athlete's expected payoffs. Thus, how peers' behavior affect strategic incentives compared to payoffs can be quite different.

A game exhibits *positive externalities* if for each v_k, and for all $x \geq x'$, $v_k(x_i, x) \geq v_k(x_i, x')$.

A game exhibits *negative externalities* if $v_k(x_i, x) \leq v_k(x_i, x')$.

Correspondingly, the game exhibits *strict* externalities (positive or negative) if the above payoff inequalities are strict whenever $x \neq x'$.

Information Structure and Equilibria:

The scenario considered throughout most of the paper is one where players know their own degree as well as the network-formation mechanism at work. In particular, the resulting network may display inter-neighbor degree correlations. In such cases, it is natural to posit that agents are aware of this feature and factor it into their decision. This amounts to assuming that agents entertain a whole set of conditional distributions.

Let the degrees of the neighbors of a player i of degree k_i be denoted by $\mathbf{k}_{N(i)}$, which is a vector of dimension k_i. A player i of degree k_i's information about the degrees of his or her neighbors is captured by a distribution $P(\mathbf{k}_{N(i)} \mid k_i)$.

We thus model the game as a Bayesian game à la Harsanyi with a type space being the degree of a player. In this setup, a strategy for player i is a mapping $\sigma_i : \{0, 1, ..., n-1\} \rightarrow \Delta(X)$, where $\Delta(X)$ is the set of distribution functions on X.

We focus on symmetric equilibria, where players follow the same strategy. We will also be interested in tracking how strategies vary with degree. Noting that σ may involve mixed strategies we say that σ is *nondecreasing* if $\sigma(k')$ first order stochastic dominates $\sigma(k)$ for each $k' > k$. Similarly, we say that σ is *nonincreasing* if the domination relationship is reversed in each case.

Given a player i of degree k_i let $\mathrm{d}\psi_{-i}(\sigma, k_i)$ denote the probability measure over $x_{N_i} \in X^{k_i}$ induced by the beliefs $P(\cdot \mid k_i)$ held by i over the degrees of neighbors in $N(i)$ composed with the strategies played via σ.

Let

$$U(x_i, \sigma, k_i) = \int_{x_{N(i)} \in X^{k_i}} v_{k_i}(x_i, x_{N(i)}) \mathrm{d}\psi_{-i}(\sigma, k_i)$$

the expected payoff to a player i with degree k_i when other players use strategy σ and i chooses action x_i.

A Bayesian equilibrium is a (Bayesian) Nash equilibrium of this game, in the standard fashion.

In addition to the beliefs $P(\cdot \mid k)$, we also keep track of $P(\cdot)$ on $\{0, 1, \dots n-1\}$ which is the distribution indicating the unconditional probability that any given node has a given degree $P(k_i)$.

Association and Domination

As formulated, our framework allows for correlation between neighbors' degrees. In particular, our formulation captures essentially all random network formation models of interest that we are aware of.

In order to keep track of such correlation, we need to keep track of how players' beliefs vary with their degree. Things are complicated in our setting by the fact that players of different degrees have different numbers of neighbors, and the degrees of the neighbors will generally be correlated with each other (for instance if the degrees of i and j are perfectly correlated, and i and k are, then j and k must be as well). To handle this, we adapt a standard definition of "association" from the statistics literature to allow for comparisons across vectors of different dimensions.

Throughout what follows, we presume that beliefs of each player across their neighborhoods treat the neighbors symmetrically. In particular, $P(k_{N(i)} \mid k_i) = P(\widetilde{k}_{N(i)} \mid k_i)$ whenever $\widetilde{k}_{N(i)}$ is a permutation of $k_{N(i)}$. This embodies a sort of anonmymity in the network formation, so that a node's name is not important in determining its expected position in the network. This is satisfied by all random network models of interest. Note that even models where networks are born over time can be viewed in this way by randomly choosing the birthdates of nodes.

Given a player with degree k_i, enumerate the degrees of i's neighbors as $k_{N(i)} = (k_1, k_2, \dots k_{k_i})$. Next, consider a function $f : \{0, 1, \dots, n-1\}^k \to I\!R$ where $k \leq k_i$. Let

$$E_{P(\cdot|k_i)}[f] = \sum_{k_{N(i)}} P(k_{N(i)} \mid k_i) f(k_1, \dots k_k).$$

Thus, this just fixes some subset of $k \leq k_i$ of i's neighbors, and then expects f operating on their degrees.

P exhibits *positive association* if, for all $k' > k$, and any nondecreasing $f : \{0, 1, \dots, n-$

$1\}^k \to I\!R$.

$$E_{P(\cdot|k')}[f] \geq E_{P(\cdot|k)}[f].$$

Analogously, P exhibits *negative association* if the reverse inequality holds for each $k' > k$ and nondecreasing f.

Association is a standard property used to keep track of correlation patterns of groups of random variables, given the complicated interdependencies that might be present. Positive association generally embodies the idea that higher levels of one variable (in this case a player's degree) correspond to higher levels of other variables (in this case the player's neighbors' degrees).

Positive and negative degree association arise naturally in well known models of network formation. For example, the classical Erdos-Renyi model of random networks gives rise to independence across the degrees of neighbors. Note that our formulation of association allows for independence as a special case. Appendix A presents results for other network formation models, the "configuration" model, the Barabasi-Albert model of preferential attachment and the Jackson-Rogers model of random plus local linking. The configuration model exhibits negative association, while the preferential attachment model and the random plus local linking model lead to positive association.

Beyond association, which makes comparisons within a given social network, we are also interested in comparisons across social networks. These are captured in the following comparison.

P' *dominates* P if for all k, and any nondecreasing $f : \{0, 1, \ldots, n-1\}^k \to I\!R$

$$E_{P'(\cdot|k)}[f] \geq E_{P(\cdot|k)}[f].$$

Again, this is a generalization of stochastic dominance relationships adapted to vectors and families of distributions.

Effectively, this captures situations where P' leads to a denser network in than P in terms of nodes' connectivities.

Degree and Strategic Interaction:

In order to make comparisons of behavior within and across networks, we have to keep track of how behavior varies with degree. The following conditions capture the essential comparisons.

Payoffs exhibit *degree complementarity* if

$$U(x_i, \sigma, k_i) - U(x_i', \sigma, k_i) \geq U(x_i, \sigma, k_i') - U(x_i', \sigma, k_i')$$

whenever $x_i > x_i'$, $k_i > k_i'$, and σ is nondecreasing. Analogously, payoffs exhibit *degree substitution* if the inequality above is reversed in each case and σ is taken to be nonincreasing.

These conditions are ones that ensure that the conditions of complementarity or substitution are carried across degrees.

Let us provide a sufficient condition for degree complementarity.

Property A $v_{k+1}(x_i, (x, 0)) = v_k(x_i, x)$ for any $(x_i, x) \in X^{k+1}$.

Under Property A, adding a link to a neighbor who chooses action 0 is payoff equivalent to not having an additional neighbor.

PROPOSITION **1** *If the game exhibits strategic complements, P exhibits positive association and Property A holds, then payoffs satisfy degree complementarity. Similarly, if the game exhibits strategic substitutes, P exhibits negative association and Property A holds, then payoffs satisfy degree substitution.*

Property A is satisfied by a number of different settings, such as situations where players care about the aggregate action of their neighbors. While Property A is sufficient for the degree complementarity/substitution, it is not necessary. This is illustrated in the following example.

EXAMPLE **1** *Payoffs Depend on the Average of Neighbors' Actions*

Let P be such that players' degrees are independent (for example, as in an Erdös-Rényi random network). For ease of exposition, suppose that $X = \{0, 1\}$ and let $\sigma(k)$ be the probability that degree k plays 1. Player i's payoff function when he or she chooses x_i and her k neighbors choose the profile $(x_1, ..., x_k)$ is:

$$v_k\left(x_i, x_1, ..., x_k\right) = x_i f\left(\frac{\sum_{j=1}^{k} x_j}{k}\right) - c(x_i), \tag{1}$$

Let Y_m be a random variable that has a binomial distribution with m draws each with probability $\sum_k P(k)\sigma(k)$. Here

$$U(x_i, \sigma, k_i) = E\left[x_i f\left(\frac{Y_{k_i}}{k_i}\right)\right] - c(x_i),$$

and thus

$$U(1, \sigma, k_i) - U(0, \sigma, k_i) = E\left[f\left(\frac{Y_{k_i}}{k_i}\right)\right] - c(1) + c(0).$$

Note that $\frac{Y_{k'}}{k'}$ is a mean-preserving spread of $\frac{Y_k}{k}$ when $k' < k$. Thus, if f is concave, then there is degree complmentarity, while if f is convex then there is degree substitution.

We now present some examples where Property A is satisfied and thus there are degree complements/substitutes depending on the specific setting.

Example 2 *Payoffs Depend on the Sum of Actions*
Player i's payoff function when he or she chooses x_i and her k neighbors choose the profile $(x_1, ..., x_k)$ is:

$$v_k(x_i, x_1, ..., x_k) = f\left(x_i + \lambda \sum_{j=1}^{k} x_j\right) - c(x_i), \tag{2}$$

where $f(\cdot)$ is non-decreasing and $c(\cdot)$ is a "cost" function associated with own effort. The parameter $\lambda \in I\!R$ determines the nature of the externality across players's actions. This example yields (strict) strategic substitutes (complements) if (assuming differentiability) $\lambda f''$ is negative (positive).

The case where f is concave, $\lambda = 1$, and $c(\cdot)$ is increasing and linear corresponds to the case of information sharing as a local public good studied by Bramoullé and Kranton (2007), where actions are strategic substitutes. In contrast, if $\lambda = 1$, but f is convex (with $c'' > f'' > 0$), then we obtain a model with strategic complements, as proposed by Goyal and Moraga-Gonzalez (2001) to study collaboration among local monopolies. In fact, the formulation in (2) is general enough to accommodate a good number of further examples in the literature such as human capital investment (Calvo-Armengol and Jackson 2004, 2005), crime networks (Ballester, Calvó-Armengol, and Zenou, 2005), some coordination problems (Ellison 1993), and the onset of social unrest (Chwe, 2000).

The following two specializations of Example 2 are also useful to keep in mind.

Example 3 *Quadratic Payoff Functions*
Here $X = [0, 1]$ and we specialize Example 2 to a case where $f(y) = \gamma y + \alpha y^2$, $y = x_i + \lambda \sum_{j \in N_i} x_j$, and $c(x_i) = \beta x_i^2$ for some $\gamma, \alpha, \beta > 0$.

Example 4 *"Best-Shot" Public Goods Games*
$X = \{0, 1\}$ and we may interpret 1 as acquiring information (or providing any local and discrete public good) and 0 as not acquiring it. We posit that $f(0) = 0$, $f(x) = 1$ for all $x \geq 1$, so that acquiring one piece of information suffices. Costs, on the other hand, are assumed to satisfy $0 = c(0) < c(1) < 1$ so that no individual can find it optimal to dispense with the information but prefers one of her neighbors to gather it. [6]

While the degree-complementarity/substitution conditions are satisfied in the above applications, there are situations where the conditions are not satisfied. In particular, if payoffs are a product of the actions of neighbors (as in Galeotti and Vega-Redondo (2005)), then the conditions may not be satisfied, depending on the specifics of the payoffs.

Existence of Equilibrium

Before entering the analysis of the model, we remark on the issue of equilibrium existence.

Proposition 2 *There exists a symmetric equilibrium. If the game has degree complements, then the equilibrium strategy σ can be chosen in pure strategies, i.e. $\sigma_i(k_i)$ places probability 1 in some element of X for all $k_i \in \mathcal{T}$.*

The proof of existence is standard and omitted. The proof of existence of a pure-strategy equilibrium under degree complements appears in the appendix, but follows along the lines of other strategic complements arguments (e.g., see Milgrom and Shannon (199?)).

[6]The Best-Shot game is a good metaphor for many situations in which there are significant spillovers between players' actions. For instance, consumers learn from relatives and friends (Feick and Price, 1987), in research and development, innovations often get transmitted between firms, and similarly in agriculture, experimentation is often shared amongst farmers (Foster and Rosenzweig, 1995, Conley and Udry, 2005). For a discussion of best shot games within the context of public good games, see Hirshleifer (1983).

3 Comparing Choices Within a Network

We first study how the position in the network affects behavior, by examining how equilibrium actions change with players' degrees.

Proposition **3** *If there is degree complementarity (substitution) then there is a symmetric equilibrium that is noncreasing (nonincreasing).*

The arguments underlying the proof are as follows. First we show that for a player, faced with a monotone strategy played by the rest of the population, there always exists a monotone best-reply. Then, since the set of monotone strategies is convex and compact, existence follows from standard arguments.

Proposition 3 raises two issues. First, one might wonder whether, under the conditions of the proposition, it can be guaranteed that *every* symmetric equilibrium is monotone. The following example shows that the answer to this question is negative. Consider a game with action set $X = \{0, 1\}$ and payoffs $v_k(\hat{x}_i, x_1, ..., x_k) = x_i \sum_{j \in N_i} x_j - cx_i$, where $0 < c < 1$. Now suppose that there is perfect degree correlation so that players are connected to others of the same degree. It is then clear that any symmetric pure strategy profile defines an equilibrium.[7]

The existence of non-monotone equilibria depends correlation in degrees. This point is highlighted by the following result, in which degrees are independent.

Proposition **4** *Suppose that payoffs satisfy Property A and that the degrees of nodes are independent. Then, under strict strategic complements (substitutes) every symmetric equilibrium is monotone increasing (decreasing).*[8]

The intuition of the proposition is as follows. Consider the strategic complements case and select a player with degree $k+1$. Suppose all of her neighbors follow a symmetric equilibrium strategy, but his or her $(k+1)$'st neighbor chooses the minimal 0 action. Property A and the

[7]Note that this example satisfies Property A and the underlying game displays strategic complements. Similarly, it is possible to construct examples of games with strategic substitutes and negative association in which there exist symmetric equilibria which are not nonincreasing.

[8]The strictness is important for the result. For instance, if players were completely indifferent between all actions, then non-monotone equilibria are clearly possible.

independence imply that his or her best response would be identical to the equilibrium best response of a degree k player. However, in any non-trivial equilibrium (where at least one player chooses an action different from 0), the $(k+1)$'st neighbor would be choosing, with positive probability, a positive action. Strict complementarities imply that the player cannot best responds with lower actions than her k degree peers. Analogous reasoning applies to the case of strict strategic substitutes.

A *second* issue is whether the nature of degree correlation as posited in Proposition 3, i.e. positive association under strategic complements, or negative association under strategic substitutes, is essential for existence of monotone equilibria. We now provide an example which illustrates the need for this pattern of association. Consider, Example 3, with $\lambda = 1$ and $\gamma, \alpha, \beta > 0$. This game exhibits strategic complements. Next suppose that the degree distribution satisfies $P(1) = P(2) = \varepsilon$ and $P(\bar{k}) = 1 - 2\varepsilon$ for some small ε and a given large $\bar{k}$. Further posit that $P(\bar{k} \mid 1) = P(2 \mid 2) = 1$, i.e. all agents with degree 1 are connected to those of degree $\bar{k}$ and all those of degree 2 are connected among themselves. Then, it is straightforward to verify that if $\bar{k}$ is large enough and ε sufficiently small, *every* interior equilibrium strategy σ is such that $\sigma_2 < \sigma_1 < \sigma_{\bar{k}}$. The induced strategy is not monotone.

We now turn to the relation between players' degrees and their payoffs. The following result identifies conditions under which payoffs satisfy a monotonicity property when the equilibrium is monotone.

PROPOSITION **5** *Suppose that Property A holds. If P is positively associated, then in every nondecreasing symmetric equilibrium, the expected payoffs are nondecreasing (non-increasing) in degree if the game displays positive externalities (negative externalities). If P is negatively associated, then in every nonincreasing symmetric equilibrium, the expected payoffs are non-decreasing (non-increasing) in degree if the game displays positive externalities (negative externalities).*

Note that the proposition does not make any assumptions about complements or substitutes, but just the nature of the equilibrium, association, and the externalities.

The intuition behind Proposition 5 is as follows. Consider the case of positive externalities, positive association and look at a player with degree $k+1$. Suppose that that all of her neighbors follow the monotone increasing equilibrium strategy, but her $(k+1)'$th neighbor chooses the minimal 0 action. Property A implies that our $(k+1)$ degree player can assure herself an expected payoff which is at least as high as that of any k degree player by simply using the strategy of the degree k player. The intuition behind the other cases is analogous.

We emphasize that under positive externalities, players with more neighbors earn higher payoffs irrespective of whether the game exhibits strategic complements or substitutes (under the appropriate monotone equilibrium). Therefore, there is a clear advantage to being well connected in terms of having more connections. In the case of strategic complements and positive association (or strategic substitutes and negative association), this comes from the fact that players with a higher degree expect higher overall actions by neighbors. In particular, this implies that if the game displays strategic substitutes and negative association, higher degree players earn a higher payoff, even if they exert a lower effort. This result says that better connected players exploit network connections to free ride on those that are less-well connected.

We conclude this section by stating a corollary of the above results which derives a monotone payoffs property in degree for the case where neighbors' degrees are independent.

PROPOSITION **6** *Suppose that payoffs satisfy Property A, there are either strict strategic substitutes or complements, and that the degrees of any two neighboring nodes are independent. Then under positive externalities the expected payoffs are non-decreasing in degree, while under negative externalities the expected payoffs are non-increasing in degree.*

The proposition shows that higher degree nodes benefit under positive externalities and suffer under negative externalities, independent of whether the game is one of complements or substitutes. The proof follows by noting first, from Proposition 4 that all symmetric equilibrium are monotone, and then applying Proposition 5 (noting that independence constitutes a special case of association).

4 Comparing Choices Across Networks: Adding Links

This section examines the effects of adding links in a network on individual behavior and social outcomes.

Proposition 7 *Suppose that payoffs satisfy Property A and strict strategic complements and P' exhibits positive association. If P dominates P', then for every nondecreasing equilibrium σ' under P' there exists a nondecreasing equilibrium σ under P which dominates it.*

To get some intuition for this result consider the case where players' choices are complements in the strict sense and let σ' be a nondecreasing equilibrium under P'. As we shift weight to higher degree neighbors by switching to the conditional degree distribution $P(\cdot \mid k)$, each player's highest best response to the original equilibrium profile would be at least as high as the supremum of her original strategy's support. We can now iterate this best response procedure. Since the action set is compact, this process converges and it is easy to see that the limit is a (symmetric) nondecreasing equilibrium which dominates the original one.

Consider now the effect of a dominance shift in the social network on welfare. The expected welfare is assessed by the expected payoff of a randomly chosen player (according to the prevailing degree distribution). Naturally, it must depend on whether the externalities are positive or negative. Suppose, for concreteness, that they are positive and let P dominate P'. Then, from Proposition 7, we know that for every monotone equilibrium σ' under P' there exists a monotone equilibrium σ under P in which players' actions are all at least as high. Hence, the expected payoff of each player is higher under P. However, since expected payoff are non-decreasing in the degree of a player, to assess welfare it is also important to consider the relation between P and P'. For example, let us further assume that P FOSD P'. Then, the above considerations imply that the *ex-ante* expected payoff of a randomly chosen player must rise when one moves from P' to P. We summarize this argument in the following result. For a monotone increasing strategy profile σ under distribution P, define $W_P(\sigma)$, as the expected payoff of a node picked at random (under P).[9]

[9] It is important to note that the relationship between an underlying degree distribution does not imply a similar relation for the conditional distribution over neighbors' degrees, even under independence. As an illustration consider a case where degrees of neighbors are independent. Consider two degree distributions P and P', where P' assigns one half probability to degrees 2 and 10 each, while distribution P assigns one half probability to degrees 8 or 10 each. Clearly P FOSD P'. When neighboring degrees are independent, the probability of having a link with a node is (at least roughly, depending on the process) proportional to the degree of that node, so that for all k, $P(k'|k) = k'P(k')/\sum P(l)l$. Let $\tilde{P}(k')$ be the neighbor's degree

Proposition 8 *Suppose that payoffs satisfy Property A and strict strategic complements and the degrees of neighbors exhibit positive association. Suppose* P *dominates* P'*, and* $P(.)$ *FOSD* $P'(.)$*. For any nondecreasing equilibrium* σ' *and the corresponding expected payoff* $W_{P'}(\sigma')$ *under* P'*, there exists a nondecreasing equilibrium* σ *under* P *such that* $W_P(\sigma) \geq W_{P'}(\sigma')$*.*

The proof follows from the above argument and is omitted. Without the FOSD comparisons on the unconditional degree distribution the result fails to be true, as illustrated in the following example.

Example 5 *The effects of adding links in games with strategic complements*

Let $X = \{0, 1\}$ and suppose that the payoffs $v(x, y)$ of a typical player only depend on her own action x and the sum of her neighbors' actions y. More specifically, suppose that $v(1, y) = 1$ if $y > 1$, $v(1, y) = -\varepsilon$ for some small $\varepsilon > 0$ if $y \leq 1$, and $v(0, \cdot) = 0$. This game exhibits strategic complements and positive externalities. Now consider two different degree distributions P and P' given by $P(1) = 3/5$ and $P(5) = 2/5$ whereas $P'(1) = P'(3) = 1/2$. Clearly, P does *not* FOSD P'. Degrees are independent and $P(1|k) = 3/13$ and $P(5|k) = 10/13$, while $P'(1|k) = 1/4$ and $P'(3|k) = 3/4$ Thus, P dominates P' and Proposition 7 applies. In particular, consider the strategy σ' given by $\sigma'(k) = 0$ for $k \leq 1$ and $\sigma'(k) = 1$ for $k \geq 2$. This strategy is an equilibrium under P' for ε small enough. On the other hand, it is clear that the only strategy σ that dominates σ' and is also an equilibrium under P is $\sigma = \sigma'$. For P' the average welfare can be approximated (for small ε) as the probability that a randomly chosen node has degree 3 and at least two of its neighbors have this degree as well. This is $W' = \frac{1}{2}\left[1 - \left[(\frac{1}{4})^3 + 3(\frac{1}{4})^2\frac{3}{4}\right]\right] \simeq 0.42$. Analogously, the average welfare W for P can be bounded above by the probability $P(5) = 2/5$ that a randomly chosen node has degree higher than one. Thus, we have that $W' > W$, which shows that a FOSD shift of the conditional degree distribution may indeed lead to a welfare loss even if the game displays strategic complements and positive externalities. ■

distribution. Under $\tilde{P}'$, the probability that a neighbor has degree 10 is 5/6, while under $\tilde{P}$, the same probability is 5/9. Thus, $\tilde{P}$ does not FOSD $\tilde{P}'$.

We now take up the study of games with strategic substitutes. Consider negative association: starting from a nonincreasing equilibrium, an increase in degrees of a neighbor (on average) implies a fall in her action (on average), which, from strategic substitutes, suggests that the best response of a player should increase. However, this increase in action of every degree may come into conflict with the expectation that neighbors must be choosing a lower action, on average. To make progress we restrict attention to *binary-action games.*

The useful feature of binary-action games with strict strategic substitutes and negative association is that there is a unique nonincreasing symmetric equilibrium strategy σ and it is fully characterized by a threshold. That is, there exists some $t \in \{1, 2, ...\}$ such that $\sigma(k_i) = 1$ for $k_i < t$, $\sigma(t_i) = 0$ for all $k_i > t$, and for $t_i = t$ the induced $\sigma(t_i)$ may be a mixture over 0 and 1. The following result shows that dominance changes in degree distributions have clear cut effects on equilibrium behavior in such games.

PROPOSITION **9** *Suppose that $X = \{0, 1\}$, payoffs satisfy Property A and strict strategic substitutes and the P' exhibits negative association. If P dominates P', then in the unique nonincreasing symmetric equilibrium, the threshold under P is at least as large as the threshold under P'; i.e., $t \geq t'$.*

The intuition behind this result is as follows. Consider a nonincreasing equilibrium under P' and let t' be its threshold. As the distribution of neighbors' degrees shifts, in the domination sense, each player believes that it is more likely that her neighbors will have higher degrees. This means that the neighbors are less likely to choose 1. Since the game is one with strategic substitutes, each player's incentives to choose 1 increases and the result follows.

We now turn to the effects on welfare. The first thing to note is that dominance shifts in the interaction structure lower the expected probability that a randomly selected neighbor of a t'-degree player (the threshold player under P') chooses 1. For otherwise, the incentives of a t'-degree player to choose 1 are lower in the new equilibrium under P, which would generate an inconsistency with the above Proposition. If the degrees of neighbors are independent, then the average effort of a randomly selected neighbor of a player i does not depend on the degree of a player, and a similar property of lower expected action from each neighbor would hold for all degrees. However, if there is negative association, matters are more complicated, as lower degree players will have higher degree neighbors who choose lower actions and so the overall effect of a first order shift in degree distribution can be positive for some degrees and negative for others. The following example illustrates this possibility.

EXAMPLE **6** *Best shot games (continued)*

Assume that $k \in \{1, 2\}$; $P'(1|1) = 1/4$, while $P'(1|2) = 1/2$ and $c = 1/6$. It follows that the only symmetric (nonincreasing) equilibrium has degree 1 players choosing action 1, while degree 2 players choose action 1 with probability 0.18. Therefore, a neighbor of degree 1 player chooses 1 with probability 0.385, while a neighbor of degree 2 player chooses 1 is probability 0.59. Now, consider a first order shift in degree distributions to $P(.|k)$ with $P(1|1) = 1/8$, while $P(1|2) = 1/4$. Now the symmetric (nonincreasing) equilibrium has degree 1 players all choosing 1, while degree 2 players choose 1 with probability 0.45. Consequently, the probability that a neighbor of a degree 1 player chooses 1 becomes 0.52, which is higher than before. In contrast, the probability that a neighbor of a degree 2 player chooses 1 will be 0.587, which is lower than before. ■

This example and the earlier observations suggest that effects of first order shifts in degrees on welfare are ambiguous in games with strategic substitutes.

5 Comparing Behavior Across Networks: Redistributing Links

The previous section compares behavior across networks where there is an increase in the density of links in the sense of domination. However, there are many cases where we might be interested in comparing networks where there is not a clear cut domination relation, but instead are changing the distribution shape in other ways. While such comparisons are difficult to make in general, we can say a great deal in the case of binary action games. For these games, we ascertain the equilibrium implications of *any* change of the degree distribution, both for the case of *strategic complements* as well as that of *strategic substitutes*. We also briefly discuss quadratic games.

In this section we maintain the presumption that Property A holds, we focus on binary action games, and we consider situations where degrees are independent. As explained above, the key feature that simplifies the analysis of binary-action games is that their symmetric equilibria are threshold equilibria – that is, the choice of action solely depends on where the player's type lies relative to a given threshold. Our analysis is summarized by the following two results, for each of the payoff scenarios under consideration: strategic complements and strategic substitutes.

In the following propositions P and P' are two different degree distributions. Let $\tilde{F}$ and $\tilde{F}'$ be the induced cumulative distribution functions of the conditional degree distributions, respectively. Let t and t' stand for the threshold types defining the (unique) threshold equilibria under P and P', respectively.

Proposition **10** *Consider a binary-action game with strict strategic complements. If* $\tilde{F}'(t) \leq \tilde{F}(t-1)$ *then there is an equilibrium with corresponding threshold type* $t' \leq t$ *whereas, conversely, if* $\tilde{F}'(t) \geq \tilde{F}(t)$ *there is an equilibrium with corresponding threshold type* $t' \geq t$. *Moreover, if the equilibrium threshold rises (falls), the probability that any given neighbor chooses* 0 *rises (falls).*

The key issue here is the change in the probability mass relative to the threshold. If the probability of types (degrees) equal or below the threshold goes down then the probability of action 1 increases and from strategic complements, the best response of threshold type t and all higher types must be 1. In other words, the threshold should fall weakly. Analogous considerations allow us to state the following result which pertains to games with strategic substitutes.

Proposition **11** *Consider an incomplete information binary-action game with strict strategic substitutes. If* $\tilde{F}'(t) \geq \tilde{F}(t)$ *then* $t' \leq t$ *whereas, conversely,* $\tilde{F}'(t) \leq \tilde{F}(t-1)$ *implies* $t' \geq t$. *Moreover, if the equilibrium threshold rises (falls), the probability that any given neighbor chooses* 1 *falls (rises).*

The novel contribution of these results is that they allow us to examine the effect of *any change of the degree distribution.* A natural and important example of such changes involves increasing the polarization of the degree distribution by shifting weights to the ends of the support of the degree distribution, as is done under a mean preserving spread of the degree distribution. In particular, the above results can be directly applied in the case of strong MPS shifts in degree distributions. We say that P is a *strong MPS* of P' if they have the same mean and there exists L and H such that $P(k) \geq P'(k)$ if $k < L$ or $k > H$, and $P(k) \leq P'(k)$ otherwise. Propositions 10 and 11 imply that, in the context of binary-action games, the equilibrium effects of any such change can be inferred from the relative values of the threshold t, and L and H.

6 Extension: Deeper Information Structures

[[add results back from complete information and contrast with incomplete information.]]

Our analysis has focused in the case where players only know their own degree and best respond to the anticipated actions of their neighbors based on the (conditional) degree distributions. We now investigate the implication of increasing the information that players possess about their local networks. As a natural first step along these lines, we examine situations where a player knows not only how many neighbors she has, but also how may neighbors each of her neighbors has.

The arguments we develop in this section extend in a natural way to general radii of local knowledge. Indeed, in the limit, as this radius of knowledge grows, we arrive at complete knowledge of the arrangement of degrees in the network. For results on this limit case see the earlier version of this paper, Galeotti et al. (2006).

Formally, the common type space $\mathcal{T}$ of every player i consists of elements of the form $(k; \ell_1, \ell_2, ..., \ell_k)$ where $k \in \{0, 1, 2, ..., n-1\}$ is the degree of the player and ℓ_j is the degree of neighbor j ($j = 1, 2, ..., k$), where (in an anonymous setup where the identity of neighbors is ignored) we may assume without loss of generality that neighbors are indexed according to decreasing degree (i.e., $\ell_j \geq \ell_{j+1}$). Given the multi-dimensionality of types in this case, the question arises as to how one should define monotonicity. In particular, the issue is what should be the order relationship $\succeq$ on the type space underlying the requirement of monotonicity. For the case of strategic complements, it is natural to say that two different types, $t = (k; \ell_1, \ell_2, ..., \ell_k)$ and $t' = (k'; \ell'_1, \ell'_2, ..., \ell'_k)$, satisfy $t \succeq t'$ iff $k \geq k'$ and $\ell_u \geq \ell'_u$ for all $u = 1, 2, ..., k'$. On the other hand, for the case of strategic substitutes, we write $t \succeq t'$ iff $k \geq k'$ and $\ell_u \leq \ell'_u$ for all $u = 1, 2, 3, ...k'$. Given any such (partial) order on $\mathcal{T}$, we say that a strategy σ is monotone increasing if for all t_i, $t'_i \in \mathcal{T}$, $t_i \geq t'_i \Rightarrow \sigma(t_i)$ FOSD $\sigma(t'_i)$. The notion of decreasing monotonicity is defined analogously.

We first illustrate the impact of richer knowledge on the nature of equilibria. To keep matters simple, we consider a case where the degrees of the neighbors are independent. Recall, from Proposition 4 that in this case all symmetric equilibrium are monotone increasing (decreasing) in the case of strategic complements (substitutes).

Example **7** *Non-monotone Equilibria with Knowledge of Neighbors' Degrees*

Consider a setting where nodes have either degree 1 or degree 2, as given by the corresponding probabilities $P(1)$ and $P(2)$. Suppose that the game is binary-action with $X = \{0, 1\}$ and displays strategic complements. Specifically, suppose that the payoff of a player only depends on her own action x_i and the sum $\bar{x}$ of her neighbors' actions as given by a function $\nu(x_i, \bar{x})$ as follows: $v(0, 0) = 0$, $v(0, 1) = 1/2$, $v(0, 2) = 3/4$, $v(1, 0) = -1$, $v(1, 1) = 1$, $v(1, 2) = 3$.

It is readily seen that, for any P with support on degrees 1 and 2, the following strategy σ defines a symmetric equilibrium: $\sigma(1; 1) = 1$; $\sigma(1; 2) = 0$; $\sigma(2; \ell_1, \ell_2) = 0$ for any $\ell_1, \ell_2 \in \{1, 2\}$. Here, two players that are only linked to each other both play 1, while all other players choose 0.

Similar non-monotonic equilibrium examples can be constructed for games with strategic substitutes. These observations leave open the issue of whether there exist *any* suitably increasing or decreasing monotone equilibria. The following result shows that a monotone equilibrium always exists if players have deeper network information.

Proposition **12** *Suppose that payoffs satisfy Property A and players know their own degree and the degrees of their neighbors. Under strategic complements and positive association (strategic substitutes and negative association) there is a symmetric equilibrium that is monotone increasing (decreasing).*

The proof of the proposition, which appears in the Appendix, extends naturally the ideas mentioned for the proof of Proposition 3, i.e. the best-reply to a monotone strategy can be chosen monotone and the set of all monotone strategies is compact and convex. A direct implication of the result is that there is always an equilibrium that, on average across the types $(k; \ell_1, \ell_2, ..., \ell_k)$ consistent with each degree k, prescribes an (average) action that is monotone in degree. Equipped with the above monotonicity result, it is also possible to recover most of the insights obtained earlier under the assumption that players only know their own degree.

The ideas here also allow us to make some comparisons across networks in the case of complete information. Let g be a network with $g_{ij} = 0$ and denote by $g' = g + ij$ the network obtained from g by adding the link between i and j.

PROPOSITION **13** *Consider a complete information game with strategic complements. For any equilibrium σ under g there exists an equilibrium σ' under g' that dominates it.*[10] *Moreover, if X is connected, the game is of strict strategic complements, and σ is interior, then there exists an equilibrium σ' under g' in which all players in the component of i and j play strictly higher actions.*

Turning now to games of strategic substitutes under complete information, in order to draw the sharpest comparison with Proposition 9, let us focus on Best-Shot games (Example 4). In these games it is the unique best response to choose action 0 if any neighbor chooses 1 and it is the unique best response to choose 1 if all neighbors choose 0. As highlighted by Bramoullé and Kranton (2005) in a related class of games, this implies that the pure-strategy equilibria of these games are related to graph-theoretic objects termed independent sets.

An *independent set* for a network g is a set $I \subseteq N$ such that for any $i, j \in I$, $g_{ij} \neq 1$; so that no two players in I are linked.

It is clear that there is a one-to-one mapping between the (pure-strategy) Nash outcomes of a Best-Shot game played in a network g and its *maximal independent sets* – i.e., independent sets that are not contained in any other independent set. Best-Shot games played

[10] In the present complete-information scenario, the general notion of dominance across strategies introduced above is understood as follows: for every $i \in N$, $\sigma'_i(g')$ FOSD $\sigma_i(g)$.

on the empty network (no links) prescribe that all players choose action 1 in the unique equilibrium, while any Best-Shot game played on the full network (in which every pair of players has a direct link) has a set of n pure equilibria characterized by exactly one player choosing action 1. As it turns out, there is a gradual and monotone transition between these two extremes as links are added to the network.

Proposition **14** *Consider a Best-Shot game played under complete information. Consider any pure strategy equilibrium σ of $g + ij$. Either σ is an equilibrium under g, or there exists an equilibrium under g in which a strict superset of players chooses 1. Moreover, if $g \neq g + ij$ then there are equilibria under g that are not equilibria of $g + ij$.*

The intuition of the proof is quite simple. Indeed, start with an equilibrium under $g + ij$. From the above discussion, this equilibrium corresponds to a maximal independent set in $g + ij$. Consider that set of nodes when the underlying network is given by g. Of course, it is still an independent set. If it is a maximal one, then it indeed identifies the original equilibrium strategies in $g + ij$ as equilibrium strategies when the prevailing network is g. Otherwise, it is a strict subset of a maximal independent set, which identifies an equilibrium under g in which a strict superset of players chooses the action 1.

This result can be interpreted as a natural complete-information counterpart of Proposition 7 for the class of Best-Shot games. Just as it was established for incomplete information, we find that the addition of new links tends to lower the prevalence of the high action at (pure-strategy Nash) equilibria under complete information.

7 Concluding Remarks

We have examined a model of social interactions in which a player's payoff depends on her own action and the actions of her neighbors in an underlying network of connections. We have investigated how location within a fixed network as well as changes in overall network structure affect individual behavior. In particular, the paper makes two innovations: we allow for a fairly general class of payoffs (which subsumes as special cases practically all the models studied so far in the literature) and we allow for incomplete information about network structures (in contrast to most existing work which assumes complete network information). Our results yield a number of insights about how network structure, location within a network, nature of the game (strategic substitutes versus complements and positive versus negative externalities), and the level of information (incomplete versus complete) shape individual behavior and payoffs.

The framework we have developed can be extended in a number of directions. We conclude by mentioning two of them. One, we have assumed that payoffs satisfy Assumption A, which rules out payoff functions where the average actions of neighbors matters. One should not expect our results to extend to the "average" case, as then degree becomes largely irrelevant, but there are situations that fall between Assumption A and the average case. Extending the analysis to cover such cases appears to be an interesting avenue for further work. Two, we have examined problems where players care only about their direct neighbors' actions. There are also contexts where the externalities in behavior extend more broadly (e.g., due to congestion), and players might care about broader play in the game.

8 Appendix: Proofs

8.1 Appendix A: Network formation mechanisms

As an illustration consider the canonical model in the modern theory of complex networks. This is the model independently proposed by Gilbert (1959) and Erdös and Ré nyi (1960), and later intensively studied by the latter two authors. Given a finite set of nodes N, every possible link g_{ij} is assumed formed independently with a fixed probability p. In general, p is best conceived as a function of n, the cardinality of the set N. Suppose the social network is actually formed as posited above. Then, it is easy to see that, the degree distribution of a node picked at random is binomial. Here, for any finite n, the degree of nodes i and j, conditional on being connected are independent, since the degree of j is the realization of $n-2$ Bernoulli random variables (the other links of j) that are completely indpendent of the realization of the the other links of i.

(ii) The Configuration model

Next, let us consider the configuration model. Here the degrees of nodes are fixed in advance, and then the nodes are connected in a manner to realize the pre-specified degrees (See Jackson (2007) for a detailed description).[11] It is clear that this results in a negative association, although it becomes independent in the limit. To see this, suppose that on four nodes, there are two nodes with degree two and two nodes of degree one. Here, being a degree two node makes it more likely that a link points to a degree one node, and vice versa.

(iii) The Barabási-Albert model

Barabási and Albert (1999) propose an explicitly dynamic model of network-formation. Nodes enter in sequence, say according to their index $i = 1, 2, .., n$. At the time of entry, every node i creates a fixed number of links, say m, to separate incumbents (i.e. nodes that entered before). For each such link, any incumbent node j is chosen with a probability proportional to j's current degree – this is what is generally known as *preferential attachment*. For large n, it can be shown that, with very high probability, the degree distribution $P_n(\cdot)$ resulting from the process is approximately "scale-free" with a decay parameter equal to 3. That is, it satisfies:

$$P_n(k) \simeq Ak^{-3} \qquad\qquad (k = m, m+1, m+2, ...) \tag{3}$$

for some suitable normalizing factor A.

Positive association of this model follows as a special case of the Jackson-Rogers model.

(iv) The Jackson-Rogers model

The model described in Jackson and Rogers (2007) is one that is dynamic as in Barabási and Albert's model, but where the link formation is a bit different. In particular, the links are formed by choosing some number m_r of nodes uniformly at random, and then linking to each one of them with an independent probability p_r, and then also searching their neighborhoods to find m_n additional nodes and linking to each one of them with an independent probability p_n. In the extreme where $p_r = 0$, this becomes a preferential attachment model, while in the other extreme where $p_n = 0$ this becomes a uniformly random attachment model as in a growing version of the Erdos-Renyi world. As these parameters vary, it spans between them. It also has additional clustering and association features not exhibited by either extreme.

This model has positive association. A first order stochastic dominance relation of the

[11] This process results in a multi-graph, which might have nodes that have multiple links between them and some nodes that are self-linked.

distribution of a single neighboring node conditional on degree is proven in Theorem 4 in Jackson and Rogers (2007), and a straightforward extension of that result establishes positive assocation as defined here (under a mean-field approximation).

8.2 Appendix B: Proof of results

Proof of Proposition 3: The proof of this proposition follows as a special case of the proof of Proposition 12. In particular, the argument is an exact replica of that used in the proof of Proposition 12, with the type space $\mathcal{T}$ simply given by the set $\{0, 1, ..., n-1\}$ of possible degrees, and the corresponding ordering $\succeq$ (here a total one) given by the natural order $\geq$ on the natural numbers. ■

Proof of Proposition 4: We present the proof for the case of strategic complements. The proof for the case of strategic substitutes is analogous and omitted. Let $\{\sigma_k^*\}$ be the strategy played in a symmetric equilibrium of the network game. If $\{\sigma_k^*\}$ is a trivial strategy with all degrees choosing action 0 with probability 1, the claim follows directly. Therefore, from now on, we shall assume that the equilibrium strategy is non-trivial and that there is some k' and some $x' > 0$ such that $x' \in \mathbf{supp}(\sigma_{k'}^*)$.

Consider any $k \in \{0, 1, ..., n\}$ and let $x_k = \sup[\mathbf{supp}(\sigma_k^*)]$. If $x_k = 0$, it trivially follows that $x_{k'} \geq x_k$ for all $x_{k'} \in \mathbf{supp}(\sigma_{k'}^*)$ with $k' > k$. So let us assume that $x_k > 0$. Then, for any $x < x_k$, the assumption of (strict) strategic complements implies that

$$v_{k+1}(x_k, x_{l_1}, ..., x_{l_k}, x_s) - v_{k+1}(x, x_{l_1}, ..., x_{l_k}, x_s) \geq v_k(x_k, x_{l_1}, ..., x_{l_k}) - v_k(x, x_{l_1}, ..., x_{l_k})$$

for any x_s, with the inequality being strict if $x_s > 0$. Then, averaging over all types, the fact that the degree of any two neighboring nodes are stochastically independent random variables together with the fact at least $x_k > 0$ implies that

$$U(x_k, \sigma^*, k+1) - U(x, \sigma^*, k+1) > U(x_k, \sigma^*, k) - U(x, \sigma^*, k).$$

On the other hand, note that from the choice of x_k,

$$U(x_k, \sigma^*, k) - U(x, \sigma^*, k) \geq 0$$

for all x. Combining the aforementioned considerations we conclude:

$$U(x_k, \sigma^*, k+1) - U(x, \sigma^*, k+1) > 0$$

for all $x < x_k$. This in turn requires that if $x_{k+1} \in \mathbf{supp}(\sigma^*_{k+1})$ then $x_{k+1} \geq x_k$, which of course implies that σ^*_{k+1} FOSD σ^*_k. Iterating the argument as needed, the desired conclusion follows, i.e., $\sigma^*_{k'}$ FOSD σ^*_k whenever $k' > k$. ■

Proof of Proposition 5: We present the proof for positive externalities; The proof for negative externalities is analogous and omitted. The claim is obviously true for a trivial equilibrium in which all players choose the action 0 with probability 1. First, suppose positive association and let σ^* be a (non-trivial) monotone increasing equilibrium strategy. Suppose $x_k \in \mathbf{supp}(\sigma^*_k)$ and $x_{k+1} \in \mathbf{supp}\left(\sigma^*_{k+1}\right)$. Property A implies,

$$v_{k+1}(x_k, x_{l_1}, ..., x_{l_k}, 0) = v_k(x_k, x_{l_1}, ..., x_{l_k})$$

for all $x_{l_1}, ..., x_{l_k}$, and since the payoff structure satisfies positive externalities, it follows that for any $x > 0$,

$$v_{k+1}(x_k, x_{l_1}, ..., x_{l_k}, x) \geq v_k(x_k, x_{l_1}, ..., x_{l_k}).$$

We now have to consider two cases. First, assume positive association and consider a monotone increasing equilibrium, then looking at expected utilities we obtain that:

$$U(x_k, \sigma^*, k+1) \geq U(x_k, \sigma^*, k).$$

Since σ^*_{k+1} is a best response in the network game being played,

$$U(x_{k+1}, \sigma^*_{-i}, k+1) \geq U(x_k, \sigma^*_{-j}, k)$$

and the result follows. Second, note that the case of negative association and monotone decreasing equilibrium strategy can be proved analogously. This concludes the proof. ■

Proof of Proposition 7: Let σ' be a monotone increasing equilibrium of the network game with underlying network characterized by P'. We first show that there exists an equilibrium in the game with degree distribution P which dominates $\{\sigma'_i(t)\}$ and is monotone increasing.

Indeed, start with the (symmetric) profile of actions prescribing each player to use her action 1 with probability 1. Now consider the best response profile for all players, placing a probability 1 on the highest possible action for each player who is indifferent. Clearly, we are left with a profile $\{\sigma_i(t)\}$ that dominates $\{\sigma_i'(t)\}$. Furthermore, from Assumption A, strategic complementarities and positive association, the profile is monotone increasing. Start with $\{\sigma_i(t)\}$ and iterate using best responses; note from complements and positive association that at each stage there is a monotone increasing best response. This sequence is decreasing, and applying compactness of the action set this process converges to a symmetric pure equilibrium profile characterized by $\{x_k\}$ (each player i uses the strategy $\tilde{\sigma}_i(t)$, where $\tilde{\sigma}_i(k) = x_k$ for all i) which dominates $\{\sigma_i'(t)\}$ and is monotone increasing. (Here we are abusing notation slightly, as σ denotes the probability distribution on actions.)

Since $\{x_k\}$ is monotonic increasing with respect to k, strategic complements and the fact that $P(.|k)$ FOSD $P'(.|k)$ implies that if players use the symmetric profile $\{x_k\}$ under $P(.|k)$, then the best response of each degree k player is at least as high as x_k. Consider the profile of best responses (and, as before, upon indifference, choose the highest best response to be played with probability 1). The new profile dominates $\{x_k\}$ and is monotone increasing. Proceeding iteratively in that way, we converge to a symmetric (monotone increasing) equilibrium profile in the network game with degree distribution $P(.|k)$ that dominates $\{x_k\}$ and so also dominates the original equilibrium $\{\sigma_i'(t)\}$. ■

Proof of Proposition 9: Suppose that the decreasing monotone equilibrium has threshold t' under P', and let m' be the probability that a player of degree t' chooses action 0. The assumption that $P(\cdot|k)$ FOSD $P'(\cdot|k)$ for all k and that players choose a monotone decreasing strategy imply that the equilibrium threshold under P cannot be lower than t'. To see this, note that for any player of degree k, the probability that $l \in \{0, 1, ..., k\}$ of her neighbors choose action 1 is given by:

$$\binom{k}{l}\left(1 - P'(t'|k)m' - \sum_{s=t'+1}^{n-1} P'(s|k)\right)^{l}\left(P'(t'|k)m' + \sum_{s=t'+1}^{n-1} P'(s|k)\right)^{k-l}.$$

Since, for all k, $P(\cdot|k)$ FOSD $P'(\cdot|k)$,

$$P(t'|k)m' + \sum_{s=t'+1}^{n-1} P(s|k) \geq P'(t'|k)m' + \sum_{s=t'+1}^{n-1} P'(s|k),$$

and so the (binomial) distribution of the number of neighbors choosing the 1 action under P' FOSD that under P. Thus, from strict strategic substitutes, the threshold t under P must be weakly higher, i.e., $t \geq t'$.

This implies that the probability of choosing action 1 increases for all types whenever $t \neq t'$. If instead $t = t'$, that probability remains equal for all types except, possibly, for t. So assume, for the sake of contradiction, that the probability for this type under P is $1 - m$ and $m > m'$. Then,

$$P(t|t)m + \sum_{s=t+1}^{n-1} P(s|t) > P'(t|t)m' + \sum_{s=t+1}^{n-1} \tilde{P}'(s|t),$$

and it would be a strict best response for t to choose 1, in contradiction to the initial hypothesis. ■

Proof of Proposition 12: Let us consider first the case of strategic complements and denote by $\sum^m$ the set of monotone strategies. The proof is based on the following two claims:

Claim 1: For any player i, if all other players $j \neq i$ use a common strategy $\sigma \in \sum^m$ there is always a strategy $\sigma_i \in \sum^m$ that is a best response to it.

Claim 2: A symmetric equilibrium exists in the strategic-form game where players' strategies are taken from $\sum^m$.

To establish Claim 1, consider a player i and let $t_i, t_i' \in \mathcal{T}$ such that $t_i' \succeq t_i$, where $\succeq$ is the partial order applicable to the case of strategic complements (see Section 6). For any $\sigma \in \sum^m$ chosen by every $j \neq i$, let $BR(\sigma, t_i)$ be the set of best-response strategies of player i to σ when her type is t_i. Let us assume that $\sigma(t_j) \neq 0$ for some $t_j \in \mathcal{T}$. (Otherwise, the desired conclusion follows even more directly, since the best-response correspondence is unaffected by being connected to a player whose strategy chooses action 0 uniformly.) By definition, for every $x_{t_i} \in BR(\sigma, t_i)$, we must have that

$$\forall x \in X, \qquad U(x_{t_i}, \sigma, t_i) - U(x, \sigma, t_i) \geq 0$$

Then, since $t_i' \succeq t_i$, the assumption of (strict) strategic complements readily implies that

$$\forall x \leq x_{t_i}, \qquad U(x_{t_i}, \sigma, t_i') - U(x, \sigma, t_i') > 0. \tag{4}$$

Indeed, this follows from a two-fold observation:

(i) From Assumption A, if $t_i = (k, \ell_1, \ell_2, ..., \ell_k)$ and $t_i' = (k', \ell_1', \ell_2', ..., \ell_k')$ and $t_i' \succeq t_i$ we can think of t_i involving k' neighbors with all neighbors indexed from $k+1$ to k' (if any) choosing the action 0;

(ii) From strict strategic complements and positive association, since $\ell_u' \geq \ell_u$ the probability distribution over actions corresponding to each of her neighbors under t_i, $u = 1, 2, ..., k$, is dominated in the FOSD sense by the corresponding neighbor under t_i'.

Let us now make use of (4) in the case where x_{t_i} is the highest best response by type t_i to σ. Then, it follows that any $x_{t_i'} \in BR(\sigma, t_i')$ must satisfy:

$$x_{t_i'} \geq \sup\{x_{t_i} : x_{t_i} \in BR(\sigma, t_i)\},$$

which establishes Claim 1.

To prove Claim 2, we can simply invoke the concavity postulated for each payoff function $v_k(\cdot, x)$ for any given $x \in X^k$ and the fact that the set of monotone strategies is compact and convex. To see the latter point, note that the monotonicity of a strategy σ is characterized by the condition:

$$\forall t_i, t_i' \in \mathcal{T}, \qquad t_i' \succeq t_i \Rightarrow \sigma(t_i') \text{ FOSD } \sigma(t_i). \tag{5}$$

Clearly, if two different strategies σ and σ' satisfy (5), then any convex combination $\hat{\sigma} = \lambda\sigma + (1-\lambda)\sigma'$ also satisfies it.

Finally, to prove the result for the case of strategic substitutes, note that the above line of arguments can be applied unchanged, with the suitable adaptation of the partial order used to define monotonicity. In this second case, as explained in Section 6, we say that $t \succeq t'$ if and only if $k \geq k'$ and $\ell_u \leq \ell_u'$ for all $u = 1, 2, ..., k'$. ■

References

[1] Ballester, C., A. Calvo-Armengol, and Y. Zenou (2006), Who's who in networks. Wanted: The Key Player, *Econometrica*, 74, 5, 1403-1417.

[2] Barabási, A.-L. and R. Albert (1999): "Emergence of scaling in random networks," *Science* **286**, 509-12.

[3] Bondonio, D (1998), Predictors of Accuracy in Perceiving Informal Social Networks, *Social Networks*, **20**, 301-330.

[4] Bramoullé, Y. and R. Kranton (2007), Strategic Experimentation in Networks, *Journal of Economic Theory*, forthcoming.

[5] Calvó-Armengol, A. and M. O. Jackson (2004), The Effects of Social Networks on Employment and Inequality, *American Economic Review*, **94(3)**, 426-454.

[6] Calvó-Armengol, A. and M. O. Jackson (2005), Like Father, Like Son: Labor Market Networks and Social Mobility, *mimeo*, Caltech and Universitat Autònoma de Barcelona.

[7] Casciaro, T. (1998), Seeing Things Clearly: Social Structure, Personality, and Accuracy in Social Network Perception, *Social Networks*, **20**, 331-351.

[8] Chwe, M. S.-Y. (2000), Communication and Coordination in Social Networks, *Review of Economic Studies* **65**, 1-16.

[9] Coleman, J. (1966), *Medical Innovation: A Diffusion Study*, Second Edition, Bobbs-Merrill. New York.

[10] Conley, T., and C. Udry (2005), Learning About a New Technology: Pineapple in Ghana, *mimeo*, Yale University.

[11] Ellison, G. (1993), Learning, Local Interaction, and Coordination, *Econometrica*, **61**, 1047-1071.

[12] Feick, L. F. and L. L. Price (1987), The Market Maven: A Diffuser of Marketplace Information, *Journal of Marketing*, **51(1)**, 83-97.

[13] Foster, A. D. and M. R. Rosenzweig (1995), Learning by Doing and Learning from Others: Human Capital and Technical Change in Agriculture, *Journal of Political Economy*, **103(6)**, 1176-1209.

[14] Glaeser, E., B. Sacredote, and J. Scheinkman (1996), Crime and Social Interactions, *Quarterly Journal of Economics*, **111**, 507-548.

[15] Galeotti, A. (2005), Consumer Networks and Search Equilibria, *Tinbergen Institute Discussion Paper 2004-75.*

[16] Galeotti, A., S. Goyal, M. O. Jackson, F. Vega-Redondo, and L. Yariv (2006), Network Games, *mimeo*, Essex and Caltech.

[17] Galeotti, A. and F. Vega-Redondo (2005), Strategic Analysis in Complex Networks with Local Externalities, *HSS-Caltech Working Paper 1224.*

[18] Glaeser, E. L. and J. Scheinkman (2003), "Non-market Interactions," in: M. Dewatripont, L.P. Hansen, and S. Turnovsky (Eds.), Advances in Economics and Econometrics: Theory and Applications, Eight World Congress, Cambridge University Press.

[19] Goyal, S. and J. L. Moraga-Gonzalez (2001), R&D Networks, *Rand Journal of Economics*, **32(4)**, 686-707.

[20] Granovetter, M. (1994), *Getting a Job: A Study of Contacts and Careers*, Northwestern University Press. Evanston.

[21] Hirshleifer, Jack (1983), From Weakest-Link to Best-Shot: The Voluntary Provision of Public Goods, *Public Choice,*, **41(3)**, 371-386.

[22] Jackson, M. O., L. K. Simon, J. M. Swinkels and W. R. Zame (2002), Communication and Equilibrium in Discontinuous Games of Incomplete Information, *Econometrica,* **70**, 1711-1740.

[23] Jackson, M. O. and L. Yariv (2005), Diffusion on Social Networks, *Économie Publique*, Numéro 16, pp 3-16, 2005/1.

[24] Jackson, M. O. and L. Yariv (2007), "Diffusion of Behavior and Equilibrium Properties in Network Games," *American Economic Review (Papers and Proceedings)*, Vol. 97, No. 2.

[25] Jackson, M. O. and B. Rogers (2007a), "Relating Network Structure to Diffusion Properties through Stochastic Dominance", *The B.E. Press Journal of Theoretical Economics*, Vol. 7, Iss. 1(Advances), Art. 6, pp. 1-13.

[26] Jackson, M. O. and B. Rogers (2007b), "Meeting Strangers and Friends of Friends: How Random are Socially Generated Networks?" *American Economic Review*, Vol. 97, No. 3.

[27] Kakade, S., M. Kearns, J. Langford, and L. Ortiz (2003), Correlated Equilibria in Graphical Games, ACM Conference on Electronic Commerce, New York.

[28] Kearns, M., M. Littman, and S. Singh (2001), Graphical Models for Game Theory, in Jack S. Breese, Daphne Koller (eds.), *Proceedings of the 17th Conference on Uncertainty in Artificial Intelligence,* 253-260, San Francisco: Morgan Kaufmann University of Washington, Seattle, Washington, USA.

[29] Krapivsky, P. L. and S. Redner (2001): Organization of growing random networks, *Physical Review E* **63**, 066123.

[30] Kumbasar, E., A. K. Romney, and W. Batchelder (1994), Systematic Biases in Social Perception, *American Journal of Sociology*, **100**, 477-505.

[31] Lopez-Pintado, D. and D. Watts (2005), Social Influence, Binary Decisions and Collective Dynamics, *mimeo*, Department of Sociology, Columbia University.

[32] Morris, S. (2000), Contagion, *The Review of Economic Studies*, 67(1), 57-78.

[33] Sundararajan, A. (2005), Local Network Effects and Network Structure, *mimeo*, Stern School of Business, New York University.

[34] Topa, G. (2001), Social Interactions, Local Spillovers and Unemployment, *Review of Economic Studies*, 68, 2, 261-295.

[35] Vega-Redondo, F. (2006): *Complex Social Networks*, Econometric Society Monograph, Cambridge University Press.