Head Starts in All-Pay Contests

Ron Siegel*

Department of Economics

Northwestern University

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Abstract

This paper studies equilibrium behavior in a class of games that models asymmetric multiprize competitions in which players' costs are non-decreasing. Such costs accommodate head starts, which can be used to model incumbency advantages, prior investments, and other asymmetries not captured by differences in players' marginal costs. Building on the results of Siegel (2009a,b), I provide an algorithm that constructs the unique equilibrium in which players do not choose weakly-dominated strategies. I apply the algorithm to study multiprize all-pay auctions with head starts.

*Email: r-siegel@northwestern.edu

1 Introduction

One type of asymmetry frequently encountered in competitions is head starts. For example, in a competition for promotions in which a worker's tenure plays a role, workers may differ in how long they have worked for the firm. In a political campaign, the incumbent may enjoy a constituency advantage over an entrant. In a lobbying scenario, some lobbyists may have a reputational advantage. In a sports competition, some competitors may enjoy a head start in the form of a "handicap". But despite the prevalence of head starts in real-world competitions, there is relatively little theoretical work analyzing models of competitions with head starts.

This paper investigates a *contest* model that accommodates many players, multiple prizes, and general asymmetries among players, which include head starts. Players compete for m identical prizes by choosing a costly score, and each of the m players with the highest scores wins one prize. Each player is characterized by his valuation for a prize and his cost function, which is non-decreasing and continuous in his chosen score. Head starts are modeled by 0 costs on some initial interval of scores. Constant cost in intermediate regions of score can be used to model "matching" technologies or differential scoring rules, which increase a player's score discretely if he reaches a certain score. The primitives of the model are commonly known, capturing players' knowledge of the asymmetries among them. This helps interpret players' payoffs as "economic rents," in contrast to "information rents" that arise in models of competition with private information. The model is a special case of Siegel's (2009a) all-pay contests.[1] Contests are defined in Section 2.

Although contests may have multiple (and even a continuum) of equilibria, Section 3 describes an algorithm that constructs the unique equilibrium in which players do not choose weakly-dominated scores, i.e., do not choose scores for which there exists higher scores with the same cost. The construction builds on the algorithm of Siegel (2009b), who analyzed a contest model in which players' costs are strictly increasing, and hence precluded head starts. Non-decreasing costs introduce several technical difficulties. One difficulty is that in equilibrium players may choose a finite number of scores, whose cost is positive, with positive probability. Another difficulty is that certain scores may be best responses for only one players. These difficulties preclude standard equilibrium analysis techniques, and also demonstrate the qualitative affects that non-decreasing costs have on equilibrium behavior. In particular, the equilibrium in which players do not choose weakly-dominated scores is in general not the limit of the unique equilibria of contests with strictly increasing costs that approach the original contest.

As an application of the algorithm, Section 1 analyzes a three-player, two-prize all-pay auction with head starts. While two-player all-pay auctions with head starts have been analyzed before (Konrad (2002,2004) and Kaplan, Luski, and Wettstein (2003)), multiple prizes pose additional challenges both in constructing the equilibrium and proving that it is unique. Player's behavior

[1] Siegel (2009a) solves for players' equilibrium payoffs, but does not solve for equilibrium.

changes qualitatively as a function of the relative magnitudes of players' head starts. If one head start is much higher than the others, then competition is "tiered" so that the weakest player stays out with some probability, competes only against the middle player with some probability, and competes against the middle and strongest player with some probability. If the two highest head starts are relatively close to each other, then the weak player takes an "all or nothing" approach, either staying out or competing against the other two players at high scores. The other two players compete only against each other with some probability.

Section 5 describes additional applications of the contest framework that are work in progress. These include contests with discontinuous costs, which can accommodate entry fees, and contests with non-convex or discrete strategy spaces. The Appendix contains the proof of results from Section 3.

2 The Model

In a *contest*, n players compete for m homogeneous prizes, $0 < m < n$. The set of players $\{1, \ldots, n\}$ is denoted by $\mathcal{N}$. Players compete by each choosing a score from $\mathbb{R}_+ = [0, \infty)$, simultaneously and independently. Each of the m players with the highest scores wins one prize. In case of a relevant tie, any procedure may be used to allocate the tie-related prizes among the tied players.

Each player i is characterized by his *valuation for a prize*, V_i, and his *cost function*, $c_i : \mathbb{R}_+ \to \mathbb{R}$. Given scores $\mathbf{s} = (s_1, \ldots, s_n)$, where s_i is player i's chosen score, *player i's payoff* is

$$u_i(\mathbf{s}) = V_i - P_i(\mathbf{s}) c_i(s_i),$$

where $P_i : \mathbb{R}^n_+ \to [0, 1]$ is *player i's probability of winning*, which satisfies

$$P_i(\mathbf{s}) = \begin{cases} 0 & \text{if } s_j > s_i \text{ for } m \text{ or more players } j \neq i; \\ 1 & \text{if } s_j < s_i \text{ for } n - m \text{ or more players } j \neq i; \\ \text{any value in } [0,1] & \text{otherwise,} \end{cases}$$

such that $\sum_{j=1}^{n} P_j(\mathbf{s}) = m$. Note that a player's probability of winning depends on all players' scores, but the cost a player incurs depends only on his chosen score.

A player's score represents his overall performance, which may depend on more than one factor. For example, in a lobbying setting a firm's score may be a function of the amount of money the firm invests, its reputation, and its political connections. The score represents an optimal choice of the different factors as one number according to which players are evaluated. Differences among players are captured by their different valuations for a prize and differences in their cost functions. The primitives of the contest are commonly known.

I make the following assumptions.

C1 c_i *is piecewise analytic and non-decreasing.*[2]

C2 $V_i > 0$, $c_i(0) = 0$, *and* $\lim_{s_i \to \infty} c_i(s_i) > V_i$.

Assumption C1 means that choosing a higher score is weakly more costly, regardless of whether a player wins or loses. This captures the "all-pay" aspect of contests. Assumption C2 means that prizes are valuable, and that sufficiently high scores are prohibitively costly. Head starts are accommodated by having costs equal to 0 on some interval of scores with lower bound 0. When all costs are linear, we have an all-pay auction (Hillman and Samet (1987), Hillman and Riley (1989), Clark and Riis (1998)).[3]

Assumption C3, which completes the description of contests, uses the following definition.

Definition 1 (i) *Player i's* reach r_i *is the highest score he can choose by expending no more than his valuation for a prize. That is,* $r_i = \max\{s_i : c_i(s_i) = V_i\}$. *Re-index players in (any) decreasing order of their reach, so that* $r_1 \geq r_2 \geq \ldots \geq r_n$.

(ii) *Player* $m+1$ *is the* marginal player*.*

(iii) *The* threshold T *of the contest is the reach of the marginal player:* $T = r_{m+1}$.

(iv) *The* power w_i *of player i is his utility if he chooses the threshold and wins:* $w_i = V_i - c_i(T)$.

In an all-pay auction, for example, a player's reach is equal to his valuation for a prize, and a player's power is equal to his valuation for a prize minus the valuation for a prize of the marginal player.

C3 (i) The marginal player is the only player with power 0. (ii) The marginal player's cost function is increasing below the threshold, i.e., for every x in $[0, T)$, $c_{m+1}(x) < c_{m+1}(T) = V_{m+1}$.

The model of contests described here is a special case of Siegel's (2009a) all-pay contest model,[4] but is not a special case of Siegel's (2009b) contest model, since Siegel (2009b) requires strictly increasing costs (which, in particular, preclude head starts). As discussed in Siegel (2009a), players' valuations for a prize and cost functions that meet Assumptions C1 and C2

[2] A function is piecewise analytic on $[0, T]$ if $[0, T]$ can be partitioned into a finite number of closed intervals such that the restriction of the function to each interval is analytic. Analytic functions include polynomials, the exponent function, trigonometric functions, and power functions. Sums, products, compositions, reciprocals, and derivatives of analytic functions are analytic (see, for example, Chapman (2002)).

[3] In an all-pay auction, $c_i(s_i) = s_i$ and ties are resolved by randomizing uniformly.

[4] Instead of Assumptions C1-C3, Siegel (2009a) makes three assumptions (A1-A3), and specifies two Generic Conditions (Cost Condition and Power Condition) on players' valuations for winning and costs of losing. Assumptions C1 and C2 imply his Assumptions A1-A3, and Assumption C3 implies his Generic Conditions. In particular, a contest here is a generic separable contest of Siegel (2009).

but not Assumption C3 can be perturbed slightly so that Assumptions C1, C2, and C3 all hold. For example, an all-pay auction satisfies Assumption C3 part (ii) because costs are strictly increasing. If the valuation for a prize of the marginal player is different from those of all other players, then Assumption C3 part (i) holds as well.

2.1 Existing Results

This subsection lists two results, Theorem 1 and Theorem 2, which I use in solving for equilibrium. They are immediate corollaries of results in Siegel (2009a).[5] The first result characterizes players' equilibrium payoffs (without solving for equilibrium).

Theorem 1 *In any equilibrium of a contest, the expected payoff of every player equals the maximum of his power and* 0*.*

In addition to giving a closed-form formula for players' equilibrium payoffs, Theorem 1 shows that players $1, \ldots, m$ have positive expected payoffs (because of Assumption C3 part (i)), and players $m+1, \ldots, n$ have expected payoffs of 0.

A player *participates* in an equilibrium of a contest if with strictly positive probability he chooses scores whose cost is strictly positive. The second result provides a sufficient condition for players $m+2, \ldots, n$ not to participate in any equilibrium.

Theorem 2 *If the normalized cost function of the marginal player is strictly lower than that of player $i > m+1$, that is*

$$\frac{c_{m+1}(x)}{V_{m+1}} < \frac{c_i(x)}{V_i} \text{ for all } x > 0,$$

then player i does not participate in any equilibrium. In particular, if this condition holds for all players $m+2, \ldots, n$, then in any equilibrium only the $m+1$ players $1, \ldots, m+1$ may participate.

Throughout the description of the algorithm in Section 3 I refer to results from Siegel (2009b).

3 Equilibrium Construction

A player's (mixed) strategy is a probability distribution over $[0, \infty)$, typically described as a cumulative distribution function (CDF), which for every score x specifies the probability the the player chooses a score lower than or equal to x. An equilibrium is a profile of strategies, one for each player, such that each player's strategy assigns probability 1 to the player's set of

[5]The results follow, respectively, from Theorem 1 and Theorem 2 in Siegel (2009a).

best-responses given the other players' strategies. I say that *player i does not choose weakly dominated scores* if his CDF assigns probability 0 to the set of scores

$$\{x : c_i(x) = c_i(y) \text{ for a score } y > x\}.$$

For an $(m+1)$-player contest, I construct the unique equilibrium in which players do not choose weakly dominated scores. I do this by extending the algorithm of Siegel (2009b), who constructs the unique equilibrium of an $(m+1)$-player contest with strictly increasing costs (so no scores are weakly dominated). His construction first identifies the probability that each player chooses 0, and then extends players' CDFs by proceeding from 0 to the threshold. All players' CDFs reach 1 precisely at the threshold. This construction relies on two equilibrium properties of contests with strictly increasing costs: (i) every positive score up to the threshold is a best response for at least two players and (ii) players' equilibrium CDFs are continuous on $(0,T)$.[6] These properties also provide a closed-form formula for the unique equilibrium when there are just two players (Theorem 3 in Siegel (2009b)).

With weakly increasing costs the two properties do not always hold. Consequently, Siegel's (2009b) equilibrium construction algorithm cannot in general be used to solve for an equilibrium of an $(m+1)$-player contest. Even the relatively simple case of two players becomes more complicated when costs are not strictly increasing. While Theorem 3 in Siegel (2009b) still specifies *an* equilibrium of a two-player contest, other equilibria may exist. Moreover, the specified equilibrium may have the undesirable property that players choose weakly dominated scores.

To deal with these difficulties, my algorithm includes three new elements. The first element deals with head starts, which imply that knowing the probability that each player chooses 0 may not suffice for continuing the definition of players' CDFs above 0. The second element deals with the possibility that the cost function of a player is "constant at x," i.e., constant on some interval of scores with lower bound x, and the player has best responses immediately above x. When this happens, it may be that no other player has best responses immediately above x. Moreover, the player may have an "atom" at some score $y > x$, i.e., choose y with positive probability, so his CDF is discontinuous above 0. The third element deals with the possibility that not all players' CDFs reach 1 precisely at the threshold.

Section 3.1 describes the algorithm and shows that the resulting profile of CDFs is indeed an equilibrium in which players do not choose weakly dominated scores.[7] Section 3.2 shows that this is the unique equilibrium in which players do not choose weakly dominated scores, and explains to what extent the construction applies to n-player contests.

[6] Lemma 1 in Siegel (2009a) and Lemma 2 in Siegel (2009b).

[7] For a two-player contest, the equilibrium constructed by the algorithm can also be derived from the equilibrium specified by Theorem 3 in Siegel (2009b) by, for any maximal interval on which a player's cost function is constant, transferring all the mass the player places on the interval to an atom at the top of the interval.

3.1 The Algorithm

The algorithm has five stages, depicted in Figure 1 below. Stages 1, 2, and 4 parallel the algorithm of Siegel (2009b). Stages 3 and 5 are novel additions that deal with the possibility that players' cost functions may be constant on some intervals. I denote by $\mathbf{G} = (G_1, \ldots, G_{m+1})$ the profile of CDFs constructed by the algorithm, where $G_i(x)$ is the probability that player i chooses a score lower than or equal to x.

The construction of $\mathbf{G}$ proceeds from 0 to T by identifying *active* players and *checkpoints.* A player is active on his best response set, and a checkpoint is a score above which the set of active players may change. Stage 1 identifies a score $x_0 \geq 0$ and the value $\mathbf{G}(x_0)$ such that $\mathbf{G}(x) = 0$ for any $x < x_0$ and the value $\mathbf{G}(x_0)$ suffices to continue the construction above x_0.[8] If $G_{m+1}(x_0) = 1$, then x_0 is the last checkpoint and the algorithm proceeds to Stage 5, which complete the definition of $\mathbf{G}$. Otherwise, x_0 is the first checkpoint and the algorithm proceeds to Stage 2. Given a checkpoint x that has been reached, Stage 2 determines a set of candidate players at x, $\mathcal{CP}(x)$, which is a superset of the set $\mathcal{A}^+(x)$ of players active immediately above x.[9] If $\mathcal{CP}(x)$ contains a player whose cost function is constant at x, then the algorithm proceeds to Stage 3. Otherwise, the cost functions of all players in $\mathcal{CP}(x)$ are "increasing at x," i.e. strictly increasing on some interval with lower bound x, and the algorithm proceeds to Stage 4. Stage 3 sets $\mathcal{A}^+(x)$ to be the set of all players in $\mathcal{CP}(x)$ whose costs are constant at x, and identifies the next checkpoint $\bar{x}$ as the upper endpoint of the longest interval with lower endpoint x on which the cost function of a player in $\mathcal{A}^+(x)$ is constant. $\mathbf{G}$ is then set to equal $\mathbf{G}(x)$ on $[x, \bar{x})$, and at most one player has an atom at $\bar{x}$. The player and the size of the atom are identified. Stage 4 uses $\mathcal{CP}(x)$ to define a function $S_{x,y}$ with a unique fixed point $H(x, y)$ for scores y immediately above x, and uses $H(x, y)$ to determine the set $\mathcal{A}^+(x)$.[10] $\mathbf{G}$ is then extended continuously to $[x, \bar{x}]$ using a closed-form formula. The checkpoint $\bar{x}$ is also identified. If the CDF of at least one player reaches 1 at $\bar{x}$ in Stages 3 or 4, then the algorithm proceeds to Stage 5. Otherwise, $\bar{x}$ is set as the next checkpoint and the algorithm returns to Stage 2. Stage 5 completes the definition of $\mathbf{G}$ by identifying, for each player whose CDF has not reached 1, a score at which he has an atom that brings his CDF to 1. Note that at every checkpoint the algorithm either returns to Stage 2 or transitions to Stage 5 and terminates. I show that the algorithm terminates after reaching a finite number of checkpoints and that the resulting $\mathbf{G}$ is indeed an equilibrium. Whenever possible, I use notation consistent with that of Siegel (2009b). An exception is my usage of checkpoints in contrast to his usage of switching points.[11]

[8] This corresponds to step 1 of Siegel's (2009b) algorithm. In his algorithm $x_0 = 0$ because all costs are strictly increasing.

[9] This corresponds to step 2, part one of Siegel's (2009b) algorithm.

[10] This corresponds to step 2, part two of Siegel's (2009b) algorithm.

[11] A switching point of Siegel (2009b) is a score above which the set of active players *does* change. Every

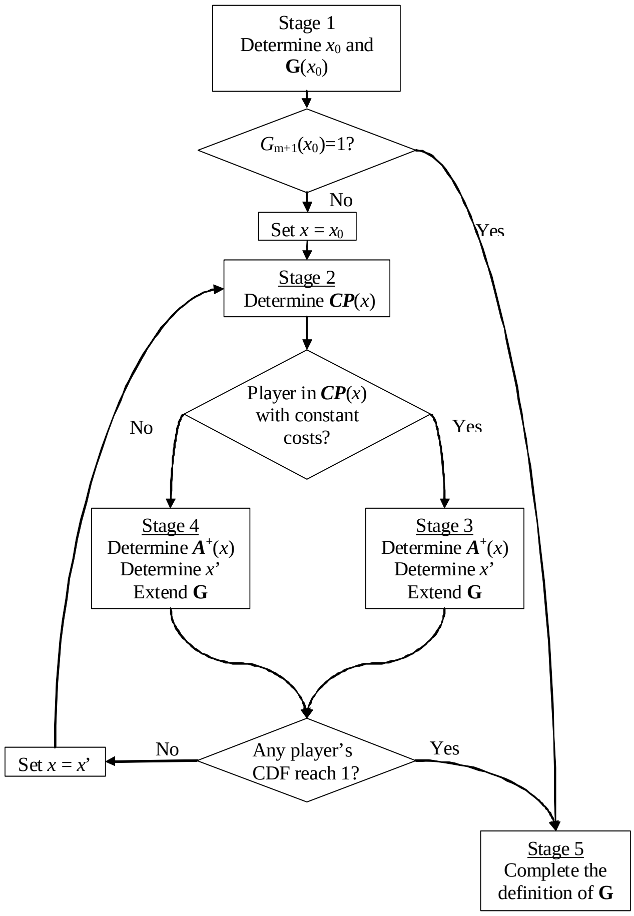

Figure 1: The equilibrium construction algorithm

Stage 1: Let $x_0 = \max\{x : c_{m+1}(x) = 0\}$. Because scores below x_0 are weakly dominated by x_0 for the marginal player, set $G_{m+1}(x) = 0$ for $x < x_0$. The following lemma determines $G_{m+1}(x_0)$ and $G_i(x)$ for every player $i < m+1$ and $x \leq x_0$. Its proof and those of the other lemmas in this section are in the Appendix.

Lemma 1 *In any equilibrium of an $(m+1)$-player contest in which players do not choose weakly dominated scores, the marginal player has an atom of size $\min_{i<m+1}\left(\frac{c_i(x_0)+w_i}{V_i}\right)$ at x_0, and the value of every other player's CDF at x_0 is 0.*

In accordance with Lemma 1, set $G_{m+1}(x_0) = \min_{i<m+1}\left(\frac{c_i(x_0)+w_i}{V_i}\right) \leq 1$ and $G_i(x) = 0$ for every player $i < m+1$ and $x \leq x_0$. If $G_{m+1}(x_0) = 1$, then set $x^L = x_0$ as the last checkpoint and proceed to Stage 5. Otherwise, set $x = x_0$ as the first checkpoint and proceed to Stage 2.

Stage 2: Suppose $\mathbf{G}$ has been defined up to $x < T$, and $G_i(x) < 1$ for every player i. We would like to identify a set of players, the "candidate players at x," who may be active

switching point of a contest with strictly increasing costs is a checkpoint, but some checkpoints may not be switching points.

immediately above x when $\mathbf{G}$ is extended above x. Consider a player i who chooses scores immediately above x when other players choose scores according to $\mathbf{G}$. Because CDFs are right-continuous and the player loses only when his score is lower than those of all other players, $1 - \Pi_{j \neq i} (1 - G_j (x)) + \varepsilon$ is the probability that the player wins a prize, for some small $\varepsilon \geq 0$. As player i chooses scores closer to x, ε approaches 0. Therefore, if

$$(1 - \Pi_{j \neq i} (1 - G_j (x))) V_i - c_i (x) < w_i, \tag{1}$$

then player i cannot be active immediately above x, because he cannot obtain his equilibrium payoff there. The candidate players at x are the other players, and for each such player i we have

$$(1 - \Pi_{j \neq i} (1 - G_j (x))) V_i - c_i (x) = w_i. \tag{2}$$[12]

We therefore denote the set of candidate players at x by

$$\mathcal{CP} (x) = \{\text{players } i \text{ for which (2) holds}\}.$$

The set $\mathcal{CP} (x)$ contains at least two players.[13] If $\mathcal{CP} (x)$ contains at least one player whose cost function is constant at x, proceed to Stage 3. Otherwise, proceed to Stage 4.

Stage 3: Suppose that $\mathcal{CP} (x)$ contains at least one player i whose cost function is constant at x. Constant costs imply that the *other* players' CDFs must remain constant immediately above x, otherwise player i would obtain more than his power immediately above x. And player i's CDF must also remain constant immediately above x, because scores immediately above x are weakly dominated by slightly higher scores. Therefore, all players' CDFs remain constant immediately above x, and the only players for whom scores immediately above x are best responses are those players in $\mathcal{CP} (x)$ whose cost functions are constant at x. Therefore, let

$$\mathcal{A}^+ (x) = \{\text{players in } \mathcal{CP} (x) \text{ whose cost functions are constant at } x\}.$$

Let $\bar{x}_i = \max \{y : c_i (y) = c_i (x)\}$ and $\bar{x} = \max \{\bar{x}_i : i \in \mathcal{A}^+ (x)\}$, and choose a player i in $\mathcal{A}^+ (x)$ for whom $\bar{x}_i = \bar{x}$. Following the previous paragraph, extend $\mathbf{G}$ to $[x, \bar{x}]$ as follows. For every player $j \neq i$ and every score y in $[x, \bar{x}]$ set $G_j (y) = G_j (x) < 1$, and for every score y in $[x, \bar{x})$ set $G_i (y) = G_i (x) < 1$. Note that $\mathcal{CP} (\bar{x})$ contains player i; set $G_i (\bar{x})$ to the lowest value such that $\mathcal{CP} (\bar{x})$ contains at least one more player. From (2), with l instead of i for some player $l \neq i$, this means that

$$G_i (\bar{x}) = \min_{l \neq i} \left\{ 1 - \frac{1 - \frac{w_l + c_l(\bar{x})}{V_l}}{\Pi_{j \neq l,i} (1 - G_j (x))} \right\} \text{ if } |\mathcal{N}| > 2 \tag{3}$$

[12] Lemma 7 in the Appendix shows that $>$ does not hold for any player.

[13] Lemma 7 in the Appendix shows this.

and

$$G_i(\bar{x}) = \frac{w_l + c_l(\bar{x})}{V_l} \text{ for } l \neq i \text{ if } |\mathcal{N}| = 2, \tag{4}$$

where $|\mathcal{S}|$ is the cardinality of a set $\mathcal{S}$. This value of $G_i(\bar{x})$ is the only value consistent with equilibrium. Indeed, setting $G_i(\bar{x})$ to a higher value implies that some player other than i can obtain more than his power immediately above $\bar{x}$. Setting a lower value means that players other than i do not have best responses immediately above $\bar{x}$ and therefore do not choose scores immediately above $\bar{x}$ with positive probability. Because player i's costs are increasing at $\bar{x}$, this means that player i too does not have best responses immediately above $\bar{x}$, so no player has best responses immediately above $\bar{x}$. This violates the following lemma.

Lemma 2 *In any equilibrium of an $(m+1)$-player contest, any score at which not all players' CDFs equal* 1 *is a best response for at least one player.*

Setting $G_i(\bar{x})$ to the lowest value such that $\mathcal{CP}(\bar{x})$ contains some player other than i implies that (i) $G_i(\bar{x}) = G_i(x)$ if there is another player $j \neq i$ for whom $\bar{x}_j = \bar{x}$ and (ii) $G_i(\bar{x}) > G_i(x)$ (so player i has an atom at $\bar{x}$) if i is the only player in $\mathcal{A}^+(x)$ for whom $\bar{x}_i = \bar{x}$. If $G_i(\bar{x}) = 1$, then set $x^L = \bar{x}$ as the last checkpoint and proceed to Stage 5. Otherwise, set $x = \bar{x}$ as the next checkpoint and proceed to Stage 2.

Stage 4: Suppose that the cost functions of all players in $\mathcal{CP}(x)$ are increasing at x. Then Step 2 part two, Step 3, and Step 4 of Siegel (2009b) can be used to uniquely identify the set $\mathcal{A}^+(x)$ of players active immediately above x, extend $\mathbf{G}$ up to the first checkpoint $\bar{x}$ above x, and identify $\bar{x}$. I describe these steps briefly; a detailed description is given in Section 2.1 of Siegel (2009b). For scores y immediately above x, let $q_i(y) = 1 - \frac{w_i + c_i(y)}{V_i} > 0$ and $\varepsilon_i(y) = -\frac{q_i'(y)}{q_i(y)} = \frac{c_i'(y)}{V_i - w_i - c_i(y)} > 0$[14] for every player i in $\mathcal{CP}(x)$, and let

$$S_{x,y}(H) = \sum_{i \in \mathcal{CP}(x)} \max\{H - \varepsilon_i(y), 0\}.$$

The function $S_{x,y}(\cdot)$ is piecewise linear and convex. Since $S_{x,y}(0) = 0$, $S'_{x,y}(0) = 0$, and $\varepsilon_i(y) > 0$ for every player i in $\mathcal{CP}(x)$, which contains at least two players, $S_{x,y}$ starts below the diagonal and reaches a slope of at least 2. It therefore has a unique positive fixed point, $H(x,y)$. Let $H(x,x) = \lim_{y \downarrow x} H(x,y)$. Siegel (2009b) shows that $H(x,\cdot)$ and $\varepsilon_i(\cdot)$ for every player i in $\mathcal{CP}(x)$ are analytic in a right-neighborhood of x, so have right-derivatives of any order at x. The lowest-order right-derivatives at x of $\varepsilon_i(\cdot)$ and $H(x,\cdot)$ that differ (beginning with the zeroth-order derivative, i.e., $\varepsilon_i(x)$ and $H(x,x)$) determine whether player i in $CP(x)$ is in $\mathcal{A}^+(x)$ ($<$ means the player is in $\mathcal{A}^+(x)$, $>$ means the player is not in $\mathcal{A}^+(x)$). (This will "generically" stop at the first derivatives.) If all derivatives are equal, then player i is in $\mathcal{A}^+(x)$. Because

[14]This follows from Lemma 8 in the Appendix and because $x < T$, so $c_i(y) < c_i(T)$ for score y immediately above x, which is shown in Lemma 3 below.

$H(x,y)$ is higher than $\varepsilon_i(y)$ for at least two players i in $\mathcal{CP}(x)$ and scores y immediately above x, $\mathcal{A}^+(x)$ contains at least two players.

Once $\mathcal{A}^+(x)$ is identified, (2) with y instead of x for every player i in $\mathcal{A}^+(x)$ and the fact that the CDFs of players not in $\mathcal{A}^+(x)$ do not increase immediately above x can be used to uniquely extend $\mathbf{G}$ to scores y in $[x,\bar{x}]$ as follows. For every player i not in $\mathcal{A}^+(x)$, set $G_i(y)=G_i(x)$. For every player i in $\mathcal{A}^+(x)$, set

$$G_i(y)=1-\frac{\Pi_{j\in\mathcal{A}^+(x)}q_j(y)^{\frac{1}{|\mathcal{A}^+(x)|-1}}}{q_i(y)\,D^{\frac{1}{|\mathcal{A}^+(x)|-1}}}, \tag{5}$$

where $D=\Pi_{j\notin\mathcal{A}^+(x)}(1-G_j(x))>0$ (if $\mathcal{A}^+(x)=\mathcal{N}$, then $D=1$) and $|\mathcal{S}|$ is the cardinality of a set $\mathcal{S}$.[15] This continuous extension of $\mathbf{G}$ guarantees that immediately above x players in $\mathcal{A}^+(x)$ obtain precisely their power and players not in $\mathcal{A}^+(x)$ do not obtain more than their power. The first checkpoint $\bar{x}$ above x is the first score above x at which G_i reaches 1 for some player i in $\mathcal{A}^+(x)$, or the cost function of a player in an $\mathcal{A}^+(x)$ is constant, or the cost functions of all players in $\mathcal{A}^+(x)$ are increasing and one of the following happens: the hazard rate of a player in $\mathcal{A}^+(x)$ drops to 0,[16] a player not in $\mathcal{A}^+(x)$ obtains his power,[17] or the score is a concatenation point of the cost function of a player in $\mathcal{A}^+(x)$ (recall that costs are piecewise-defined functions). If $G_i(\bar{x})=1$ for some player i in $\mathcal{A}^+(x)$, then set $x^L=\bar{x}$ as the last checkpoint and proceed to Stage 5. Otherwise, set $x=\bar{x}$ as the next checkpoint and proceed to Stage 2.

Stage 5: Because some player's CDF reaches 1 at x^L, no player j has best responses at scores y for which $c_j(y)>c_j(x^L)$. Therefore, for every player i, let $x_i^{\max}=\max\{y:c_i(y)=c_i(x^L)\}$, set $G_i(y)=G_i(x^L)$ for every score y in $(x^L,x_i^{\max})$, and set $G_i(y)=1$ for every score $y\geq x_i^{\max}$. This completes the construction of $\mathbf{G}$.

The following lemma shows that the algorithm always stops.

Lemma 3 *The algorithm reaches Stage 5 at a checkpoint $x^L\leq T$ via a finite number of checkpoints.*

Theorem 3 summarizes the construction. Its proof and that of Theorem 3.2 below are in the Appendix.

Theorem 3 *For any $(m+1)$-player contest the algorithm constructs an equilibrium $\mathbf{G}$ in which players do not choose weakly-dominated scores. The equilibrium is characterized by a partition into a finite number of intervals of positive length, on the interior of which the set of active players remains constant.*

[15] If $q_i(x')=0$, which happens if $c_i(x')=c_i(T)$, then set $G_i(x')=\lim_{y\uparrow x'}G_i(y)$.

[16] Ignoring any players whose hazard rate is identically 0 starting at x.

[17] That is, (2) holds for a player i not in $\mathcal{A}^+(x)$.

To gain some intuition for how the algorithm works, let us apply it to the two-player all-pay auction depicted in Figure 2. The common value of the prize is 1 and player 1 has a head start captured by zero costs up to $a_1 < 1$.[18]

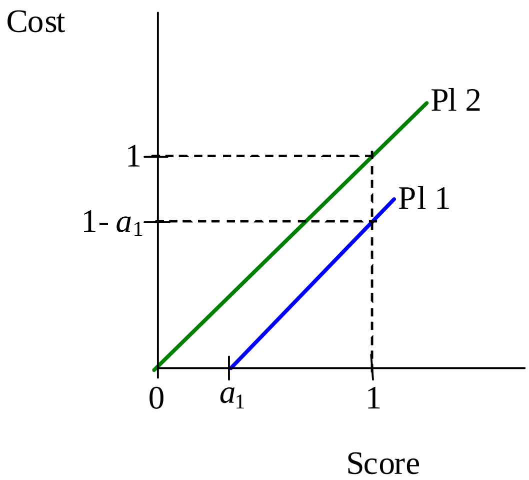

Figure 2: Players' costs in a two-player all-pay auction with a head start for player 1

Player 1's power is a_1, and player 2's power is 0. Because player 2 is the marginal player and his costs are increasing at 0, we have that $x_0 = 0$. By Lemma 1, $G_2(0) = a_1 < 1$, and Stage 2 shows that $\mathcal{CP}(0) = \{1,2\}$. Because player 1's costs are constant at 0, the algorithm proceeds to Stage 3, which specifies that $\mathcal{A}^+(0) = \{1\}$. Therefore, the first checkpoint above 0 is $\bar{x} = a_1$ and no player's CDF increases in $(0, a_1)$. From (4) we have $G_1(a_1) = a_1$. The algorithm proceeds to Stage 2, and we have that $\mathcal{CP}(0) = \{1,2\}$. Because both players' costs are increasing at a_1, the algorithm proceeds to Stage 4 $\mathcal{A}^+(a_1) = \{1,2\}$. From (5) we have that $G_1(y) = 1 - q_1(y)\, q_2(y) / q_1(y) = 1 - q_2(y) = y$ and $G_2(y) = a_1 + y - a_1 = y$ for scores y in $[a_1, \bar{x}]$, where $\bar{x} = 1$ is the score at which the players' CDFs reach 1. We then have $x^L = 1$ and by Stage 5 both players' CDFs equal 1 starting from 1. Players's atoms and densities in the equilibrium are depicted in Figure 3.

[18] If $a_1 \geq 1$, then player 2 has an atom of size 1 at 0, and the algorithm proceeds to Stage 5, which specifies that player 1 has an atom of size 1 at a_1.

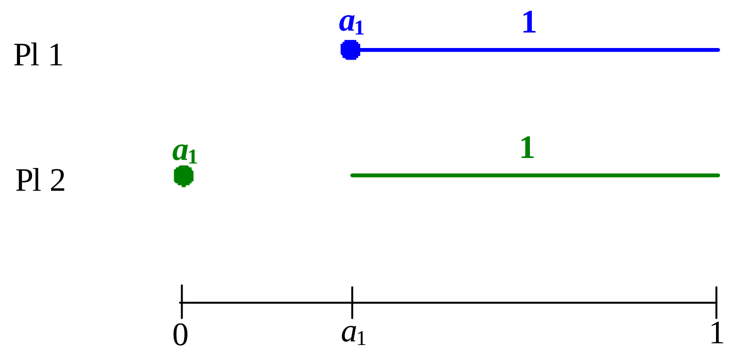

Figure 3: Players' atoms and densities in the equilibrium of the contest depicted in Figure 2

When the marginal player has a head start, x_0 is positive. For an illustration, consider the two-player contest depicted in Figure 4. The common value of the prize is 1, player 1 has marginal costs $1 - a_1$ starting from 0, and player 2 has zero costs up to $a_1 < 1$ and marginal costs $1/(1 - a_1)$ starting from a_1.

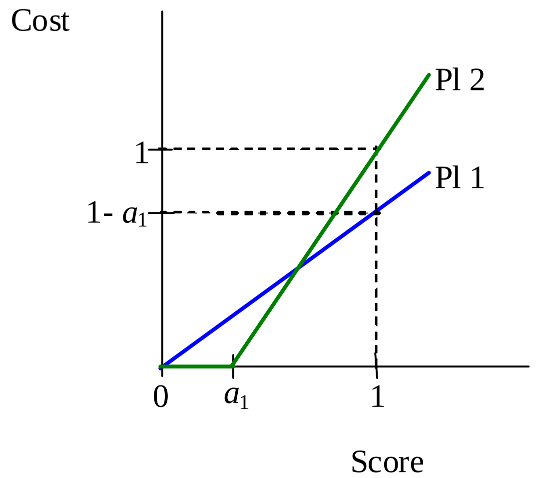

Figure 4: Players' costs in a two-player contest with differing marginal costs and a head start for player 2

Player 1's power is a_1, and player 2's power is 0. Because player 2 is the marginal player, by definition $x_0 = a_1$, the highest score for which player 2's costs are 0. By Lemma 1, $G_2(a_1) = (1 - a_1) a_1 + a_1 = 2a_1 - a_1^2 < 1$, and Stage 2 shows that $\mathcal{CP}(0) = \{1, 2\}$. Because both players' costs are increasing at a_1, the algorithm proceeds to Stage 4 and we have that $\mathcal{A}^+(a_1) = \{1, 2\}$. From (5) we have that $G_1(y) = 1 - q_2(y) = c_2(y) = (y - a_1)/(1 - a_1)$ and $G_2(y) = a_1 + c_1(y) = a_1 + y(1 - a_1)$ for scores y in $[a_1, \bar{x}]$, where $\bar{x} = 1$ is the score at which players' CDFs reach 1. We then have $x^L = 1$ and by Stage 5 both players' CDFs equal 1 starting from 1. Players's atoms and densities in the equilibrium are depicted in Figure 5.

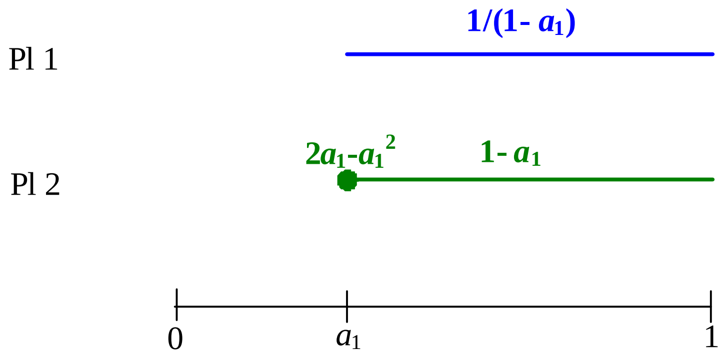

Figure 5: Players' atoms and densities in the equilibrium of the contest depicted in Figure 4

To see the importance of Stage 5, consider the two-player contest depicted in Figure 6. The common value of the prize is 1, player 1 has marginal costs 1 up to a_1, marginal costs 0 in $[a_1, 1]$, and marginal costs 1 above 1, and player 2 has marginal costs 1 starting from 0.

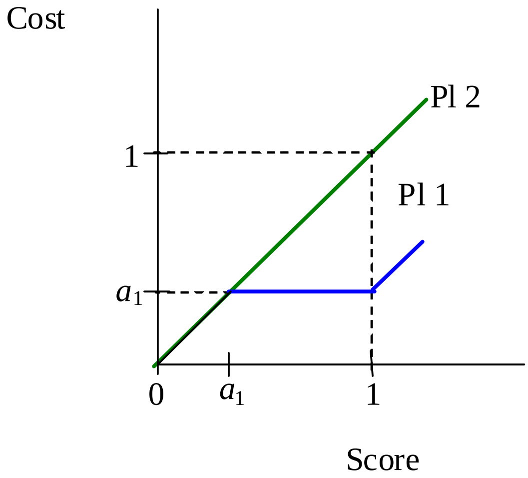

Figure 6: Players' costs in a two-player contest in which player 1 has an intermediate region with marginal costs 0

Player 1's power is $1 - a_1$, and player 2's power is 0. Because player 2 is the marginal player and his costs are increasing at 0, we have that $x_0 = 0$. By Lemma 1, $G_2(0) = 1 - a_1 < 1$, and Stage 2 shows that $\mathcal{CP}(0) = \{1, 2\}$. Because both players' costs are increasing at 0, the algorithm proceeds to Stage 4 and we have that $\mathcal{A}^+(a_1) = \{1, 2\}$. From (5) we have that $G_1(y) = 1 - q_2(y) = y$ and $G_2(y) = 1 - a_1 + y$ for scores y in $[a_1, \bar{x}]$, where $\bar{x} = a_1$ is the first score at which player 1's costs are constant. Because player 2's CDF reaches 1 at a_1, we have that $x^L = a_1$. The algorithm proceeds to Stage 5, which specifies that player 1's remaining probability mass is expended at 1, the highest score whose cost for player 1 is equal to player 1's cost of choosing a_1. Both players' CDFs equal 1 starting from 1. Players's atoms and densities in the equilibrium are depicted in Figure 7.

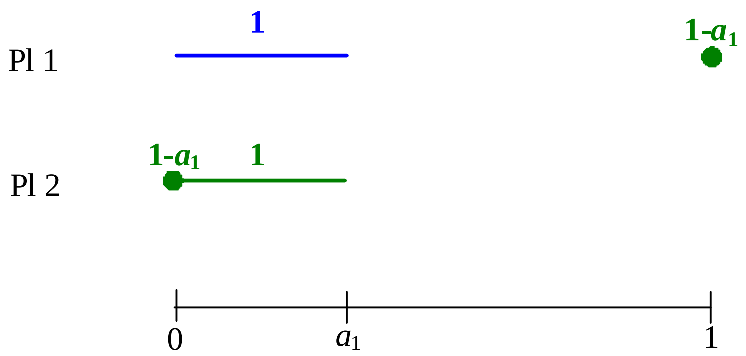

Figure 7: Players' atoms and densities in the equilibrium of the contest depicted in Figure 6

3.2 Equilibrium Uniqueness and n-player contests

As discussed above, an $(m + 1)$-player contest may have many, even a continuum, of equilibria. The following result, which is a modified version of Theorem 6 in Siegel (2009b), shows that the equilibrium $\mathbf{G}$ constructed by the algorithm is the unique equilibrium in which players do not choose weakly-dominated strategies. The proof uses the property of $\mathbf{G}$ that every player's best response set is a finite union of disjoint intervals to show that every equilibrium in which players do not choose weakly-dominated strategies coincides with $\mathbf{G}$.

Theorem 4 *For any $(m + 1)$-player contest the algorithm constructs the unique equilibrium of the contest in which players do not choose weakly-dominated scores.*

A contest with more than $m + 1$ players may have multiple equilibria in which players do not choose weakly-dominated strategies.[19] Uniqueness is guaranteed, however, if the costs of the marginal player are lower than those of player $m + 2, \ldots, n$. This is the content of the following result.[20]

Theorem 5 *If the conditions of Theorem 2 hold for every player $m + 2, \ldots, n$, then the contest has a unique equilibrium in which players do not choose weakly-dominated strategies. In this*

[19]See Example 3 in Siegel (2009a).

[20]This result is somewhat weaker than Theorem 8 in Siegel (2009b). A result that parallels Theorem 7 in Siegel (2009b) can also be stated.

equilibrium, players $m+2,\ldots,n$ *choose* 0 *and players* $1,\ldots,m+1$ *behave as in the equilibrium constructed by the algorithm of the reduced contest with players* $1,\ldots,m+1$*.*

Proof. By Theorem 3.2, the only candidate equilibrium is the one described in the statement of the theorem. The proof of Theorem 2 in Siegel (2009a) shows that if a player $i > m+1$ could obtain a positive payoff by when players $1,\ldots,m+1$ play according to the equilibrium constructed by the algorithm, then the marginal player could do the same, contradicting Theorem 1. ■

Note that the conditions of Theorem 2 do not place any restrictions on how the cost functions of players $\mathcal{N}\setminus\{m+1\}$ relate to each other.

4 All-Pay Auctions with Head Starts

There are $n \geq 2$ players who compete for $m < n$ identical prizes of common value V. Each player i is characterized by his head start a_i, and $0 \leq a_n \leq \cdots \leq a_1$. Players compete by making investments, each player's investment is added to his head start, and the m players with the highest sum win one prize each. Relevant ties are broken using any tie-breaking rule.

To model this game as a contest, let $c_i(x) = \begin{cases} 0 & x \leq a_i \\ x-a_i & x > a_i \end{cases}$. To meet condition C3, assume $a_{m+2} < a_{m+1} < a_m$. Player i's reach is $a_i + V$, so the contest's threshold is $V + a_{m+1}$. Theorem 1 tells us that in any equilibrium, player i's payoff is equal to his power, $\min\{a_i - a_{m+1}, V\}$, if $i < m+1$ and 0 otherwise. Not every equilibrium of the contest, however, is an equilibrium of the original game. This is because in the contest player i can choose a score lower than a_i, whereas in the original game an investment of 0 leads to the score a_i. Nevertheless, it is easy to see that every equilibrium of the original game corresponds to an equilibrium of the contest in which players do not choose weakly dominated scores, and vice versa. We therefore have the following corollary of Theorem 5.

Corollary 1 *An all-pay auction with head starts in which* $a_{m+2} < a_{m+1} < a_m$ *has a unique equilibrium. Players' behavior in the equilibrium is as in the equilibrium of the corresponding contest in which players do not choose weakly dominated strategies. In particular, players* $m+2,\ldots,n$ *and every player* i *for whom* $a_i \geq a_{m+1}+V$ *choose their respective head starts.*

Proof. The conditions of Theorem 2 hold for players $m+2,\ldots,n$ in the contest corresponding to the original game. Therefore the contest has a unique equilibrium in which players do not choose weakly dominated strategies, and in this equilibrium players $m+2,\ldots,n$ do not participate. Because scores lower than a player's head start are weakly dominated by the head start, players $m+2,\ldots,n$ choose their respective head starts. The power of a player i for whom $a_i \geq a_{m+1}+V$

equals V, so he cannot choose scores higher than a_i in the equilibrium. Scores lower than a_i are weakly dominated by a_i, so he must choose a_i in the equilibrium. ■

A useful property of the corresponding contest is that

$$\varepsilon_i = \frac{c_i'(y)}{V_i - w_i - c_i(y)} = \frac{1}{V - (a_i - a_{m+1}) - (y - a_i)} = \frac{1}{V + a_{m+1} - y}$$

for every player i for whom $a_i < a_{m+1} + V$ and every score $y \geq a_i$. That ε_i for these players and scores does not depend on i means that if the algorithm proceeds to Stage 4 at a checkpoint x, then $\mathcal{CP}(x) = \mathcal{A}^+(x)$. This makes applying the algorithm relatively simple, as the following example demonstrates.

Consider three players and two prizes,[21] with $a_3 = 0$ and $V = 1$. If $a_2 \geq 1$, then every player i chooses a_i and players 1 and 2 each win a prize with certainty.[22] If $a_2 < 1 \leq a_3$, then player 1 chooses a_1, and players 2 and 3 compete as in the two-player contest of Figure 3.[23]

If $a_1 < 1$, then players' behavior is more complicated and depends on how close a_2 is to a_3. To see this, suppose that $a_1 = 3/4$ and consider the equilibrium as a function of a_2 in $(0, 3/4)$. Player's powers are $3/4$, a_2, and 0. Stage 1 specifies that $x_0 = 0$ and $G_3(0) = a_2$. Stage 2 shows that $\mathcal{CP}(0) = \{2, 3\}$. Because player 2's costs are constant at 0, the algorithm proceeds to Stage 3, which gives $\mathcal{A}^+(0) = \{2\}$. Therefore, the first checkpoint above 0 is $\bar{x} = a_2$ and no player's CDF increases in $(0, a_2)$. From (3) we have

$$G_2(a_2) = \min\left\{1 - \frac{1 - w_1 - c_1(a_2)}{1 - G_3(0)}, 1 - \frac{1 - w_3 - c_3(a_2)}{1 - G_1(0)}\right\}$$

$$= \min\left\{\frac{\frac{3}{4} - a_2}{1 - a_2}, a_2\right\} = \begin{cases} a_2 & a_2 \leq \frac{1}{2} \\ \frac{\frac{3}{4} - a_2}{1 - a_2} & a_2 > \frac{1}{2} \end{cases}.$$

This implies that

$$\mathcal{CP}(0) = \begin{cases} \{2, 3\} & a_2 < \frac{1}{2} \\ \{1, 2, 3\} & a_2 = \frac{1}{2} \\ \{1, 2\} & a_2 > \frac{3}{4} \end{cases}.$$

Case 1, $a_2 < \frac{1}{2}$: Because the cost functions of players 2 and 3 are increasing at a_2, the algorithm proceeds to Stage 4. Because $\mathcal{A}^+(a_2) = \mathcal{CP}(a_2) = \{2, 3\}$, from (5) we have that $G_2(y) = 1 - q_2(y) q_3(y) / q_2(y) = 1 - q_3(y) = y$ and $G_3(y) = a_1 + y - a_1 = y$ for scores y in $[a_2, \bar{x}]$, where $\bar{x}$ is the first score at which player 1 obtains his power:

$$1 - (1 - \bar{x})^2 = \frac{3}{4} \Rightarrow \bar{x} = \frac{1}{2}.$$

[21] Any additional players $4, \ldots, n$ do not participate in the equilibrium.

[22] Stage 1 of the algorithm specifies that $x_0 = 0$ and $G_3(0) = 1$, and Stage 5 of the algorithm specifies that $G_2(a_2) = 1$ and $G_1(a_1) = 1$.

[23] The algorithm proceeds as explained in the paragraph preceding Figure 3, with player 1 instead of player 2 and player 2 instead of player 3. Stage 5 specifies that $G_1(a_1) = 1$.

Proceeding to Stage 2, we have that $\mathcal{CP}(1/2) = \{1,2,3\}$, and because the cost function of player 1 is constant at $1/2$, the algorithm proceeds to stage 3, which gives $\mathcal{A}^+(1/2) = \{1\}$. Therefore, the next checkpoint above $1/2$ is $a_1 = 3/4$ and no player's CDF increases in $(1/2, 3/4)$. From (3) we have

$$G_1(3/4) = \min\left\{1 - \frac{1 - w_2 - c_2\left(\frac{3}{4}\right)}{1 - G_3\left(\frac{1}{2}\right)}, 1 - \frac{1 - w_3 - c_3\left(\frac{3}{4}\right)}{1 - G_2\left(\frac{1}{2}\right)}\right\} = \min\left\{\frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2}.$$

Proceeding to Stage 2, we have that $\mathcal{CP}(3/4) = \{1,2,3\}$, and because all players' costs are increasing at $3/4$ the algorithm proceeds to Stage 4. Because $\mathcal{A}^+(3/4) = \mathcal{CP}(3/4) = \{1,2,3\}$, from (5) we have

$$G_i(y) = 1 - \frac{\Pi_{j\in\{1,2,3\}} q_j(y)^{\frac{1}{|\{1,2,3\}|-1}}}{q_i(y)\, D^{\frac{1}{|\{1,2,3\}|-1}}} = 1 - \frac{(1-y)^{\frac{3}{2}}}{1-y} = 1 - (1-y)^{\frac{1}{2}} \tag{6}$$

for every player i and score y in $[3/4, \bar{x}]$, where $\bar{x} = 1$ is the score at which players' CDFs reach 1. Stage 5 then specifies that players' CDFs remain 1 above 1. The equilibrium is depicted in Figure 8.

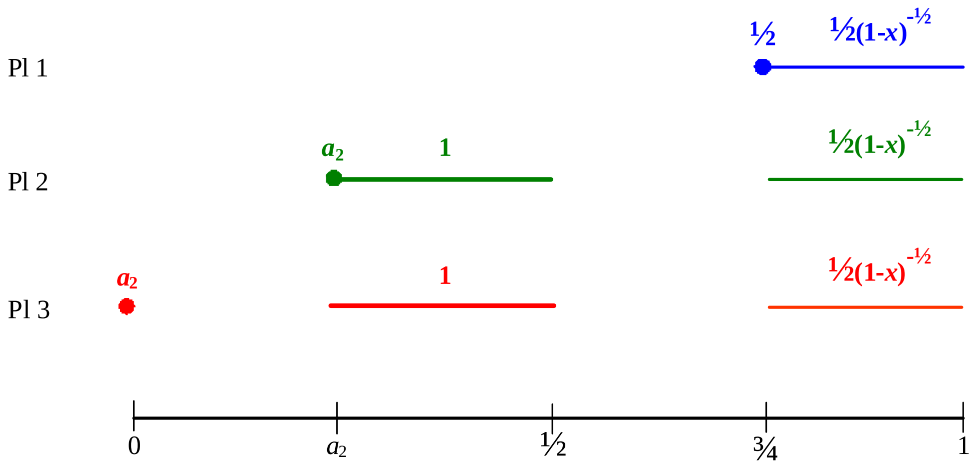

Figure 8: Players' atoms and densities in the equilibrium when $a_2 < \frac{1}{2}$

Case 2, $a_2 = \frac{1}{2}$: Because the cost function of player 1 is constant at $1/2$, the algorithm proceeds to stage 3, which gives $\mathcal{A}^+(1/2) = \{1\}$. Therefore, the next checkpoint above $1/2$ is $a_1 = 3/4$ and no player's CDF increases in $(1/2, 3/4)$. The equilibrium starting from $3/4$ is as in the case $a_2 < 1/2$: $G_1(3/4) = 1/2$, and $G_i(y)\, 1 - (1-y)^{1/2}$ for every player i and score y in $[3/4, 1]$. All players' CDF reach 1 at 1. The equilibrium is depicted in Figure 9 .

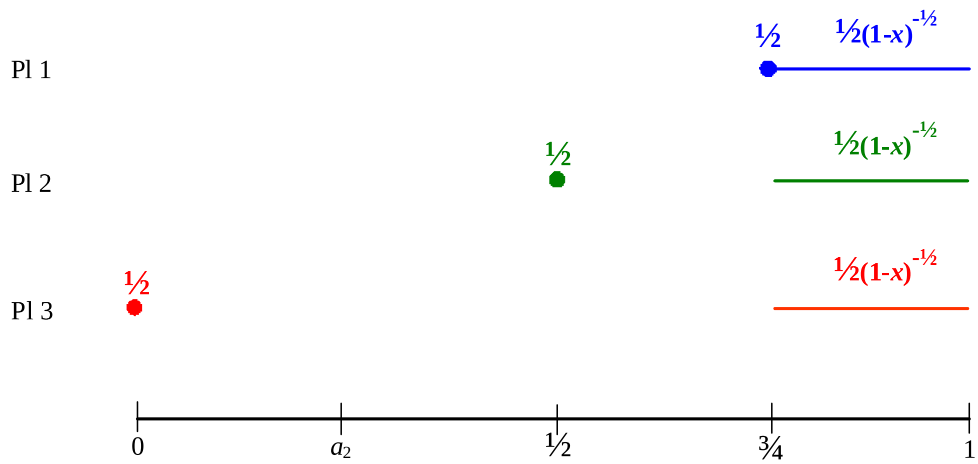

Figure 9: Players' atoms and densities in the equilibrium when $a_2 = \frac{1}{2}$

Case 3, $a_2 > \frac{1}{2}$: Because the cost function of player 1 is constant at a_2, the algorithm proceeds to stage 3, which gives $\mathcal{A}^+(a_2) = \{1\}$. Therefore, the next checkpoint above a_2 is $a_1 = 3/4$ and no player's CDF increases in $(a_2, 3/4)$. From (3) we have

$$G_1(3/4) = \min\left\{1 - \frac{1 - w_2 - c_2\left(\frac{3}{4}\right)}{1 - G_3(a_2)}, 1 - \frac{1 - w_3 - c_3\left(\frac{3}{4}\right)}{1 - G_2(a_2)}\right\} = \min\left\{\frac{\frac{3}{4} - a_2}{1 - a_2}, a_2\right\} = \frac{\frac{3}{4} - a_2}{1 - a_2}.$$

Proceeding to Stage 2, we have that $\mathcal{CP}(3/4) = \{1, 2\}$, and because both players' costs are increasing at 3/4 the algorithm proceeds to Stage 4. Because $\mathcal{A}^+(3/4) = \mathcal{CP}(3/4) = \{1, 2\}$, from (5) we have that

$$G_1(y) = 1 - \frac{q_1(y)\, q_2(y)}{q_1(y)\left(1 - G_3\left(\frac{3}{4}\right)\right)} = \frac{y - a_2}{1 - a_2} = 1 - \frac{q_1(y)\, q_2(y)}{q_2(y)\left(1 - G_3\left(\frac{3}{4}\right)\right)} = G_2(y)$$

for scores y in $[3/4, \bar{x}]$, where $\bar{x}$ is the first score at which player 3 obtains his power:

$$1 - \left(1 - \frac{\bar{x} - a_2}{1 - a_2}\right)^2 - \bar{x} = 0 \Rightarrow \bar{x} = 2a_2 - a_2^2 > \frac{3}{4}.$$

Proceeding to Stage 2, we have that $\mathcal{CP}(\bar{x}) = \{1, 2, 3\}$, and because all players' costs are increasing at $\bar{x}$ the algorithm proceeds to Stage 4. Because $\mathcal{A}^+(\bar{x}) = \mathcal{CP}(\bar{x}) = \{1, 2, 3\}$, (6) tells us that $G_i(y) = 1 - (1 - y)^{1/2}$ for every player i and score y in $[\bar{x}, 1]$, since players' CDFs reach 1 at 1. The equilibrium is depicted in Figure 10.

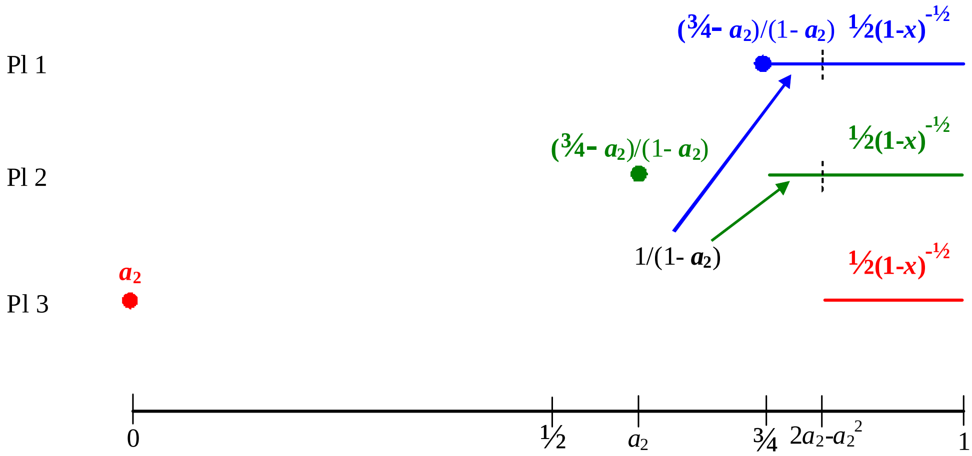

Figure 10: Players' atoms and densities in the equilibrium when $a_2 > \frac{1}{2}$

Consider how players' equilibrium behavior changes as a_2 increases from 0 to $3/4$. For a_2 in $(0, 1/2)$, the size of player 2 and 3's atoms at their respective head starts is a_2. The probability that player 3 does not participate increases in a_2, and when he does participate he chooses whether to compete only against player 2 or to compete against players 1 and 2. Player 2 takes advantage of this by increasing the probability he does not invest and by competing only against player 3 some of the time. But from player 1's perspective, the value of a_2 in $(0, 1/2)$ does not matter: with probability $1/2$ he does not invest, capitalizing on the fact that players 2 and 3 do not compete against him some of the time, and all three players' behavior in the region in which all three players compete is independent of a_2. When $a_2 = 1/2$, part of the time players 2 and 3 do not invest, but when they do invest, they do so in the region in which all three players compete. For a_2 in $(1/2, 3/4]$, as a_2 increases the size of player 1 and 2's atoms decreases and the players compete more aggressively against each other in the region immediately above $3/4$. When player 3 participates, he competes only in the upper region in which players 1 and 2 compete. As a_2 increases, all three players become more aggressive, conditional on investing. In particular, when a_2 is large enough player 3 competes only at the very top, conditional on participating. Contrast this "all or nothing" behavior with player 3's "tiered competition" that results for lower values of a_2.

Now compare this contest to a three-player, two-prize all-pay auction in which players do not have head starts but have different valuations for a prize. Suppose that player i's valuation is $V_i = 1 + a_i$. Then every player i's power, and therefore payoff, is a_i in both contests. What about players' equilibrium expenditures? In the all-pay auction, a stronger player expends more than a weaker one, i.e., player 1's expenditures are the highest, then player 2's, and then player 3's. This follows from Corollary 5 in Siegel (2009b), which states that the equilibrium CDF of a player first-order stochastically dominates (FOSD) the CDFs of all weaker players. In the all-pay auction with head starts considered above, this ranking of expenditures is reversed. To see this, consider players' equilibrium strategies in Figure 8. Player 1 and 2's CDFs coincide on $[3/4, 1]$, so on this interval player 1's expenditures are lower (because of his head start), and player 2 also makes additional expenditures in the interval $[a_2, 1/2]$. Player 2 and 3's CDFs coincide on $[a_2, 1]$, so on this interval player 2's expenditures are lower (because of his head start), and players 2 and 3 do not make additional expenditures outside this interval. The same arguments apply to the equilibrium depicted in Figure 9. These arguments also show that player 1's expenditures are lower than those of player 2 in the equilibrium depicted in Figure 10. To compare player 2's expenditures to those of player 3 in Figure 10, note that player 3's cost of every score in $[2a_2 - a_2^2, 1]$ is a_2 more than that of player 2. Since both players choose scores in $[2a_2 - a_2^2, 1]$ with probability $1 - a_2$, and their CDFs conditional on choosing a score in $[2a_2 - a_2^2, 1]$ are the same, player 3 expends $a_2 (1 - a_2)$ more than player 2 in this interval. Player 2 also expends

$$\left(\frac{\frac{3}{4} + 2a_2 - a_2^2}{2} - a_2\right) \cdot \frac{1}{1 - a_2}\left(\left(2a_2 - a_2^2\right) - \frac{3}{4}\right)$$

in the interval $[3/4, 2a_2 - a_2^2]$, which can be shown to be less than $a_2 (1 - a_2)$ for every a_2 in $[1/2, 3/4]$. Therefore player 2's expenditures are lower than those of player 3.

Regarding all-pay auctions with head starts more generally, I have the following conjecture.

Conjecture 1 *The following statements hold for the equilibrium* $\mathbf{G}$ *described in Corollary 1.*

1. *Whenever player j is active at a score $y > a_i$ for $i < j$, then player i is also active and $G_i(y) = G_j(y)$.*

2. *G_i FOSD G_j for all $i < j$. This implies that every player i wins a prize with higher probability than every player $j > i$.*

3. *Players' expenditures are ranked, so that player i's expenditures are lower than those of player j if and only if $i < j$.*

4. *All players are active on $[x, V + a_{m+1}]$ for some $x < V + a_{m+1}$. The probability that players choose scores in this interval increases in V and approaches 1 as V grows to infinity.*

5 Additional Applications

The equilibrium construction result can be used to address various issues in contest design. Examples include optimal handicaps, scoring rules, and number and value of prizes. Each specification of these parameters corresponds to a contest, and optimizing over the range of parameters can guide a designer how to maximize some desired function of effort or score. The contests considered need not, of course, be limited to linear costs or identical valuations.

The framework can also be used to model discontinuities in players' costs and non-convexities in their strategy spaces. A finite number of discontinuities in a player's cost function can be accommodated by "inserting" a segment of scores for every player at every point of discontinuity and extending the players' cost functions so that the resulting cost functions are continuous. A player for whom the point is not a point of discontinuity has constant costs on the inserted interval. Taking the limit as the length of these segments goes to 0 provides the tie-breaking rule necessary to support the limiting equilibrium. Discontinuous costs can model entry fees and fixed costs incurred at various levels of investment. They can also model a contest designer who ranks contestants coarsely based on their output, so that an interval of output level is lumped together. Discrete strategy spaces may be modeled similarly.

A Appendix

I denote by $\widetilde{G}=\left(\widetilde{G}_1,\ldots,\widetilde{G}_{m+1}\right)$ an equilibrium of an $(m+1)$-player contest, where $\widetilde{G}_i$ is player i's strategy, so $\widetilde{G}_i(x)$ is the probability that player i chooses a score lower than or equal to x.

A.1 Proof of Lemma 1

The proof relies on the following lemma.

Lemma 4 *Suppose that in $\widetilde{G}$ no player chooses weakly dominated scores, and denote by $s_{\inf}$ the infimum of the union of the best response sets of players $1,\ldots,m$. Then $\widetilde{G}_{m+1}(s_{\inf})>0$, $\widetilde{G}_i(s_{\inf})=0$ for every player $i<m+1$, and $s_{\inf}=x_0$.*

Proof. By definition of $s_{\inf}$, there exists a player $i<m+1$ and a set of scores $\{x_k\}_{k=1}^{\infty}$ that are best responses for player i such that $x_k\to s_{\inf}$. Because player i's power is positive, his probability of winning when choosing any score x_k in the set is bounded away from 0, so by right-continuity of players' CDFs there is a player $j\neq i$ with $\widetilde{G}_j(s_{\inf})>0$. Suppose in contradiction that $\widetilde{G}_{m+1}(s_{\inf})=0$. Then $j\neq m+1$, and because $\widetilde{G}_j(x)=0$ for every $x<s_{\inf}$, player j has an atom at $s_{\inf}$. The same argument for player j instead of player i shows that there is another player, in addition to player j, with an atom at $s_{\inf}$. Because $\widetilde{G}_{m+1}(s_{\inf})=0$, not all players with an atom at $s_{\inf}$ win a prize with probability 1 when choosing $s_{\inf}$, so by the Tie Lemma of Siegel (2009a) (henceforth: Tie Lemma) they win a prize with probability 0 and have a payoff of 0 when choosing $s_{\inf}$. But players $1,\ldots,m$ have a positive payoff, a contradiction. Thus, $\widetilde{G}_{m+1}(s_{\inf})>0$. Because $\widetilde{G}_{m+1}(x)=0$ for $x<x_0$, we have that $x_0\leq s_{\inf}$. If $x_0=s_{\inf}$, then the marginal player has an atom at $s_{\inf}$, so by the Tie Lemma $\widetilde{G}_i(s_{\inf})=0$ for $i<m+1$. Therefore, to complete the proof it suffices to rule out $x_0<s_{\inf}$. Suppose in contradiction that $x_0<s_{\inf}$. By choosing a score x in $(x_0,s_{\inf})$ the marginal player wins a prize with probability 0 and incurs a positive cost, so no score in $(x_0,s_{\inf})$ is a best response for him. Therefore, if $\widetilde{G}_{m+1}(s_{\inf})>\widetilde{G}_{m+1}(x_0)$ then the marginal player has an atom at $s_{\inf}$, which implies that $s_{\inf}$ is a best reply for him. This means that another player has an atom at $s_{\inf}$ (because $c_{m+1}(s_{\inf})>c_{m+1}(x_0)=0$), which contradicts the Tie Lemma. If $\widetilde{G}_{m+1}(s_{\inf})=\widetilde{G}_{m+1}(x_0)<1$, consider a player $i<m+1$ such that (i) the player has best responses that approach $s_{\inf}$ and (ii) no other player $j<m+1$ has an atom at $s_{\inf}$ (the Tie Lemma guarantees that such a player i exists). Then $s_{\inf}$ is a best response or player i. But the CDFs of players other than i do not increase on $(x_0,s_{\inf}]$, so all scores in $(x_0,s_{\inf})$ are at least as good as $s_{\inf}$ for player i, contradicting the definition of $s_{\inf}$. Finally, if $\widetilde{G}_{m+1}(s_{\inf})=\widetilde{G}_{m+1}(x_0)=1$ then all scores in $(x_0,s_{\inf}]$ are at least as good as $s_{\inf}$ for any player i who has best responses that approach $s_{\inf}$, contradicting the definition of $s_{\inf}$. ■

To complete the proof, it remains to show that $\widetilde{G}_{m+1}(x_0) = \min_{i<m+1}\left(\frac{c_i(x_0)+w_i}{V_i}\right)$. If $\widetilde{G}_{m+1}(x_0)$ were larger, some player $i < m+1$ would obtain more than his power at scores immediately above x_0. If $\widetilde{G}_{m+1}(x_0)$ were smaller, then by Lemma 4 and right-continuity of $\widetilde{G}$ no player $i < m+1$ would have best responses in $[x_0, x_0+\varepsilon]$ for some $\varepsilon > 0$. This would contradict $s_{\text{inf}} = x_0$.

A.2 Proof of Lemma 2

Suppose in contradiction that $\widetilde{G}_j(y) < 1$ for some player j at a score y that is not a best response for any player. This implies that no player has an atom at y, so by continuity of players' cost functions and CDFs at y, no player has best responses in some neighborhood of y. Denote by BR_y^k the set of player k's best responses above y, and by $s_{\text{inf},y}$ the infimum of $\cup_{k=1}^{m+1} BR_y^k$. Because $G_j(y) < 1$, BR_y^j is not empty, so $s_{\text{inf},y} > y$ is well defined and no player has best responses in $[y, s_{\text{inf},y})$. If $s_{\text{inf},y}$ were not a best response for at least one player, then by the argument above no player would have best responses in a neighborhood of $s_{\text{inf},y}$, contradicting the definition of $s_{\text{inf},y}$. Consider a player k for whom $s_{\text{inf},y}$ is a best response. Because $c_k(s_{\text{inf},y}) \geq c_k(y)$ and y is not a best response for player k, he must win a prize with higher probability when choosing $s_{\text{inf},y}$ than when choosing y. This means that some player other than k has an atom at $s_{\text{inf},y}$, because no player has best responses in $[y, s_{\text{inf},y})$. This implies that $s_{\text{inf},y}$ is a best response for that other player, which requires another player to have an atom at $s_{\text{inf},y}$. Therefore, there are at least two players with an atom at $s_{\text{inf},y}$. By the Tie Lemma, either all the players with an atom at $s_{\text{inf},y}$ win a prize with probability 0 or all win a prize with probability 1 when choosing $s_{\text{inf},y}$. If the former is true, then y is a best response for the players with an atom at $s_{\text{inf},y}$, a contradiction. If the latter is true, then denote by N' the set of players l for whom $\widetilde{G}_l(s_{\text{inf},y}) = 1$. This set is non-empty, otherwise no player could win a prize with probability 1 by choosing $s_{\text{inf},y}$. Moreover, every player in N' has an atom at $s_{\text{inf},y}$, otherwise $\widetilde{G}_l(y) = 1$ for a player l in N' so scores slightly below $s_{\text{inf},y}$ would be best-responses for every player with an atom at $s_{\text{inf},y}$, a contradiction. Now consider the positive-probability event that the $(m+1) - |N'|$ players not in N' choose scores strictly higher than $s_{\text{inf},y}$ and the $|N'|$ players in N' choose $s_{\text{inf},y}$.[24] Conditional on this event, $m - ((m+1) - |N'|) = |N'| - 1$ prizes are divided among the $|N'|$ players in N', so the players in N' cannot all win a prize with probability 1 when choosing $s_{\text{inf},y}$, a contradiction.

A.3 Proof of Lemma 3

The proof requires two lemmas.

Lemma 5 *Suppose that a checkpoint $x < T$ has been reached with $G_i(x) < 1$ for every player i,*

[24] If $|N'| = m+1$ then this event is simply the event that all players choose $s_{\text{inf},y}$.

so Stage 2 is executed. If the algorithm proceeds to Stage 3, then the next checkpoint $\bar{x}$ satisfies (i) $\bar{x} < T$ and (ii) $G_i(x) \leq G_i(\bar{x}) \leq 1$ for every player i.

Proof. By definition of the checkpoint $\bar{x}$ identified in Stage 3, $\mathcal{A}^+(x)$ contains at least one player i for whom $c_i(x) = c_i(\bar{x})$. Because player i is in $\mathcal{CP}(x)$, (2) holds. Because $G_j(x) < 1$ for every player j, (2) implies that $V_i - c_i(x) > w_i$. By definition, $w_i = V_i - c_i(T)$, so $c_i(x) < c_i(T)$ and consequently $c_i(\bar{x}) < c_i(T)$, which implies that (i) holds. For (ii), recall that Stage 3 specifies to set $G_j(\bar{x}) = G_j(x)$ for every player $j \neq i$. To see that $G_i(x) \leq G_i(\bar{x}) \leq 1$, note that $\bar{x} < T$ implies $V_j - c_j(\bar{x}) \geq w_j$ for every player j. Therefore, the lowest value of $G_i(\bar{x})$ such that a player other than i is in $\mathcal{CP}(\bar{x})$ is at most 1. Moreover, because for every player $\tilde{\imath}$ either (1) or (2) holds with $\tilde{\imath}$ instead of i, $c_{\tilde{\imath}}(\bar{x}) \geq c_{\tilde{\imath}}(x)$, and $G_j(\bar{x}) = G_j(x)$ for every player $j \neq i$, the lowest value of $G_i(\bar{x})$ such that a player other than i is in $\mathcal{CP}(\bar{x})$ is at least $G_i(x)$. ■

Lemma 6 *Suppose that a checkpoint $x < T$ has been reached with $G_i(x) < 1$ for every player i, so Stage 2 is executed. If the algorithm proceeds to Stage 4, then the next checkpoint $\bar{x}$ satisfies (i) $\bar{x} \leq T$, (ii) G_i is non-decreasing on $[x, \bar{x}]$ and $G_i(\bar{x}) \leq 1$ for every player i, and (iii) if $x = T$ then $G_i(\bar{x}) = 1$ for some player i.*

Proof. The extension of $\mathbf{G}$ to $[x, \bar{x}]$ in Stage 4 is such that the hazard rate $-\frac{(1-G_i(y))'}{G_i(y)}$ (where the derivative is a right-derivative) of every player i in $\mathcal{A}^+(x)$ at scores y immediately above x equals $H(x, y) - \varepsilon_i(y) \geq 0$. This shows that $\mathbf{G}$ is non-decreasing immediately above x. The checkpoint $\bar{x}$ is chosen so that if player i's hazard rate is positive immediately above x, then his hazard rate remains positive up to $\bar{x}$. Therefore, $\mathbf{G}$ is non-decreasing in $[x, \bar{x}]$. Also, if $\bar{x} \leq T$ then $G_i(\bar{x}) \leq 1$ for every player i, because $\bar{x}$ is chosen so that when $\mathbf{G}$ is extended to $[x, \bar{x}]$ no player can obtain more than his power by choosing a score in $[x, \bar{x}]$. If $\bar{x} \geq T$, then the CDF of at least one player is 1 at T when $\mathbf{G}$ is extended as specified in Stage 4, because for a player to obtain his power by choosing T he must win a prize with probability 1, and every player in $\mathcal{CP}(x)$ obtains his power in $[x, \bar{x}]$. Thus $\bar{x} \geq T$ implies $\bar{x} = T$. ■

To prove Lemma 3, first suppose that $G_{m+1}(x_0) = 1$. Then Stage 1 is followed by Stage 5. By Condition C3 part (ii), $c_{m+1}(T) > c_{m+1}(0) = 0$, so $x^L = x_0 < T$. For the remainder of the proof, suppose that $G_{m+1}(x_0) < 1$, and recall that by definition $G_i(x_0) = 0$ for every player $i \neq m+1$. Lemmas 5 and 6 suggest two possibilities. The first possibility is that $x^L \leq T$ with $G_i(x^L) = 1$ for some player i is reached after Stages 3 and 4 are executed a finite number of times. In this case, the statement of the lemma holds. The second possibility is that Stage 3 or 4 (or both) are executed infinitely many times. To complete the proof, it suffices to rule out this possibility. That Stage 3 is executed at most a finite number of times follows because the checkpoint identified in Stage 3 is the endpoint of a maximal interval on which a player's cost function in constant, and every player's cost function is constant on at most a finite number of such intervals because cost functions are piecewise analytical. If Stage 4 is executed an infinite number of times, then there exists a sequence of consecutive checkpoints $x_1 < x_2 < \ldots$, all identified via iterations of Stage 4. By Lemmas 5 and 6, and because Stage 5 is not reached, the value of every player's CDF at every checkpoint in the sequence is less than 1, and $\lim_{k\to\infty} x_k \leq T$. The proof of Lemma 7 in Siegel (2009b) shows that there is only a finite number of checkpoints x_k in the sequence for which $\mathcal{A}^+(x_k) \neq \mathcal{A}^+(x_{k+1})$. This means that there is an infinite subsequence of consecutive checkpoints $x_{i_1} < x_{i_2} < \ldots$ in some subinterval $[a,b]$ of $[0,T]$ for which $\mathcal{A}^+(x_{i_1}) = \mathcal{A}^+(x_{i_2}) = \ldots$. This, together with (5) and the fact that the CDFs of players not in $\mathcal{A}^+(\cdot)$ do not increase, means that $\mathbf{G}$ is piecewise analytic on $[a,b]$.[25] And because players' cost functions are piecewise analytic, the criteria specified in Stage 4 for identifying the next checkpoint mean that there can only be a finite number of checkpoints in $[a,b]$, a contradiction.

A.4 Lemmas 7 and 8

Lemma 7 *At every checkpoint x that has been reached, $\mathcal{CP}(x)$ contains at least two players, and for every player i not in $\mathcal{CP}(x)$ (1) holds.*

Proof. The proof is by induction on the number of checkpoints that have been reached. The first checkpoint is x_0, and $\mathbf{G}(x_0)$ is set to the lowest value such that $\mathcal{CP}(x_0)$ contains player $m+1$ and at least one other player, so for every player i not in $\mathcal{CP}(x_0)$ (1) holds. For the induction step, consider two consecutive checkpoints $x < \bar{x}$. By the induction hypothesis, $\mathcal{CP}(x)$ contains at least two players, and for every player i not in $\mathcal{CP}(x)$ (1) holds. The first possibility is that $\mathcal{CP}(x)$ contains at least one player i whose cost function is constant at x. Then $\bar{x}$ is identified in Stage 3, and $\mathbf{G}(\bar{x})$ is set to the lowest value such that $\mathcal{CP}(\bar{x})$ contains player i and at least one other player, so for every player i not in $\mathcal{CP}(\bar{x})$ (1) holds with $\bar{x}$ instead of x. The second possibility is that the cost functions of all players in $\mathcal{CP}(x)$ are increasing at x. Then $\bar{x}$ is identified in Stage 4, and $\mathbf{G}$ is defined on $[x,\bar{x}]$ so that $\mathcal{A}^+(x) \subseteq \mathcal{CP}(\bar{x})$, and for every player i not in $\mathcal{CP}(\bar{x})$ (1) holds with $\bar{x}$ instead of x. Because $\mathcal{A}^+(x)$ identified in Stage 4 contains at least two players, we are done. ■

Lemma 8 *If c_i is increasing at $x < T$, then $q_i(y) > 0$ and $\varepsilon_i(y) > 0$ for every y in some open right-neighborhood of x.*

Proof. Because $q_i(y) = 1 - \frac{w_i + c_i(y)}{V_i}$, for the first inequality it suffices to show that $w_i + c_i(y) < V_i$, or $c_i(y) < c_i(T)$. Because c_i is increasing at x, $c_i(x) < c_i(T)$, so by continuity the same is true for scores y immediately above x. For the second inequality, because $\varepsilon_i(y) = \frac{c_i'(y)}{V_i - w_i - c_i(y)}$ and we have seen that $w_i + c_i(y) < V_i$, it suffices to show that $c_i'(y) > 0$. If this were not true in some right-neighborhood of x, then by analyticity c_i would be constant on some right-neighborhood of x, contradicting the fact that c_i is increasing at x. ■

[25] A proof similar to that of Lemma 4 in Siegel (2009b) shows that $\mathbf{G}$ is analytic at b.

A.5 Proof of Theorem 3

To prove that $\mathbf{G}$ is an equilibrium, it suffices to show three things: (i) $\mathbf{G}$ is a profile of CDFs (non-decreasing, right-continuous, and $\lim_{x\to\infty}\mathbf{G}(x) = 1$), (ii) no player can obtain a payoff higher than his power by choosing any score when the other players choose scores according to $\mathbf{G}$, and (iii) every player i assigns G_i-measure 1 to scores that give him a payoff equal to his power when the other players choose scores according to $\mathbf{G}$. (i) follows from the construction of $\mathbf{G}$, Lemma 3, and part (ii) of Lemmas 5 and 6. For (ii), by construction of $\mathbf{G}$ no player can obtain more than his power by choosing a score in $\left[0, x^L\right]$, where x^L is the last checkpoint identified by the algorithm. Also, by definition of power no player can obtain more than his power above T. It therefore suffices to show that no player can obtain more than his power by choosing a score in $\left(x^L, T\right)$. The following lemma shows that if $x^L < T$, then $G_{m+1}\left(x^L\right) = 1$ and $c_j\left(x^L\right) = c_j(T)$ for every player $j < m+1$. This is enough: by Stage 5, the marginal player cannot increase his probability of winning a prize by increasing his chosen score from x^L to a score in $\left(x^L, T\right)$, and every other player obtains his power by choosing a score in $\left(x^L, T\right)$.

Lemma 9 *If $x^L < T$, then $G_{m+1}\left(x^L\right) = 1$ and $c_j\left(x^L\right) = c_j(T)$ for every player $j \neq i$.*

Proof. By definition, $G_i\left(x^L\right) = 1$ for some player i, and x^L is identified in Stage 1, 3, or 4. If x^L is identified in Stage 1, then $x^L = x_0$ and

$$1 = G_{m+1}(x_0) = \min_{i<m+1}\left(\frac{c_i(x_0) + w_i}{V_i}\right) = \min_{i<m+1}\left(1 - \left(c_i(T) - c_i(x_0)\right)\right),$$

which implies that $c_j\left(x^L\right) = c_j(T)$ for all $j < m+1$. If x^L is identified in Stage 3 or 4, then consider a player i for which $G_i\left(x^L\right) = 1$. Player i is in $\mathcal{A}^+(x)$, where x is the checkpoint preceding x^L, and is therefore in $\mathcal{CP}\left(x^L\right)$. Suppose $i < m+1$. In Stage 3 this means that the marginal player obtains strictly more than his power immediately above x^L (Assumption C3 part (ii)), so $G_i\left(x^L\right)$ can be set lower so that the marginal player is in $\mathcal{CP}\left(x^L\right)$. This contradicts the definition of $G_i\left(x^L\right)$ in Stage 3 as the lowest value such that $\mathcal{CP}\left(x^L\right)$ contains at least two players. In Stage 4 this means that the marginal player can obtain strictly more than his power immediately below x^L (by continuity of the extension of $\mathbf{G}$ in Stage 4), contradicting the property of the extension of $\mathbf{G}$ in Stage 4 that no player can obtain more than his payoff in $\left[x, x^L\right]$. Given that $G_{m+1}\left(x^L\right) = 1$, the same contradictions showed for the marginal player hold for any player $j < m+1$ with $c_j\left(x^L\right) < c_j(T)$. ■

For (iii), note that a player's CDF increases continuously only in Stage 4, and $\mathbf{G}$ is extended in Stage 4 in such a way that all players whose CDFs increase obtain their power (because they are all in $\mathcal{A}^+(x)$). It remains to show that a player obtains precisely his power whenever he has an atom. By construction, a player can be assigned an atom in Stage 1, 3, or 5. Stage 1 assigns an atom only to the marginal player, at x_0, and the marginal player obtains a payoff of 0 (equal to his power) when choosing x_0. Each execution of Stage 3 identifies a checkpoint $\bar{x}$

and assigns an atom to at most one player, at $\bar{x}$. That player obtain his power at $\bar{x}$, because he is in $\mathcal{CP}(\bar{x})$ and there is no tie at $\bar{x}$ (because no other player has an atom there). For Stage 5, consider two cases. If $x^L < T$, then Lemma 9 shows that Stage 5 assigns atoms to a subset of players $1, \ldots, m$ at scores at which they win with probability 1 and whose cost equals their cost of choosing the threshold, so they obtain their power by choosing these scores. If $x^L = T$, then x^L is identified in Stage 4 (part (i) of Lemma 5). Because the extension of $\mathbf{G}$ in Stage 4 is continuous, the players whose CDF reaches 1 in Stage 4 at x^L do not have an atom at $x^L = T$. Thus, any player j with an atom at $x_j^{\max} \geq T$ wins a prize with probability 1 by choosing $x_j^{\max}$. And by Stage 5, $c_j\left(x_j^{\max}\right) = c_j\left(x^L\right) = c_j(T)$. Therefore player j obtains his power by choosing $x_j^{\max}$.

It remains to show that players do not choose weakly dominated scores in $\mathbf{G}$. Recall that a weakly-dominated score for player i is a score x such that $c_i(x) = c_i(y)$ for a score $y > x$. Consider the stages of the algorithm in which a player's CDF may increase. If a player's CDF increases in Stage 4, then his cost function is increasing (the cost functions of all candidate players in Stage 4 are increasing). If a player's CDF increases in Stages 1, 3, or 5, then he has an atom, and by construction that atom is at the upper endpoint of a maximal interval in which the player's cost function is constant. In both cases the player does not choose weakly-dominated scores.

A.6 Proof of Theorem 4

Suppose that in $\widetilde{G}$ no player chooses weakly dominated scores. I will show that $\mathbf{G} = \widetilde{G}$. The following lemma shows that if $\mathbf{G}$ and $\widetilde{G}$ differ, they differ at scores lower than x^L, the last checkpoint in $\mathbf{G}$.

Lemma 10 *If $\mathbf{G}\left(x^L\right) = \widetilde{G}\left(x^L\right)$, then $\mathbf{G}(y) = \widetilde{G}(y)$ for all $y > x^L$.*

Proof. Because $G_i\left(x^L\right) = \widetilde{G}_i\left(x^L\right) = 1$ for some player i (which also implies $G_i(y) = \widetilde{G}_i(y) = 1$ for all $y > x^L$) no other player j has best responses in $\mathbf{G}$ or $\widetilde{G}$ at scores y for which $c_j(y) > c_j\left(x^L\right)$. So because in $\widetilde{G}$ no player chooses weakly dominated scores, $\widetilde{G}_j(y) = \widetilde{G}_j\left(x^L\right)$ for every score y in $\left(x^L, x_j^{\max}\right)$, and $\widetilde{G}_j(y) = 1$ for every score $y \geq x_j^{\max}$, where $x_j^{\max}$ is defined as in Stage 5. Therefore, $\mathbf{G}(y) = \widetilde{G}(y)$ for all $y > x^L$. ■

For the next step, divide the checkpoints in $\mathbf{G}$, excluding x^L, into "constant checkpoints" and "increasing checkpoints." A constant checkpoint leads to Stage 3, i.e., it is a checkpoint at which at least one active player's cost function is constant. An increasing checkpoint leads to Stage 4, i.e., it is a checkpoint at which all active players' cost functions are increasing. The next lemma shows that if $\mathbf{G}$ and $\widetilde{G}$ coincide at a constant checkpoint, then they coincide at all scores up to the following checkpoint. Denote by x_k and x_{k+1} two consecutive checkpoints.

Lemma 11 *If x_k is a constant checkpoint and $\mathbf{G}(x_k) = \widetilde{G}(x_k)$, then $\mathbf{G}(y) = \widetilde{G}(y)$ for all y in $[x_k, x_{k+1}]$.*

Proof. By construction, because x_k is a constant checkpoint, there is a player i in $\mathcal{CP}(x_k)$ for which $c_i(x_k) = c_i(x_{k+1})$. This means that $\widetilde{G}_j(y) = \widetilde{G}_j(x_k)$ for every other player j and y in $[x_k, x_{k+1}]$, otherwise player i would obtain more than his power in $\widetilde{G}$. Because no player chooses weakly dominated scores in $\widetilde{G}$, $\widetilde{G}_i(y) = \widetilde{G}_i(x_k) = G_i(x_k) = G_i(y)$ for every y in $[x_k, x_{k+1})$. It remains to show that $G_i(x_{k+1}) = \widetilde{G}_i(x_{k+1})$. This is true because $G_i(x_{k+1}) < \widetilde{G}_i(x_{k+1})$ implies that some player other than i obtains more than his power immediately above x_{k+1}, and $G_i(x_{k+1}) > \widetilde{G}_i(x_{k+1})$ implies that Lemma 2 is violated immediately above x_{k+1}, as explained in the paragraph preceding the statement of Lemma 2. ■

Suppose that $\mathbf{G} \neq \widetilde{G}$ and x_k is the highest checkpoint such that $\mathbf{G}(y) = \widetilde{G}(y)$ for every y in $[0, x_k]$ (Lemmas 1 and 10 show that x_k and x_{k+1} are well defined). Lemma 11 shows that x_k is an increasing checkpoint. The following lemma shows that $\widetilde{\mathcal{CP}}(x) \neq \mathcal{CP}(x)$ for some x in (x_k, x_{k+1}), where is $\widetilde{\mathcal{CP}}(x)$ the set of players i for (2) holds with $\widetilde{G}_i$ instead of G_i (the players for whom x is a best response in $\widetilde{G}$).

Lemma 12 *Suppose that $\widetilde{G}(x_k) = \mathbf{G}(x_k)$ and $\widetilde{\mathcal{CP}}(x) = \mathcal{CP}(x)$ for every x in (x_k, x_{k+1}). Then $\widetilde{G}(x) = \mathbf{G}(x)$ for every $x \in [x_k, x_{k+1}]$.*

Proof. By construction, $\mathcal{CP}(x) = \mathcal{A}^+(x_k)$ for every x in (x_k, x_{k+1}). This implies that for every player i not in $\mathcal{A}^+(x_k)$ and every x in (x_k, x_{k+1}) we have $\widetilde{G}_i(x) = \widetilde{G}_i(x_k) = G_i(x_k) = G_i(x)$, because player i does not have best responses in (x_k, x_{k+1}) so his CDF does not increase (in both equilibria). For every player i in $\mathcal{A}^+(x_k)$ and every x in (x_k, x_{k+1}) (2) holds with $\widetilde{G}_i$ instead of G_i, so $\widetilde{G}_i$ is given by (5). This shows that $\widetilde{G}(x) = \mathbf{G}(x)$ for every $x \in [x_k, x_{k+1})$. To see that $\widetilde{G}(x_{k+1}) = \mathbf{G}(x_{k+1})$, suppose first that $x_{k+1} < x^L$. In this case, by construction $\mathbf{G}$ is continuous at x_{k+1}, so the same must true of $\widetilde{G}$ otherwise a player in $\mathcal{A}^+(x_k)$ would get too high a payoff in $\widetilde{G}$. Therefore, $\widetilde{G}(x_{k+1}) = \mathbf{G}(x_{k+1})$. Now suppose that $x_{k+1} = x^L$. By construction, $G_i\left(x^L\right) = 1$ and G_i is continuous at x^L for some player i, so the same is true of $\widetilde{G}$. Consider another player j. Because in both $\widetilde{G}$ and $\mathbf{G}$ players do not choose weakly-dominated scores, if player j's cost function is constant at x^L then he does not have an atom at x^L, so $\widetilde{G}_j\left(x^L\right) = \lim_{x \to x^L} \widetilde{G}_j(x) = \lim_{x \to x^L} G_j(x) = G_j(x)$. If player j's cost function is increasing at x^L, then in both $\widetilde{G}$ and $\mathbf{G}$ he does not have best responses above x^L (he wins with probability 1 by choosing x^L), so $\widetilde{G}_j\left(x^L\right) = G_j(x) = 1$. ■

I conclude the proof by showing that $\widetilde{\mathcal{CP}}(x) \neq \mathcal{CP}(x)$ is impossible. That $\widetilde{\mathcal{CP}}(x) \subseteq \mathcal{CP}(x)$ follows from Lemma 9 in Siegel (2009b), which applies here.[26] This inclusion means that in both

[26] In the proof of the lemma replace $v_i(x)$ with $V_i - c_i(x)$ for every player i, and note that here we have $\bar{G}(x_k) = \mathbf{G}(x_k)$ by definition of x_k.

equilibria only the CDFs of players in $\mathcal{A}^+(x_k)$ increase on (x_k, x_{k+1}). Because x_k is an increasing checkpoint, the costs of all players in $\mathcal{A}^+(x_k)$ are increasing at every x in (x_k, x_{k+1}). Therefore, in both equilibria the CDFs of these players are continuous on (x_k, x_{k+1}).[27] Lemmas 10 and 11 in Siegel (2009b) now imply that $\widetilde{\mathcal{CP}}(x) \supseteq \mathcal{CP}(x)$.

[27] If a player had an atom in (x_k, x_{k+1}), then because no player's CDF reaches 1 before x_{k+1}, no other player would choose scores immediately below x_k. But then the player would be better off by shifting the mass of his atom to a lower score. The proof of Lemma 1 in Siegel (2009a) provides a more detailed argument.

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