# INTERGENERATIONAL EQUITY AND THE DISCOUNT RATE FOR COST-BENEFIT ANALYSIS

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Abstract. We develop a simple method to evaluate small policy changes affecting several generations, by reducing the dynamic problem to a static one. A necessary condition is time-invariance, which is satisfied by any common solution concept in an overlapping generations model with exogenous growth. The method is applied to derive the discount rate for cost-benefit analysis under two different utilitarian welfare functions: traditional and relative. It is only under relative utilitarianism that the discount rate is well-defined for a heterogeneous society, is corroborated by an independent argument on the value of human life, and equals the growth rate of per capita consumption, thus falling in the range suggested by the U.S. Office of Management and Budget.

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## Contents

## 1. Introduction

1.1. **Motivation.** The usual difficulty in evaluating policies affecting yet-to-be-born generations is two-fold. First, the choice of the relevant criterion is non-trivial, sparking ethical debates on intergenerational equity; second, the evaluation itself — 'comparative dynamics' in infinite economies — is often done using numerical methods,[1] the direct computation being intractable.

The tool we develop solves the last problem by reducing the evaluation of small policy changes to the commonly-practiced present-value calculation with a fixed discount rate. More precisely, the derivative of welfare with respect to policy perturbations is the product of a static and a dynamic component, the latter being an exponential function of time. The key requirement for the result, time-invariance, is satisfied, for example, at an equilibrium selection by local uniqueness in the neighbourhood of a balanced growth equilibrium (BGE), in an overlapping generations economy with exogenous growth (OG).

Rather than entering ethical debates on intergenerational equity and the related choice of policies, we derive a discount rate for cost-benefit analysis—i.e., for small (and non-permanent) changes—, using two different welfare functions, traditional and relative utilitarian. In the latter case individual utilities are 0-1 normalised over the space of feasible policies, and then summed.[2] Although our approach is not applicable for "big" policy changes, e.g., those that might affect global climate, we can test the two different utilitarian criteria by their implications for small changes, where the differential approach is applicable.

Hence the intent is to focus on the choice of the underlying social objectives rather than on relative merits of different policies. This echoes the suggestion of Stern (2006, p. 23) to "go back to the first principles from which the standard marginal results are derived." An additional benefit of doing so is that the methods apply then also to problems involving dynamics and risk in heterogeneous societies. In particular, the main result can be used in a general OG model satisfying the minimal set of assumptions needed for balanced growth, see sect. 3.

1.2. **Difficulties with the traditional approach.** First, consider the standard derivation of the discount rate, closely related to the one suggested by Arrow and Kurz (1970), and used, e.g., in the analysis of Stern (2006). Take a discrete time model where individuals live for just one period, with utility function $U(c_t) = \frac{c_t^{1-\rho}}{1-\rho}$ where $\rho > 0$. The economy is on a balanced growth path with per-capita consumption growing exponentially at rate $\gamma > 0$, production being 'black-boxed' for now. The status-quo per-capita consumption at time $t$ is $c_0 e^{\gamma t}$, with $c_0 > 0$. Consider a policy that involves a variation in aggregate

[1] See e.g. de la Croix and Michel (2002) for an overview.

[2] See Dhillon and Mertens (1999) for axiomatisation.

consumption $\delta C_t$ for each generation $t$. It is to be evaluated at time 0, using the traditional criterion, $W = \sum_t e^{-\beta t} N_t U(c_t)$, where $N_t$ is the number of individuals at time $t$. Then the net (social) benefit equals

$$\sum_t e^{-\beta t} N_t U'\big(c_0 e^{\gamma t}\big) \cdot \frac{\delta C_t}{N_t} = \sum_t c_0^{-\rho} e^{-(\rho\gamma+\beta)t} \delta C_t$$

Thus, the traditional approach implies $\beta + \rho\gamma$ as (real) discount rate. This conclusion is robust, see cor. 4. To identify the problems associated with this approach, we start by interpreting each parameter.

$\gamma$ is the growth of real per-capita GDP, say 2% per year.[3]

Recall, $\beta$ is the discount rate applied to *utilities*, and the only reason a true utilitarian, wishing to treat all generations equally,[4] would accept $\beta > 0$ is uncertainty about future 'world' existence. Note that, in contrast with the representative agent models, $\beta$ has nothing to do with the prevailing interest rate, nor with individual intertemporal preferences (individuals live for 1 period). It is a purely normative parameter, so there is hardly a way to estimate it from observed behaviour.[5]

The parameter $\rho$ admits three rather distinct interpretations. In the report by Stern (2006) it stands for *the elasticity of the social marginal utility of consumption*. In this case individual utilities are assumed purely ordinal, so that, as in Arrow (1963), the value of $\rho$ must stem from normative considerations.[6] If on the contrary one adopts a utilitarian approach, then $\rho$ is an individual characteristic, either standing for the *income elasticity of the individual marginal utility of income*, or, as in Harsanyi (1955), for the coefficient of relative risk aversion.[7]

Even if the estimates for this discount rate, $\beta + \rho\gamma$, were to fall into an acceptable range, which is hardly true as shown in sect. 6, a serious conceptual problem still remains. How should one proceed in determining the discount rate for a society consisting of individuals with different risk attitudes or different income elasticities?

More importantly, if we are to look at a wider class of policies that alter individual wealth distribution and, possibly, capital accumulation in the economy, will the above approach produce a meanigful result? Clearly, a direct calculation is tedious, especially if one is to take into account an anticipation of the policy change: the allocations in the

[3]For the US, e.g., according to Johnston and Williamson (2007), average till 2006 is 2.1% since 1950, 1.9% since 1900 or 1850, 2% since 1869, the first year where data become reliable (*loc. cit.*), especially for growth computations since by then both colonial expansion and the immediate aftermath of the Civil War are over.

[4]The premise stressed in Sidgwick (1874) and Ramsey (1928), cf. sect. 1.3.

[5]$\beta$ is, in principle, identifiable from the policy decisions of an actual government (e.g., cf. the second paragraph of sect. 6).

[6]In this case the social welfare functional is a map from individual preferences to societal preferences, so that individual utilities are purely ordinal. This differs thus from the utilitarian point of view in this paper, where utilities are cardinal.

[7]Estimates for $\rho$ are discussed in sect. 6.

economy might respond long before the policy is enacted. Similarly, an effect of even a small temporary change in policy can be propagated in time years after the policy is back to the 'status-quo'. Hence, we ought to analyse an economy where time is the real line: any artificial 'start' or 'end' point will eliminate a potentially important stretch of time when the affected generations are alive.

Good news is, under time invariance, no tedious calculations are necessary, as the main result shows. Surprisingly, the 'back of the envelope' calculation provided above is still valid, with only one component missing. The derivative of a wide class of welfare functions, we show, reduces to the net present value of the policy change with a discount rate that can be determined from the primitives of the model, the static component of the evaluation depends on 'the direction' of the change and is left as a black-box here. The latter, however, does not prevent us from stating the 'bad news' for the traditional utilitarian welfare criterion.

The main result implies that under the traditional utilitarian welfare function in the OG model of sect. 3, for a society populated by individuals whose utility is homogeneous of the same degree $(1-\rho)$, the discount rate is still $\beta+\rho\gamma$, which brings back the problem of finding sensible estimates for these parameters discussed above. In addition, if preferences are heterogeneous, all the weight is put asymptotically on the least risk-averse individuals, cf. cor. 3. This, in particular, invalidates the present value calculation altogether, as the social discount rate is not well-defined in this case.

Nevertheless this does not imply that one should abandon cost-benefit analysis in OG models with growth. For example, with a relative utilitarian criterion, the discount rate is well defined, independent of $\rho$, even with heterogenous risk attitudes, and is simply $\beta+\gamma$, cor. 7.

In the absence of utility discounting ($\beta = 0$), this implication of relative utilitarianism is corroborated in sect. 5 by an independent argument about the value of human life.

1.3. **Related literature.** The issue of discounting and, more broadly, intergenerational justice, has been controversial in the literature since, probably, Sidgwick (1874).[8] Ramsey (1928) (p. 543) presents discounting future utility ('enjoyments') as a "practice which is ethically indefensible and arises merely from the weakness of the imagination." He then suggests a way to overcome 'technical' difficulties of constructing

[8] "How far we are to consider the interests of posterity when they seem to conflict with those of existing human beings? It seems, however, clear that the time at which a man exists cannot affect the value of his happiness from a universal point of view; and that the interests of posterity must concern a Utilitarian as much as those of his contemporaries, except in so far as the effect of his actions on posterity — and even the existence of human beings to be affected — must necessarily be more uncertain." (p. 414)

a discount-free utilitarian social welfare criterion based on the difference between actual and 'bliss' level of utility. Utility discounting is not required per se in our case either, as we evaluate *temporary policy changes*, and thus aggregate utility *differences* from a status-quo.

Aggregation of utility differences is also why Ramsey's egalitarianism and strong Pareto can be combined here, avoiding the impossibility results of e.g. Basu and Mitra (2003); Crespo, Núñez, and Rincón-Zapatero (2008). The literature in welfare economics and social choice offers diverse ways to construct welfare criteria by weakening one of the two desiderata. Koopmans (1960) axiomatises discounting utilities, or 'social impatience'. Several authors are concerned with incorporating intergenerational justice principles in a social welfare criterion. Chichilnisky (1996) offers the 'no dictatorship of the past' and 'no dictatorship of the future' axioms (describing 'sustainable preferences') and shows that the resulting welfare criterion is inconsistent with a sum of discounted utilities. d'Aspremont (2006) and Asheim, Mitra, and Tungodden (2006) show existence of welfare functions satisfying some of Koopmans' (1960) postulates and principles of intergenerational equity, in particular, Chichilnisky's (1996) axioms. For alternative formulations of ethically acceptable allocations see, e.g., Asheim (1991), Fleurbaey and Michel (2003).

The "need to specify a social welfare function which embodies more definite judgements" in cost-benefit analysis was stressed in Drèze and Stern (1987, p. 49), distinguishing this approach from an examination of "potential improvements"[9] stemming from a project. Formulating a social welfare function, the authors argue, provides greater transparency to the cost-benefit analysis, assures consistency of related choices and avoids a special preference for inaction.

1.4. **Basic heuristic.** Many dynamic models have some form of stationary structure, hence natural solution concepts exhibit the same. This can be exploited for local comparative statics; for instance, if one looks at perturbations around a balanced growth equilibrium, and *if* there is a selection by local uniqueness in its neighbourhood, then this selection 'should' satisfy the same invariance: indeed, time-shifts map the balanced growth path to itself, so neighbouring paths are mapped in its neighbourhood, hence, by local uniqueness, the selection is mapped to itself.[10] Such a selection gives (locally) a map from policies to outcomes, that is the basis for comparative statics. In this

[9]See Mishan (1976) for an in-depth discussion of "potential Pareto improvements" (traced back to Pigou (1932)) and their application to cost-benefit analysis. For a more recent overview of cost-benefit criteria see Coate (2000).

[10]Clearly this also needs some form of stability, else as the amount of shifting grows, the corresponding equilibria might slowly get out of the specified neighbourhood. However, as shown in sect. 7 there are other reasons why much more stringent stability properties are needed anyway.

paper we abstract away the exact nature of such an 'outcome map', and retain only its time-invariant structure — in order to be able to 'blackbox' the policy space.

However, as our chief purpose is to determine the discount rate for cost-benefit analysis implied by intergenerational equity (in sect. 4.2.2), we need to apply those methods to a specific economic model. The answer being the growth rate of per-capita real consumption, we need a growth model, and take growth as exogenous for simplicity. For intergenerational equity, we clearly need generations; and those should overlap, lest there be no reason for savings, and hence capital accumulation and growth. Finally, to distinguish the answer from the interest rate (e.g., under a golden rule equilibrium), we need non-zero population growth. Subject to that general structure, the only additional thing we need is existence of balanced growth solutions, as is clear from the above heuristic; for the rest, the model is formulated in the "general equilibrium" fashion (in sect. 3.1).

Time starting at $-\infty$ rather than 0 seems crucial for our argument, as well as for unbiased comparative statics: the policy perturbation must be as fully anticipated as the status-quo balanced-growth equilibrium.[11] This poses novel questions concerning the above model, especially how to specify correctly initial conditions at $-\infty$. This is addressed in sect. 3.1 too, because those initial conditions are crucial to our argument (in prop. 1); informally, there can be no balanced growth in presence of natural resources.

1.5. **Outline of the paper.** Sect. 2 presents the basic tool for evaluating policy reforms. In sect. 2.2, the 'outcome map' is fully abstracted, as a map from policies to social welfare (like in decision theory); so this would also cover models with a single decision-maker, or "infinitely-lived agent". In sect. 2.3, this is applied to a model with a bit more structure, more appropriate for an economy with finitely-lived agents: the map associates to each policy a full profile of individual utilities (as in social choice theory), and the aggregation is done explicitly, enabling use of the previous result. For further usage, the results are particularised to the classical aggregation in sect. 2.4 (thm. 2).

Sect. 3.1 describes an overlapping generations economy with exogenous growth, and sect. 3.2 defines then outcome maps for this model as having still more structure, being now maps from policies to allocations. The time-invariance requirement on them is carefully justified by exhibiting an automorphism of the economy (uniquely) associated with time-shifts. The result of sect. 2 is then applied to this economy in sect. 4, to derive the discount rates implied by the traditional (sect. 4.1)

[11]Another argument for considering fully anticipated policies is that one cannot surprise everybody everyday, pretend every day is a new big-bang (date 0).

and relative utilitarian (sect. 4.2) criteria, with quite different implications. In each case, we first compute the derivative of welfare w.r.t. policy variations on an abstract policy space, and then apply this to a specific policy space of lump sum taxes and subsidies, thought of as representing the monetised value of public projects, to derive the discount rate for cost-benefit analysis.

Sect. 5 presents an alternative derivation, based on the value of a human life, of the same discount rate $\gamma$, thus confirming independently the relative utilitarian conclusion. Merits of the two criteria are then discussed in sect. 6. Concluding remarks in sect. 7 address the issues of evaluating the static component of the derivative of welfare and of non-vacuity of the results.

Proofs are in App. B.

## 2. Differentiating welfare w.r.t. policy variations

**Notation.** $\overline{\mathbb{R}}$ is the extended real line, i.e., $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$; $E^\star$ denotes the dual of a topological vector space $E$.

### 2.1. The basic model.

2.1.1. *Policies.* Basic policies are time-independent specifications of government actions. They belong to a Banach space, since already income tax schedules with brackets indexed to per-capita income are in a function space. The status-quo is some basic policy kept constant over time. A policy (reform) is a temporary deviation from the status-quo.

Continuous time is probably only a matter of convenience here, or of greater transparency of the model; but note it starts at $-\infty$ rather than at 0, which is the only way to model fully anticipated policy changes.

**Definition 1.**
- (i) Let $t_h \colon t \mapsto t + h$ be the translation by $h$ on $\mathbb{R}$; and $\mathbb{S}_h \colon \xi \mapsto \xi \circ t_{-h}$ be the *time-shift* on functions of time.
- (ii) $(B, \bar\pi)$ is the set $B$ of *basic policies*, open in a Banach space $E$, together with some point $\bar\pi \in B$, called the *status-quo* policy.
- (iii) $K_E$ is the space of infinitely differentiable functions $\varphi \colon \mathbb{R} \to E$ with compact support. A sequence $\varphi_n \in K_E$ converges to 0 if $\exists h \in \mathbb{R} \colon |x| \geq h \Rightarrow \varphi_n(x) = 0$ for all $n$, and $\varphi_n$ and its successive derivatives converge uniformly to 0. $K_E^\star$ is the space of linear functionals $\psi$ on $K_E$ s.t. $\psi(\varphi_n) \to 0$ when $\varphi_n \to 0$ in $K_E$.[12]
- (iv) $F$ is a topological vector space of $E$-valued functions s.t. $\mathbb{S}_h F \subseteq F$ and s.t. $K_E$ embeds continuously as a dense subset of $F$.[13]
- (v) $P$ is the set of *policies* $\pi \colon t \mapsto \pi(t) \in B$ s.t. $\delta\pi = \pi - \bar\pi \in F$.

The policy space $P$ is shift-invariant, as is $F$; i.e., policies can be shifted in time. Def. 2 below implies this shift must be meaningful; so we have to think about a basic policy as expressed in "time-invariant"

[12] $K [= K_{\mathbb{R}}]$ is defined in Schwartz (1957-59) or Gel'fand and Shilov (1959).

[13] E.g., the space of continuous functions with compact support and the sup norm.

terms. This implies, in particular, that a basic policy has to be *unit-free* and *non-discriminatory*, not prescribing date-specific actions or special treatment of particular individuals or generations, to be applicable at any time. An example would be a full description of government expenditures, including public goods, regulation and taxation, expressed in per-capita terms, as fractions of per-capita income.

A more precise statement to this effect is that a basic policy, if kept constant over time, should lead to balanced growth (lemma 7).

2.1.2. *Objective function.*

**Definition 2.** $W\colon P \to \overline{\mathbb{R}}$ is an invariant welfare function (IWF) if, $\forall h \in \mathbb{R}$, $\exists$ Lebesgue-measurable $a_h, b_h > 0\colon W \circ \mathbb{S}_h = a_h + b_h W$.[14]

**Lemma 1.** *For an IWF $W$ there exist constants $\zeta$ and $A \in \mathbb{R}$ s.t. $a_h$ and $b_h$ in def. 2 can be taken as $a_h = A\frac{e^{\zeta h}-1}{e^{\zeta}-1}$, $b_h = e^{\zeta h}$, the ratio being defined by continuity at $\zeta = 0$. Such a $\zeta$ is unique if $W$ takes at least 2 different real values. $\zeta$ is called the parameter of the IWF.*

2.2. **The main tool.** Recall a map is Gateaux-differentiable if it has directional derivatives in every direction, which form a continuous linear function of the direction. It is the weakest sense of differentiability.

**Theorem 1.** *Assume (i) $\forall f \in F\, \exists \varepsilon_0 > 0\colon |\varepsilon| < \varepsilon_0 \Rightarrow \bar{\pi} + \varepsilon f(t) \in B$ $\forall t$,[15] and (ii) $\forall q \in E^\star, f \mapsto \int e^{\zeta t}\langle q, f(t)\rangle dt$ belongs to $F^\star$. If $W$, an IWF with parameter $\zeta$, is Gateaux-differentiable on $P$ at $\bar{\pi}$ (so $W(\bar{\pi}) \neq \pm\infty$), then its differential equals $\int e^{\zeta t}\langle q, \delta\pi(t)\rangle dt$ for some $q \in E^\star$.*

The theorem justifies discounting, i.e., shows that the time-component of the derivative of welfare is exponential in time, with a time-independent valuation $q$ of the current policy change $\delta\pi(t)$.

IWFs have by definition full domain, but for their use in thm. 1 it suffices that the domain intersects every straight line through $\bar{\pi}$ in a neighbourhood of $\bar{\pi}$, and this is how they are obtained in fact (cf. 3.2.4). One can even allow them to be a correspondence, as in thm. 2.

An example of how to prove the differentiability assumption can be found in Mertens and Rubinchik (2008).

Next, we show how to express the discount rate, $\zeta$, in terms of the parameters of an overlapping generations model. The first step in this direction is to move from an objective of a single decision maker to a welfare aggregator over individual utilities.

2.3. **Constructing an IWF.** This section offers sufficient conditions for a welfare function to be an IWF in an OG model.

[14]I.e., the von Neumann-Morgenstern preferences on $P$ are shift-invariant.

[15]Just to ensure the Gateaux-differential is unambiguously defined.

2.3.1. *Population.* Individuals differ by type $\tau \in \{1, \dots, \Theta\}$ and by birth-date, $x \in \mathbb{R}$. They have life-length $T_\tau$, and $N_x^\tau dx = N_0^\tau e^{\nu x} dx$ is the number of births in $(x, x + dx)$.

2.3.2. *Utilities over policies.*

**Definition 3.** A profile $v$ of real-valued functions $v_x^\tau$, defined on $P$, is a *valuation*, if it is weakly shift-invariant, i.e., $\forall h \in \mathbb{R}\ \exists$ Lebesgue-measurable $a_h \in \mathbb{R}^\Theta, b_h > 0\colon v \circ \mathbb{S}_h = a_h + b_h \mathbb{S}_h \circ v$.[16] The profile is a *strict valuation* if it is shift-invariant, i.e., if $a_h = 0, b_h = 1$.

There is a simple translation of a valuation into a strict one:

**Lemma 2.** *For a valuation $v$ there exist constants $A \in \mathbb{R}^\Theta$ and $\varrho$ s.t. $u_x^\tau = A^\tau \frac{1-e^{-\varrho x}}{e^\varrho - 1} + e^{-\varrho x} v_x^\tau$ is a strict valuation, with $x$ for the ratio at $\varrho = 0$. Such $\varrho$ and $A$ are unique except if $\forall \tau, v_x^\tau(\pi)$ is constant in $x$ and $\pi$. $\varrho$ is called the parameter of the valuation.*

**Corollary 1.** *For a valuation $v$ and a constant policy $\pi$, $v_x^\tau(\pi)$ is of the form $e^{\varrho x} v^\tau(\pi) + C$.*

*Proof.* For a strict valuation, this follows from the definition; apply then lemma 2. ∎

Thus the parameter $\varrho$ is the rate of growth of individual utility scales over policies. It will be further disentangled in prop. 2 and 3 into effects stemming from growth and effects of the utility functions.

2.3.3. *Aggregation.*

**Definition 4.** A *social welfare aggregator* (SWA) is an $\overline{\mathbb{R}}$-valued function $V$ defined all profiles of utilities $\mathbb{R}^{\Theta \times \mathbb{R}}$, such that $\forall h \in \mathbb{R}$, $\exists$ Lebesgue-measurable $a_h, b_h > 0\colon V \circ \mathbb{S}_h = a_h + b_h V$, i.e., $V$ is weakly shift-invariant.[17]

As for lemma 1, we obtain now:

**Lemma 3.** *$a_h$ and $b_h$ in def. 4 can be taken as $a_h = a\frac{e^{ch}-1}{e^c-1}$, $b_h = e^{ch}$, the ratio being defined by continuity at $c = 0$. Such $c$ is unique if $V$ takes at least 2 different real values. $c$ is called the parameter of the SWA.*

Given the goal to evaluate policy changes from the status quo, it is natural to aggregate individual utility *differences* from the status quo:

**Lemma 4.** *Take a valuation $v$ with parameter $\varrho$, and a SWA $V_{c,r}$ with parameter $c$, positively homogeneous of degree $r$.*

*Then $W(\cdot) \stackrel{\text{def}}{=} V_{c,r}(v(\cdot) - v(\bar{\pi}))$ is an IWF with $\zeta = \varrho r + c$.*

*If the valuation is strict, homogeneity is not needed, and $\zeta = c$.*

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[16] I.e., shifts preserve interpersonal comparisons of utility differences.

[17] It may sometimes be more convenient to use as domain a specific shift-invariant subset $\mathcal{U}$ of $\mathbb{R}^{\Theta \times \mathbb{R}}$ (e.g., continuous functions), restricting valuations to be $\mathcal{U}$-valued. We will not need this embellishment here.

With this, we could continue and use general SWA's throughout (homogeneous in sect. 4.1); however, for concreteness, and to have an explicit parameter $c$, we concentrate henceforth on the classical case, and first summarise for future use our results for that case.

2.4. **The utilitarian aggregator.** The two *social welfare functions* (SWF) used in sect. 4 are based on the same utilitarian aggregator. It is however not necessarily a map, so some additional care is required.

**Definition 5.** The *utilitarian aggregator* $S$ maps a valuation $v$ to $S(v) = \int_{-\infty}^{\infty} e^{-\beta x} \sum_\tau N_x^\tau (v_x^\tau - v_x^\tau(\bar\pi))dx$, understood as the interval between the lower and upper Denjoy-integrals (e.g., Gordon, 1994).[18]

**Definition 6.** An $\overline{\mathbb{R}}$-valued correspondence $\Gamma$ with domain in $F$ is *G-differentiable* at $x$ iff every $f\colon F \to \overline{\mathbb{R}}$, s.t. $f(y) \in \Gamma(y)$ when $\Gamma(y)$ is defined and non-empty, is s.t. $f(x) \in \mathbb{R}$, and Gateaux-differentiable at $x$. Their (common) Gateaux-differential is then the *G-differential* of $\Gamma$ at $x$.

**Lemma 5.** *For a valuation $v$ with parameter $\varrho$, and $S^*$ the upper bound of $S$ (the upper-integral), $S^*(v)$ is an IWF with $\zeta = \varrho + \nu - \beta$.*

**Theorem 2.** *Let $v$ be a valuation with parameter $\varrho$. If $W = S(v)$ is G-differentiable on $P$ at $\bar\pi$ and $\forall q \in E^\star$, $f \mapsto \int e^{(\varrho+\nu-\beta)t}\langle q, f(t)\rangle dt$ belongs to $F^\star$, then the G-differential of $W$ at $\bar\pi$ equals $\int e^{(\varrho+\nu-\beta)t}\langle q, \delta\pi(t)\rangle dt$ for some $q \in E^\star$.*

We obtain thus $\beta - \nu - \varrho$ as discount rate for policies. Our purpose in the next two sections is to identify the last parameter, $\varrho$, in terms of economic primitives in a growing economy.

## 3. A growing economy

Valuations with their built-in time-invariance might seem confined to stationary economies, but they also arise naturally in models with exogenous growth. We impose only the minimal conditions required for existence of balanced growth: homogeneity of utility functions with respect to consumption, constant returns to scale in production, absence of land and natural resources, and labour-saving technological growth.

3.1. **The economy.**

[18] The basic reason for using Denjoy integration is the capital-accumulation equation in sect. 3.1 below, to be sure the meaning of its differential equation form is the same as that of the integral form, and then to systematically use always the same integration theory on $\mathbb{R}$. No harm is done by sticking with the most encompassing one; in particular in this case, where a requirement of absolute summability would have no economic meaning whatsoever.

3.1.1. *Consumption and labour.* Instantaneous consumption is a non-negative bundle of $n$ consumption goods and $h$ fractions of total time allocated to $h$ different types of labour. Individual preferences over lifetime streams of time allocation and consumption bundles are represented by a utility function $U^\tau$, homogeneous of degree $1-\rho^\tau$ in consumption. By fixing consumption prices and relative wages, labour income varies linearly with a 'numeraire' wage by the homogeneity, so an individual indirect utility function can be viewed as a function of (labour-)income at those fixed consumption prices and relative wages. It allows thus for two interpretations of the parameter $\rho^\tau$: (1) relative risk aversion coefficient, (2) income elasticity of the marginal utility of income, as mentioned in the introduction.

The fraction of time, $z_i^\tau(s,t)$, devoted at date $t$ to activity $i$ by an agent of type $\tau$ and age $s$ is multiplied by a non-negative and integrable efficiency factor $\varepsilon_i^\tau(s)$, to form effective time. Effective time devoted at date $t$ to any activity is multiplied by $e^{\gamma t}$ to form effective labour input, $e^{\gamma t}\varepsilon_i^\tau(s)z_i^\tau(s,t)$, thus representing labour-saving technological progress.

*Example.* With $\gamma = 0$ and $\varepsilon(s) = 1$ in the first part of life and zero thereafter the model is a continuous-time reinterpretation of the standard OG model, as presented in Gale (1973), Samuelson (1958).

3.1.2. *Production.* There are $m$ capital goods, and a corresponding investment good for each, linked by the usual capital accumulation equation, $K^{i\prime}(t) = I^i(t) - \delta^i K^i(t)$,[19] with $K^i \geq 0$ capital and $I^i$ investment of type $i$, and $\delta^i$ the depreciation rate. Consumption and investment goods are manufactured instantaneously by production firms from (the services of) capital and effective labour (and, possibly, from investment), with as instantaneous production set a closed convex cone $Y \subseteq \mathbb{R}^h_- \times \mathbb{R}^m_- \times \mathbb{R}^m \times \mathbb{R}^n$ of production vectors $(-L, -K, I, C)$, satisfying the classical free-disposal and irreversibility ($Y$ contains no straight line) conditions. An investment firm of type $i$ acquires capital $K^i(t_0)$ at time $t_0$, chooses investment flows, rents out accumulated capital to production firms, and sells $K^i(t_1)$ at time $t_1 > t_0$.

Investment goods can be viewed both as outputs and inputs. To model a storable good, for example, one can introduce a corresponding investment good and a capital good ("the good in storage"). A production firm creates the storable investment good, purchased by an intermediary investment ("storage") firm that transforms it into the corresponding capital good, which typically has no use in production. At the time of consumption, the investment firm disinvests and sells the corresponding investment good to a production ("marketing") firm, that transforms it one to one into the corresponding consumption good.

[19]Assumed to hold a.e., and implying the conditions for it to be meaningfull: that $K_t^i$ is assumed locally a Denjoy primitive and $I_t^i$ locally Denjoy-integrable.

So allow all investment firms to disinvest as well as invest in all goods; restrictions on disinvestment are described by $Y$.

To incorporate consumer durables in this model, introduce the corresponding investment and capital goods. A production firm creates the durable investment good, purchased by an intermediary investment firm, which then rents the capital good out to a leasing production firm, that produces with this capital the consumption good (services), purchased by consumers.

The rest of this subsection is to ensure that the production set of the economy (set of feasible input and output paths) is well-defined.

To ensure its boundedness, assume capital can not reproduce itself:

**Assumption 1** (No-rabbit economy)**.** $(0, -K, I, 0) \in Y \Rightarrow I \leq 0$.

*Remark* 1. Observe that although production of durables, as described before, involves a production of consumption good with only capital and no labour input, it does not violate our assumption on $Y$ that no *investment good* can be produced without some form of labour input. Similarly production activities (as for storable goods) transforming investment goods one to one into consumption goods, without any capital or labour input, do not violate this assumption.

To see why such restrictions on the instantaneous production set are needed consider the following "rabbit economy":

*Example.* Assume a single good, a single type of labour, a CES production function $(AK^\alpha + BL^\alpha)^{1/\alpha}$, $A^{1/\alpha} \geq R$ with $R = \gamma + \nu + \delta$. In order to get an upper bound on capital and investment consider a path with all agents working full-time and consuming nothing. Note that $L_t = L_0 e^{(\gamma+\nu)t}$, so for $D = BL_0^\alpha$, $K'(t) = (AK^\alpha(t) + De^{\alpha(\gamma+\nu)t})^{1/\alpha} - \delta K(t)$; or with $x(t) = K(t)e^{-(\gamma+\nu)t}$, $x'(t) = (Ax^\alpha(t) + D)^{1/\alpha} - Rx(t) \geq D^{1/\alpha} > 0$. Since $x(t) \geq 0$, there is no solution, i.e., the upper bound of $K(t)$ is infinity. And even if $B = 0$, the solutions are $x(t) = Ce^{(A^{1/\alpha} - R)t}$, with $C \geq 0$ arbitrarily large, so $K(t)$ is unbounded in this case too.

As for any differential equation, initial conditions are needed. Their natural form is that the capital stock $K_t$ converges at $-\infty$ to given initial values. Note that such initial values of land and resources[20] are thus part of the description of the technology; any feasible path must converge to the specified values. But, for balanced growth, those initial values must be zero, thus ruling out land and resources:

**Assumption 2** (Initial condition)**.** Let $\delta = \mathsf{min}_i \, \delta^i$. Then $e^{\delta t} K_t$ converges exponentially fast to 0 along some sequence $t \to -\infty$.

Also assume $R \overset{\text{def}}{=} \gamma + \nu + \delta > 0$.

[20]Non-null initial values can occur only for capital goods with $\delta^i = 0$, corresponding to land and resources. Disinvestment is crucial to describe resource extraction.

**Lemma 6.** *(i)* $K^i(t) = \int_{-\infty}^{t} e^{\delta^i(s-t)} I^i(s) ds$, *where the* $L_1$*-norms of all feasible integrands are bounded by a constant times* $e^{(\gamma+\nu)t}$*; in particular, the integral is a Lebesgue integral.*

*(ii) Let* $i_t = e^{-(\gamma+\nu)t} I_t$, $k_t = e^{-(\gamma+\nu)t} K_t$. *There exists* $\bar{K}$ *s.t. along any feasible path,* $\int_a^b \|i_t\| dt \le \bar{K}(b-a+1)$ *for any pair* $a \le b$*, and (hence)* $\|k_t\| \le \bar{K}$.

*Remark* 2. As explained and addressed in appendix C, the initial condition is a bit too stringent conceptually, requiring exponential convergence to 0 instead of just plain convergence. This is not crucial in this paper: land and natural resources being anyway ruled out by the need for balanced growth, it is natural to expect all $\delta^i > 0$, so just $K_t$ bounded at $-\infty$ already ensures exponential convergence to 0.

3.2. **Time-invariant solution concepts.** Solution concepts map an economy and policy pair to an allocation. To induce a valuation they have to be single-valued and satisfy time-invariance, in which case they are called outcome functions (def. 8).

3.2.1. *Isomorphism between Arrow-Debreu economies.* To motivate the definitions below, define isomorphism between two Arrow-Debreu economies with finitely many goods and individuals. They are isomorphic if there is a linear map $\zeta$ from the commodity space of one economy to that of the other and there are one-to-one mappings from the sets of individuals and of firms of one to those of the other such that:

(i) the consumption set, preferences, and endowment of any agent in the first economy are mapped by $\zeta$ to those of the corresponding agent in the second economy;
(ii) the production set of each firm in one economy is mapped by $\zeta$ to that of the corresponding firm in the second;
(iii) shareholdings are preserved.

When consumption sets are the non-negative orthant, $\zeta$ must map the commodities' names in the first economy one-to-one to those in the second, together with appropriate re-scalings (changes of unit).

Another aspect of isomorphism, which is more familiar with a continuum of agents, is to multiply the population measure by a positive constant $C$. "Shareholdings" refer then for each firm to a probability distribution over the agents; and the one-to-one mapping of agents has to be understood to be measurable as well as its inverse, and such that the induced map on measures maps the first population measure to $\frac{1}{C}$ times the second. Further, the firms' production sets, as well as points therein, are multiplied by $C$ (in addition to the above re-scalings).

When production has constant returns to scale, as here (capital-accumulation equations are linear, and the instantaneous production sets, cones), shareholdings become irrelevant (profits being zero), and multiplication by $C$ maps the production set onto itself.

The isomorphism is equivalently described by a single linear bijection (with the required structure) between allocation spaces (product of all consumption sets and production sets) of both economies. For the isomorphism property, suffices then that it maps allocations *to and onto* allocations, endowments to endowments, preserves preferences, and that population measures are mapped to each other by the induced map of agents and the multiplication by $C$, obtaining $C$ from how the map behaves on production sets as compared to consumption sets.

We will use this below with a new twist, in that indeed the mass of each agent is multiplied by $C$, but with as final effect to *preserve* the population measure, it being $\sigma$-finite.

3.2.2. *Time-invariance in the OG model.* Consider next particular case of such isomorphisms: it maps an agent of type $\tau$ born at time $t$ to an agent of the same type born at $t + h$, multiplying his mass by $e^{\nu h}$, and maps any good dated $t$ to the same good dated $t + h$, multiplying non-labour quantities by $e^{\gamma h}$, and labour quantities by 1. Individual time is not a good (not marketed), so the map of sect. 3.2.1 is applied using the equivalent vector of effective time. The map of allocations is thus:

**Definition 7.** The transformation $\mathbb{T}_h$

(i) applies $\mathbb{S}_h$ to allocations;
(ii) multiplies individual consumption bundles by $e^{\gamma h}$;
(iii) multiplies production plans — aggregate effective labour, capital, investment, and consumption— by $e^{(\gamma+\nu)h}$.

*Remark* 3. Equivalently, $\mathbb{T}_h$ shifts the origin of time back by $h$, multiplies population measure and thus all aggregates by $e^{\nu h}$, and divides *units* of all non-labour goods by $e^{\gamma h}$.

**Proposition 1.** *$\mathbb{T}_h$ is an automorphism of the economy.*

Since $\mathbb{T}_h$ maps the economy to itself $h$ units of time later, and since policies are unit-free, the corresponding operation on policies is a pure time-shift, without rescaling. Time-invariance requires thus that solutions for $\mathbb{S}_h\pi$ be obtained via $\mathbb{T}_h$ from the solutions of $\pi$. We impose in addition single-valuedness.

**Definition 8.** An *outcome function* $\varsigma$ is a map from policies to individual allocations, that is invariant under all $\mathbb{T}_h$: $\mathbb{T}_h \circ \varsigma = \varsigma \circ \mathbb{S}_h$.

3.2.3. *Examples of outcome functions.* The first example is the maximisation of a time-invariant *social welfare function*, say, a utilitarian one, provided the maximum is unique.

Other examples are equilibrium-based. An outcome function should map a policy perturbation $\pi$ to a locally unique equilibrium close to that stemming from the status-quo policy $\bar{\pi}$.

One way to model *policy surprises* is to assume the contracts signed ("at the beginning of time") in anticipation of the status-quo policy can not be changed, so in the wake of an unexpected policy change, individuals sign additional contracts taking their status-quo consumption as new endowment. The initial equilibrium being a balanced growth equilibrium, the economy with that endowment also satisfies time-invariance, and the resulting map from policies is again an outcome function if the final allocation is locally unique. At least when policies are lump sum taxes and benefits (endowment perturbations), this case is particularly simple, as net individual demand under the base-line prices is zero, so income effects disappear and the variation in individual utility depends just on the value of the endowment perturbation.

3.2.4. *The case of indeterminacy.* Even if dealing with a situation that does not guarantee local uniqueness, one can choose the closest prices to those in the initial equilibrium in terms of the $L_\infty$ norm:[21] $\sum_i \|\ln p_i(t) - \ln \bar{p}_i(t)\|_\infty$, where $\bar{p}(t)$ is the price vector at the initial equilibrium and $p(t)$ at a perturbed economy. Though the price system does not necessarily specify an equilibrium, it does specify the individual utility levels, which is sufficient for welfare analysis. The logarithms make the distance independent of price normalisation, hence induce a distance between price-rays: for any multiple of $\bar{p}_i$, the minimum, over all multiples of $p_i$, will be achieved at the corresponding multiple, and the value of the minimum is independent of this multiple, and remains the same when permuting the roles of $\bar{p}_i$ and $p_i$. Finally, the $L_\infty$ norm being shift-invariant, the selection will be time-invariant. If the set of minimisers is not a singleton, their correspondence can be expected to be sufficiently 'thin' that hopefully any outcome function obtained as an invariant selection (axiom of choice) generically satisfies the differentiability requirement — e.g., as in Mertens and Rubinchik (2008), discussed in sect. 7. Finally, since thm. 2 already allows for a correspondence, one could similarly extend def. 8, to obviate the need to appeal to the axiom of choice in such cases.

But this is only one example of how to possibly construct outcome functions in case of indeterminacy (which we do not expect to occur in the model of sect. 3.1); one expects a continuum of such outcome functions then. Since our results below hold for any of them, conceivably with a linear functional $q$ depending on the chosen outcome function, the discount rate is established even then.

3.2.5. *Balanced growth.*

**Definition 9.** A *balanced growth path* is a $\mathbb{T}$-invariant allocation.

On a balanced growth path individual labour is independent of the birth-date, individual consumption grows at rate $\gamma$, and all aggregate

[21] Or equivalent ones, e.g., the $L_\infty$-norm of the $\ell_2$-norm over $i$ of the $\ln$ differences.

inputs and outputs at rate $\gamma+\nu$, as in the standard (1 type, 1 good) case (e.g., Arrow and Kurz, 1970; King, Plosser, and Rebelo, 2002).

The following sharpens the interpretation of basic policies, see cor. 1:

**Lemma 7.** *The outcome of a constant policy is a balanced growth path.*

*Proof.* By def. 8, it is mapped to itself by any $\mathbb{T}_h$. ■

## 4. The discount rate

The discount rate for cost-benefit analysis depends on the social welfare function. We consider both relative and traditional utilitarianism.

Let $v = U \circ \varsigma$ the profile of utility functions on $P$ induced by the profile of utility functions $U$ and the outcome function $\varsigma$.

### 4.1. The traditional utilitarian approach.

In sect. 4.1 we assume:

**Assumption 3.** $\forall \tau, \forall q \in E^\star$; $f \mapsto \int e^{((1-\rho^\tau)\gamma+\nu-\beta)t}\langle q, f(t)\rangle dt$ belongs to $F^\star$.

#### 4.1.1. *Evaluating policies.*

**Proposition 2.** *Assume all types have the same parameter $\rho$. Then $v$ is a valuation with parameter $\varrho = (1-\rho)\gamma$.*

Prop. 2 and thm. 2 imply now:

**Corollary 2.** *Assume all types have the same parameter $\rho$. If $W = S(v)$ is G-differentiable on $P$ at $\bar{\pi}$, then its differential equals, for some $q \in E^\star$, $\int e^{(\nu-\beta+(1-\rho)\gamma)t}\langle q, \delta\pi(t)\rangle dt$.*

In a society with type-dependent $\rho$, traditional utilitarianism leads to questionable implications; besides, it invalidates discounting:

**Corollary 3.** *The weight in the welfare function of the types with smallest $\rho$ approaches one as time goes to $+\infty$.*

There are other ways to express the same idea; e.g., that along any balanced growth path, in an optimal redistribution of consumption goods (keeping the rest fixed) the fraction allocated to the agents with the smallest $\rho$ converges to 1.

#### 4.1.2. *The discount rate for cost-benefit analysis.*

In cost-benefit analysis, the effects of a variation in public policy are traditionally first "monetised", i.e., expressed as an equivalent perturbation of individual endowments of consumption goods, here initially 0.

Let thus $E$ be the Banach space $M$ of measures[22] on age-groups and types — i.e., on $\cup_\tau(\{\tau\}\times[0,T_\tau])$ — with values in $\mathbb{R}^n$ (space of consumption bundles), with $\bar{\pi}=0$ as status-quo, where $b \in B$ determines the endowment perturbation $\omega(t) = e^{(\nu+\gamma)t}b$. Equivalently, express basic policies unit-free as fractions of status-quo aggregate consumption.

[22]Or $L_1$, or the measures with continuous densities.

Policies are thus arbitrary (smooth) endowment perturbations, representing arbitrary flows of lump-sum real taxes and benefits. Then we get $\beta+\rho\gamma$ as discount rate, confirming our rough calculation in sect. 1.2:

**Corollary 4.** *Assume all types have the same parameter $\rho$. If $W = S(v)$ is G-differentiable on $P$ at 0, then its differential equals, for some $q \in M^\star$, $\int e^{-(\beta+\rho\gamma)t}\langle q, \omega(t)\rangle dt$.*

*Proof.* By construction, $\bar{\pi} = 0$, so $\delta\pi = \pi$ and $\pi(t) = e^{-(\nu+\gamma)t}\omega(t)$. ■

*Remark* 4. Clearly, choosing a different growth rate in the definition of $\omega$ would lead to the same corollary with a different discount rate. That statement would however be empty, since there can be no outcome function: $B$ being a neighbourhood of 0, choose a negative measure $b$ s.t. $\forall s \in [0,1], sb \in B$, and let $\psi = b\phi$ for some $\phi \in K$ with values in $[0,1]$. Then $\psi$ is a policy, yet when it is shifted sufficiently to $\pm\infty$, the feasible set under that policy becomes empty, by lemma 6.

### 4.2. The Relative Utilitarian approach.

In sect. 4.2 we assume:

**Assumption 4.** $\forall q \in E^\star$, $f \mapsto \int e^{(\nu-\beta)t}\langle q, f(t)\rangle dt$ belongs to $F^\star$.

As an alternative, we suggest to apply relative utilitarianism (RU),[23] the social welfare functional where individual von Neumann-Morgenstern utilities are normalised between zero and one, and then summed. It is stressed in Dhillon and Mertens (1999) that the RU-normalisation of individual utilities has to be done on some universal set $A$ of acceptable alternatives, not specific to the problem under consideration, and representing the constraints both of feasibility and of justice.

**Assumption 5.** The set $A$ of *acceptable* policies is shift-invariant and each individual utility is bounded on $A$.

The boundedness is a minimal implication of justice; as to the shift-invariance, it is clearly a property of feasibility, but in relation to justice it has a strong meaning, that physical units (like calories per day) are irrelevant. And without it RU might lead to quite different conclusions. But it is straight in the spirit of exogenous growth models — that (acceptable) policies affect only the height of the growth path, not the growth rate; and it is arguably justified in a world described by such a model: e.g., if calories per-day matter, utilities can't be homogeneous.

Assume thus utility functions are von Neumann-Morgenstern, and that $\varsigma$ is defined on $A$ — and hence $v$ too, by the definitions at the beginning of this section. Let $\mathbb{M}_A$ denote the RU-*normalisation*, i.e., the operation on a profile such that each individual utility is normalised as to have range of size 1 on $A$.

**Definition 10.** The relative utilitarian SWF is $W = S(\mathbb{M}_A(v))$.

[23] The axiomatisation (Dhillon and Mertens, 1999) is for a finite set of agents.

RU's anonymity axiom implies $\beta = 0$ in the specification of $S$, def. 5. However, to allow for a richer model incorporating a probability of the world ending tomorrow, $\beta$ is not restricted.

4.2.1. *Evaluating policies.* In a growing economy the RU-normalisation yields shift-invariance, hence strict valuations:

**Proposition 3.** *$\mathbb{M}_A(v)$ is a strict valuation.*

**Corollary 5.** *The* RU*-normalised utility of an agent of type $\tau$ on a balanced growth path is independent of his birth-date.*

*Proof.* Apply prop. 3 to the case where the basic policy space is a singleton, that doesn't do anything, and where the outcome function maps to the chosen balanced growth path. ■

**Corollary 6.** *If $W = S(\mathbb{M}_A(v))$ is G-differentiable on $P$ at $\bar{\pi}$, then its differential equals $\int e^{(\nu-\beta)t}\langle q, \delta\pi(t)\rangle dt$ for some $q \in E^\star$.*

*Proof.* Prop. 3 and thm. 2. ■

4.2.2. *The discount rate for cost-benefit analysis.* As in sect. 4.1.2, one gets now, using cor. 6, the discount rate $\beta + \gamma$ implied by relative utilitarianism, well-defined even for a population with variable $\rho^\tau$:

**Corollary 7.** *If $W = S(\mathbb{M}_A(v))$ is G-differentiable on $P$ at 0, then its differential equals $\int e^{-(\beta+\gamma)t}\langle q, \omega(t)\rangle dt$ for some $q \in M^\star$.*

Restricting basic policies $b$ to have all the same distribution over age-groups and types, and setting $\beta = 0$, yields then the main result in Mertens and Rubinchik (2006).

The derived discount rate, $\gamma$, differs generically from the interest rate, even at the 'golden rule' equilibrium if $\nu$ is non-zero.

## 5. A value of life argument

One touch-stone is the case $\beta = 0$, no discounting of utilities. Do the presciptions of the theory then indeed correspond to the intuitive meaning of treating individuals of different generations equally?

A compelling implication of equal treatment is to give equal weight to individual lives,[24] so, in cost-benefit analysis, to their monetised values, i.e., the change in real consumption which is equivalent for the individual to an extension of his life.

[24]See, e.g.: "Morally speaking, there is no difference between current and future risk. Theories which, for example, attempt to discount effects on human health in twenty years to the extent that they are equivalent to only one-tenth of present-day effects in cost-benefit considerations are not acceptable." Wildi, Appel, Buser, Dermange, Eckhardt, Hufschmied, and Keusen (2000)

The monetised value of life, according to any criteria [e.g., each of the four in Mishan's (1971) introduction, or even judicial criteria in assessing damages], is proportional to the individual's life-time income.[25]

This is also formally true in the above economic model, when allowing for a variable life-span: individual life-time utility is homogeneous, so willingness to pay to extend life is proportional to life-time income. Indeed, one can evaluate the (relative) increase in utility from augmenting individual life-time, $T_\tau$, assuming that during this additional stretch of time the person's consumption is in fixed proportion to his previous consumption. By homogeneity, an identical increase in utility can be generated by a proportional increase in life-time consumption, i.e., a proportional increase in life-time income.

Note that along a balanced growth path (the outcome of a basic policy), this income is proportional to $e^{\gamma t}$. So, to treat individuals of all generations equally, future incomes have to be discounted *exactly* at rate $\gamma$, as implied by relative utilitarianism (corollary 7).

## 6. The choice of social welfare function

Since traditional and relative utilitarianism have so different implications for discounting, we wish to discuss some of the underlying principles of equitable treatment of different generations incorporated in each of the criteria. This should provide a better guidance as to which of the two is better suited for evaluating long-term projects.

Interestingly, the implication of relative utilitarianism is consistent with accepted public policy: Circular A-4 of the U.S. Office of Management and Budget (2003) requires that all executive agencies and establishments conduct a "regulatory analysis" for any new proposal, and more specifically (pp. 33–36), a cost-benefit analysis, at the rates of both 3% and 7%.[26] The circular does refer explicitly to the requirement of equity vis-à-vis future generations, and acknowledges it by requiring, for projects with substantial long-term impact, a further analysis at a "still lower but positive" discount rate (p. 36), but more specific suggestions are hard to find.[27] The rate based on the relative utilitarian criterion, $\approx 2\%$, falls exactly in this range.

Remarkably, relative utilitarianism is also consistent with the 'balanced generational policy' as presented in Kotlikoff (2002, p. 1905), requiring "...that the generational accounts [lifetime net tax burdens] of all future generations are equal, except for a growth adjustment."

[25] Even a claim that from the point of view of society, it would be proportional to average life-time income at his time would leave our argument below intact.

[26] Both rates are rationalised there as "the interest rate": the first one relative to private savings, the second one relative to capital formation and/or displacement, i.e., as the gross return on capital.

[27] It is indeed this glaring absence which is it at the origin of the present paper.

The discount rate $\beta + \rho\gamma$ based on traditional utilitarianism (cor. 4) is well-known in the applied literature. Even if individual risk attitudes were identical (far from the empirical findings, see Einav and Cohen (2007)), to get acceptable conclusions, one has to set $\rho$ close to unity (e.g., Stern, 2007, pp. 6–11). So, the choice of a discount rate involves imposing individual risk preferences and contradicts fully any utilitarian foundation, since only the indifference map is retained as individual characteristic. Without forcing $\rho$ to unity, based on reported estimates (Drèze, 1981; Einav and Cohen, 2007), the implied discount rate would be far above the range suggested by the OMB. Note that forcing $\rho$ to 1 means forcing the discount rate implied by RU.

The main reason $\rho$ enters the calculation of the discount rate under traditional utilitarianism is the presumption that the marginal utility of income is independent of the environment surrounding the individual. In particular, a 1% increase in real income of any of our contemporaries has the same effect as it would 100 years ago for the same individual *with the same real income*.

In contrast, relative utilitarianism, in the context of a growing economy, implies that to compare individual utility differences, the utilities have first to be normalised over the space of feasible policies (consumption paths). As a consequence, even in the presence of economic growth, the social value of a 1% increase in real income of an individual *at a given quantile* of the income distribution is independent of the date. Forcing logarithmic utilities, as in Stern (2006), amounts to choose the best possible approximation to this under traditional utilitarianism.

## 7. Concluding remarks

7.1. **The linear functional $q$.** The problem of evaluating policy changes, i.e., finding the derivative of social welfare with respect to policy variations, is reduced by the above results to the static problem of evaluating $q$, a linear functional on $B$, the space of basic policies. If $B$ is finite-dimensional, for example, $q$ can be computed by a finite number of evaluations (direct computations) of the derivative in the direction of each of the $b \in B$ constituting a basis of $B$. Arguably, the effect of a perturbation in the direction of a *constant* policy should be easier to compute, especially in view of the main result assuring the derivative should be of the form $q(b) \int e^{\zeta t} \phi(t) dt$, where $\phi \in K$ describes the intensity of the perturbation away from the status-quo in the direction of $b$.[28]

7.2. **The differentiability assumption.** Applicability of the results hinges on the existence of differentiable outcome functions, as illustrated by rem. 4. For the case of endowment perturbations (cor. 4 and 7), this would be a straight extension of Debreu's 1976 classical generic

[28]Mertens and Rubinchik (2008) contains an example of such a computation, for a one-dimensional basic policy space.

regularity theorem. There are, however, several aspects that make such an extension highly non-trivial. First, it is well-known that overlapping generations models can give rise to indeterminacy, see e.g., Kehoe and Levine (1984), Geanakoplos and Brown (1985). Next, even if regularity is assured, already for the welfare function to be well-defined, the equilibrium has to be stable: the perturbed equilibrium has to converge sufficiently fast back to the unperturbed solution, both at $+\infty$ and at $-\infty$.

This program was successfully completed for a very particular case in Mertens and Rubinchik (2008), ensuring thus at least non-vacuity of our results. It seems a crucial aspect there too is to start time at $-\infty$.

7.3. **Permanent changes.** For evaluating permanent policy changes, one should re-interpret welfare functions in this paper as normalised, e.g. in per-capita terms, like $\lim_{T\to\infty} \frac{1}{N_T}\int_{-T}^{T} e^{\nu x}\sum_\tau N_0^\tau U_x^\tau dx$ (where $N_T = 2\sinh(\nu T)/\nu$). Welfare per-capita is our preferred interpretation of a social welfare function, as would have been Harsanyi's, if we are reading him correctly, e.g. when thinking of it as the expected utility of an un-identified individual. Welfare viewed as a sum, as in this paper, is then only an additional higher order term (of order $\frac{1}{N_T}$) in the expansion of the above (w.r.t. $N_T$), to fine-tune transitions. In fact, our restriction to temporary policy deviations was just to make sure the other terms vanish. But this whole area remains to be explored; we offer even no idea about what might be the form of an asymptotic expansion.

Observe that under RU, for constant policies $\pi$, $U_t^\tau(\pi)$ is independent of $x$ (cor. 1 and prop. 3), so the above average gives trivially the SWF over constant policies (asymptotically constant likely too, generically, but proof is certainly non-obvious). Note this is independent of $\beta$, while in general the above linear functional $q$ depends on $\beta$: e.g., policies favouring the old will come out better with high discounting, since the old were born earlier. This is why an approach like in Mertens and Rubinchik (2008) seems more promising to get analytically a handle on $q$.

## Appendix A. An equation with shift operators

**Lemma 8.** *Let $E$ be a set, with maps $\mathbb{S}_h\colon E\to E$ s.t. $\mathbb{S}_{h_1}\circ\mathbb{S}_{h_2}=\mathbb{S}_{h_1+h_2}$. Let also $V$ be the space of functions of time with values in $\overline{\mathbb{R}}^n$. Assume $\varphi\colon E\to V$ is s.t. $\forall h\ \exists a_h\in\mathbb{R}^n$, $b_h\in\mathbb{R}_{++}$, both Lebesgue-measurable in $h$:*

$$\varphi\circ\mathbb{S}_h = a_h + b_h\mathbb{S}_h\circ\varphi$$

*Then $\exists\zeta\in\mathbb{R}$ and $A\in\mathbb{R}^n$ such that $\forall h$, one can take $b_h=e^{\zeta h}, a_h = A\frac{e^{\zeta h}-1}{e^\zeta-1}$, the fraction being defined by continuity if $\zeta=0$.*

*$\zeta$ is not unique iff $\exists\alpha\in\mathbb{R}^n\colon \varphi_e^i(t)\in\mathbb{R}\Rightarrow\varphi_e^i(t)=\alpha^i$, the superscript denoting the coordinate. When $\zeta$ is unique, $A^i$ is unique iff $\exists e,t\colon \varphi_e^i(t)\in\mathbb{R}$.*

*Proof.* Let $h=h_1+h_2$. Then $\varphi\circ\mathbb{S}_h = a_{h_2}+b_{h_2}[\mathbb{S}_{h_2}\circ(a_{h_1}+b_{h_1}\mathbb{S}_{h_1}\circ\varphi)]$. So $a_h+b_h\mathbb{S}_h\circ\varphi = a_{h_2}+b_{h_2}a_{h_1}+b_{h_2}b_{h_1}\mathbb{S}_h\circ\varphi$.

If, for some pair $h_1,h_2$, $b_h\neq b_{h_1}b_{h_2}$, then whenever $(\mathbb{S}_h\circ\varphi)_e^i(t)\in\mathbb{R}$, the above equation determines its value, say $\alpha^i$. The same obviously holds then for $\varphi$ itself. For such a $\varphi$, one can set $a_h = 0$, $b_h = 1$ $\forall h$; thus we can always assume $b_{h_1+h_2} = b_{h_1} b_{h_2}$. Since $b_h > 0$, taking logarithms reduces it to $f(x+y) = f(x)+f(y)$, of which it is well-known that any Lebesgue-measurable solution is linear (Fréchet, 1913). Thus $b_h = e^{\zeta h}$.

As to $a_h$, for each $i$, if $\exists e, t$: $(\mathbb{S}_h \circ \varphi)^i_e(t) \in \mathbb{R}$, then our above equation simplifies, after substituting the $b$'s, to $a^i_{h_1+h_2} = a^i_{h_2} + a^i_{h_1} e^{\zeta h_2}$, and else one can set $a^i_h = 0$ $\forall h$, so again we can assume the above equation holds always. The same argument as above implies then the result in the case $\zeta = 0$. And for $\zeta \neq 0$, we get $a_{h_2} + a_{h_1} e^{\zeta h_2} = a_{h_1} + a_{h_2} e^{\zeta h_1}$, i.e., $a_{h_1}(e^{\zeta h_2} - 1) = a_{h_2}(e^{\zeta h_1} - 1)$. This implies first $a_0 = 0$, hence the result for $h = 0$, and next that, for all $h_1$, $h_2$ different from 0, $\frac{a_{h_i}}{e^{\zeta h_i}-1}$ is independent of $i$, so $\frac{a_h}{e^{\zeta h}-1}$ is constant over all $h \neq 0$. Since $\zeta \neq 0$, we can write this constant as $\frac{A}{e^{\zeta}-1}$, thus finishing the proof, the uniqueness part being elementary. ■

## Appendix B. Proofs

*Remark* 5. $K_E = \{\varphi : \mathbb{R} \to E \mid \forall f \in E^*, f \circ \varphi \in K_{\mathbb{R}}\}$.

*Proof.* Obviously, for $\varphi \in K_E$, $f \circ \varphi \in K_{\mathbb{R}}$. Conversely, since $f \circ \varphi$ is $C^\infty$ for all $f \in E^*$, $\varphi$ is $C^\infty$ with values in $E$ (e.g., Edwards, 1965, Ex. 8.14 p. 609). Since further each $f \circ \varphi$ has compact support, it is elementary that $\varphi$ has compact support. ■

*Proof of lemma 1.* By lemma 8, identifying values of $W$ with constant $\overline{\mathbb{R}}$-valued functions of time. ■

*Proof of thm. 1.* If $\zeta$ is not uniquely determined, lemma 1 implies $W$ is, on every straight line through $\bar{\pi}$, constant in a neighbourhood of $\bar{\pi}$. Letting thus $q = 0$ makes the result true for any $\zeta$.

Else there exists, by definition of a Gateaux-differential, $\mu \in K^*_E$ s.t.

$$DW_{\bar{\pi}}(\delta\pi) = \lim_{\varepsilon \to 0} \frac{W(\bar{\pi} + \varepsilon\delta\pi) - W(\bar{\pi})}{\varepsilon} = \langle \mu, \delta\pi \rangle \tag{1}$$

Start with the particular case $E = \mathbb{R}$ and $F = K$, using $K$ for $K_{\mathbb{R}}$.

By lemma 1, $W \circ \mathbb{S}_h = e^{\zeta h} W + A \frac{e^{\zeta h}-1}{e^{\zeta}-1}$, hence by constancy of $\bar{\pi}$ ($\mathbb{S}\bar{\pi} = \bar{\pi}$), and (1),

$$\langle \mu, \mathbb{S}_h(\delta\pi) \rangle = e^{\zeta h} \langle \mu, \delta\pi \rangle$$

Since $B$ is a neighbourhood of $\bar{\pi}$, every $\varphi \in K$ is a multiple of some $\delta\pi$, hence the following holds for all $h \in \mathbb{R}$ and all $\varphi \in K$:

$$\langle \mu, \varphi - e^{-\zeta h} \mathbb{S}_h \varphi \rangle = 0$$

Dividing by $h$ and taking the limit (in $K$!) as $h \to 0$ yields

$$\langle \mu, \varphi' + \zeta\varphi \rangle = 0$$

The definition of the derivative of a generalised function, $\mu \in K^*$,

$$\langle \mu', \varphi \rangle = - \langle \mu, \varphi' \rangle, \ \forall \varphi \in K$$

yields then

$$\langle \zeta\mu - \mu', \varphi \rangle = 0, \ \forall \varphi \in K$$

By Gel'fand and Shilov (1959, p. 53) the equation $\zeta\mu - \mu' = 0$ has $\mu = qe^{\zeta t}$ for some $q \in \mathbb{R}$ as only solutions in $K^*$, so,

$$DW_{\bar{\pi}}(\delta\pi) = \langle qe^{\zeta t}, \delta\pi\rangle = q\int e^{\zeta t}\delta\pi_t dt,\ \forall\delta\pi \in K \tag{2}$$

Next step is to extend the result to any Banach space $E$ and $F = K_E$.

**Lemma 9.** *Any function $\varphi \in K_E$ can be approximated in $K_E$ by functions with finite-dimensional range.*

*Proof.* Let $D_n = \{\varphi^{(i)}\big(\frac{j}{n!}\big) \mid 0 \le i \le n,\ j \in \mathbb{Z}\}$. $D_n$ is an increasing sequence of finite subsets of $E$. Let $F_n$ be the subspace spanned by $D_n$ and $p_n$ a projector from $E$ to $F_n$ (i.e., $p_n\colon E \to F_n$ is the identity on $F_n$). Its existence follows from Hahn-Banach, $F_n$ being finite-dimensional. Then $\varphi_n = p_n \circ \varphi \in K_{F_n}$ and converges in $K_E$ to $\varphi$. ∎

Consider policy variations $\delta\pi \in K_E$ of the form $b\phi\colon t \mapsto b\phi(t)$ with $b \in E$ and $\phi \in K$. By (1) and (2), $\forall b \in E\ \exists q_b \in \mathbb{R}$ s.t. $\langle\mu, b\phi\rangle = q_b\int\phi(t)e^{\zeta t}dt$ $\forall\phi \in K$. So, for $I_\phi = \int\phi(t)e^{\zeta t}dt \neq 0$, the map $b \mapsto q_b = \frac{\langle\mu,b\phi\rangle}{I_\phi}$ is in $E^*$, i.e., $q_b = \langle q, b\rangle$ with $q \in E^*$.

So, for any $\varphi$ of the form $b\phi$,

$$\langle\mu, \varphi\rangle = \int\langle q, \varphi(t)\rangle e^{\zeta t}dt \tag{1}$$

Since any $\varphi \in K_E$ with finite-dimensional range is a sum of policy variations of the form $b\phi$, (1) remains true by linearity for them. They being dense in $K_E$ by lemma 9, (1) extends by continuity to $K_E$.

Finally we extend the result to arbitrary $F$.

Since $K_E$ embeds continuously in $F$, $P^{K_E} \subseteq P^F$, and Gateaux-differentiability on $P^F$ implies that on $P^{K_E}$. Thus the assumptions of the theorem hold on $K_E$ too. So the differential is a continuous linear functional on $F$, given on $K_E$ by the formula $\int\langle q, \varphi(t)\rangle e^{\zeta t}dt$. This being by assumption continuous on $F$, the differential on $F$ must coincide with it, $K_E$ being dense in $F$. ∎

*Proof of lemma 2.* By lemma 8, with $n = \Theta$, $b_h = e^{\varrho h}$ and $a_h = A\frac{e^{\varrho h}-1}{e^{\varrho}-1}$ for some constants $A$ and $\varrho$, the ratio being $h$ for $\varrho = 0$. The result follows then by a straightforward verification. ∎

*Proof of lemma 4.* Since $\mathbb{S}_h\bar{\pi} = \bar{\pi}$, $W(\mathbb{S}_h\pi) = V_{c,r}(v(\mathbb{S}_h\pi) - v(\mathbb{S}_h\bar{\pi}))$; by def. 3, lemma 2, homogeneity of $V_{c,r}$, def. 4 and lemma 3 resp.,

$$\begin{aligned} W(\mathbb{S}_h(\pi)) &= V_{c,r}(e^{\varrho h}\mathbb{S}_h(v(\pi) - v(\bar{\pi}))) = e^{\varrho hr}V_{c,r}(\mathbb{S}_h(v(\pi) - v(\bar{\pi}))) \\ &= a_h e^{\varrho hr} + e^{(\varrho r+c)h}V_{c,r}(v(\pi) - v(\bar{\pi})) = a'_h + e^{(\varrho r+c)h}W(\pi) \end{aligned}$$ ∎

*Proof of lemma 5.* $S^*(v) = W$ of lemma 4, using the SWA, with degree $r = 1$ and parameter $c = \nu - \beta$, $V_{c,r}\colon u \mapsto \int^* e^{-\beta x}\sum_\tau N_0^\tau e^{\nu x}u_x^\tau dx$. ∎

*Proof of thm. 2.* G-differentiability of $W$ implies assumption (i) of thm. 1 holds, and the other is assumed. It also implies $S^*$ is Gateaux-differentiable, hence its differential from thm. 1 is the G-differential of $W$. ∎

*Proof of lemma 6.* Since the closed convex cone $Y$ is pointed (irreversibility), there exists a linear functional $\alpha$ whose unique maximiser on $Y$ is 0. Then $\langle \alpha, y\rangle \le -\varepsilon\|y\|$ on $Y$, i.e., by rescaling $\alpha$, $\langle \alpha, y\rangle \le -\|y\|$. Observe too that free disposal implies $\alpha \gg 0$. Write $\alpha$ as $(\alpha^L, \alpha^K, \alpha^I, \alpha^C)$.

First step is to establish the bound on $K$ sub (ii).

Fix a vector $\bar{L} \in \mathbb{R}^h$ s.t. any feasible vector of labour inputs $L_t \le \bar{L}e^{(\gamma+\nu)t}$ (i.e., to compute a given coordinate of $\bar{L}$, assume all agents spend 100% of their time on that activity).

Allow perfect substitution at rates $\alpha^I$ between all investment goods and between all capital goods: let $F\colon \mathbb{R}^2_+ \to \mathbb{R}_+\colon (\bar\kappa, \lambda) \mapsto \mathsf{sup}\{\langle \alpha^I, I\rangle \mid \exists K \ge 0, \langle \alpha^I, K\rangle \le \bar\kappa, (-\lambda\bar{L}, -K, I, 0) \in Y\}$.

The $\mathsf{sup}$ is finite, since $\langle \alpha^I, I\rangle \le \langle \alpha^K, K\rangle + \lambda\langle \alpha^L, \bar{L}\rangle$ and $\langle \alpha^K, K\rangle$ is bounded on the compact set $K \ge 0, \langle \alpha^I, K\rangle \le \bar\kappa$ (recall $\alpha \gg 0$). Further the $\mathsf{sup}$ is achieved, the sets $\{y \in Y \mid \langle \alpha, y\rangle \ge -M\}$ being compact (since then $\|y\| \le M$), so that the $\mathsf{sup}$ is effectively over a compact set. Clearly $F$ is positively homogeneous of degree 1 and concave, and is continuous again because locally everything happens within a compact subset of $Y$.

Thus by homogeneity $F(\bar\kappa, \lambda) = \lambda\varphi(\frac{\bar\kappa}{\lambda})$, where $\varphi(x) = F(x, 1)$ is concave, $\ge 0$ and continuous. Further by the "no-rabbit" assumption, $F(\bar\kappa, 0) = 0$, so, by continuity of $F$, $\frac{\varphi(x)}{x} \to 0$ at $\infty$.

For any feasible path $(L_t, K_t, I_t, C_t)$, let $\tilde\iota_t = \langle \alpha^I, I_t\rangle$ and $\tilde\kappa_t = \langle \alpha^I, K_t\rangle$. Then, since $\delta \le \delta^i$ and $K \ge 0$, the capital-accumulation equation implies $\tilde\kappa'_t \le \tilde\iota_t - \delta\tilde\kappa_t$; further $L_t \le \bar{L}e^{(\gamma+\nu)t}$ implies (free-disposal) $\tilde\iota_t \le e^{(\gamma+\nu)t}\varphi(e^{-(\gamma+\nu)t}\tilde\kappa_t)$. So with $(l_t, \kappa_t, \iota_t, c_t) = e^{-(\gamma+\nu)t}(L_t, \tilde\kappa_t, \tilde\iota_t, C_t)$:

$$\kappa'_t \le \varphi(\kappa_t) - R\kappa_t$$

To bound $\|k_t\|$, it suffices to prove from this that $\kappa_t$ is bounded by some constant independent of the feasible path, since $\alpha^I \gg 0$.

Also the initial condition yields that $e^{Rt}\kappa_t$ converges exponentially fast to 0 at $-\infty$, i.e., since $R > 0$, there exists $\varepsilon\colon 0 < \varepsilon < R$ such that, with $r = R - \varepsilon > 0$, $e^{rt}\kappa_t \to 0$ at $-\infty$ along a subsequence. Since $\frac{\varphi(x)}{x} \to 0$ at $\infty$, there exists $A$ s.t., $\forall x$, $\varphi(x) \le A + \varepsilon x$; so $\kappa'_t \le A - r\kappa_t$.

Next step is to prove from this that $\kappa_t \le \bar{K}$, with $\bar{K} = \frac{A}{r}$.

Else $\kappa_{t_0} > \bar{K}$ for some $t_0$; since $\kappa'_t < 0$ for $\kappa_t > \bar{K}$, this implies that $\kappa_t > \bar{K}$ and is decreasing for $t \le t_0$. Define $y$ by $y'_t = A - ry_t$ with the prescribed terminal value $\kappa_{t_0}$ at $t_0$. The relations for $\kappa$ and $y$ are equivalent to $\frac{r\,d}{de^{rt}}(e^{rt}\kappa_t) \le A$ and $\frac{r\,d}{de^{rt}}(e^{rt}y_t) = A$, so, since $\frac{de^{rt}}{rdt} > 0$, $\frac{d}{dt}(e^{rt}\kappa_t) \le \frac{d}{dt}(e^{rt}y_t)$: for $t \le t_0$, $\kappa_t \ge y_t = \frac{A}{r} + (\kappa_{t_0} - \frac{A}{r})e^{r(t_0-t)}$, contradicting that $e^{rt}\kappa_t \to 0$ at $-\infty$ along a subsequence.

Hence the uniform bound on $\|k_t\|$ .

Next, $\langle \alpha, y\rangle \le -\|y\|$ yields $\int_a^b \|i_t\|dt \le \int_a^b (\langle \alpha^L, l_t\rangle + \langle \alpha^K, k_t\rangle - \langle \alpha^I, i_t\rangle - \langle \alpha^C, c_t\rangle)dt$. The last inner product is non-negative, and the capital-accumulation equation yields $k'^j_t = i^j_t - R^j k^j_t$ with $R^j = \gamma + \nu + \delta^j$ as before, so that $\int_a^b \langle \alpha^I, i_t\rangle dt = \int_a^b \langle \alpha^I, k'_t\rangle dt + \sum_j \alpha^I_j R^j \int_a^b k^j_t dt = \langle \alpha^I, k_b - k_a\rangle + \sum_j \alpha^I_j R^j \int_a^b k^j_t dt \ge -\langle \alpha^I, k_a\rangle$, since $R^j > 0$. Thus our bounds on $k_t$ and $l_t$ imply $\int_a^b \|i_t\|dt \le \bar{K}(b - a + 1)$ for some constant $\bar{K}$.

Thus point (ii). For (i), $e^{-(\gamma+\nu)t}\int_{-\infty}^{t} e^{\delta^j(s-t)}|I_s^j|ds = \int_0^\infty e^{-R^j x}|i_{t-x}^j|dx \le \sum_{n=0}^\infty e^{-R^j n}\int_n^{n+1}|i_{t-x}^j|dx$ is uniformly bounded by (ii). For $M_t^i = e^{\delta^i t}K_t^i$, the differential equations become $M_t^{i\prime} = h_t^i$, with $h_t^i \stackrel{\text{def}}{=} e^{\delta^i t}I_t^i$, hence, by the integrability, $M_t^i = M_{-\infty}^i + \int_{-\infty}^t h_s^i ds$. And the initial condition yields $\lim_{t\to-\infty} M_t^i = 0$, so $M_{-\infty}^i = 0$, hence (i). ■

*Proof of prop. 1.* Comparing the rescaling of consumption goods in the consumption sets (ii) and in the production set (iii) shows that the mass of any agent is to be multiplied by $C = e^{\nu h}$. For labour goods, this ratio is correct too, given the labour saving technological growth included in aggregate effective labour. By (i), the "induced map of agents" maps an individual of type $\tau$ born at time $t$ to an individual of the same type born at $t+h$, so the population of the new economy at time $t$ is that of the old at time $t-h$ multiplied by $e^{\nu h}$, hence equals that of the original economy at $t$: the population measure is preserved. Remains thus only to prove that preferences are preserved and that allocations are mapped to allocations: the one-to-one and onto aspect will then follow from the same property for the inverse $\mathbb{T}_{-h}$. Consumption sets are unchanged: at any $t$ non-negativity constraints are preserved by the re-scalings (ii), besides, time fractions are not re-scaled, so the constraint that their sum be $\le 1$ is preserved too. Preferences are homogeneous in the consumption goods, so are preserved by re-scaling (ii). As for production, capital accumulation equations are linear in capital and investment, so are preserved given (iii), as well as the initial condition (also in its weak form of prop. 4): both convergence to 0 and exponential convergence to 0 are preserved under shifting and multiplication by a constant. And $Y$ is unchanged under the scaling by $e^{(\gamma+\nu)h}$ (iii). ■

*Proof of prop. 2.* Let $(c,l) \stackrel{\text{def}}{=} \varsigma(\pi)$. By the time-invariance of $\varsigma$, $\varsigma(\mathbb{S}_h\pi) = \mathbb{T}_h(c,l) = (e^{\gamma h}\mathbb{S}_h(c), \mathbb{S}_h(l))$. So, by homogeneity of $U$, $v\circ\mathbb{S}_h = e^{(1-\rho)\gamma h}\mathbb{S}_h(v)$. Thus $v$ is a valuation with $a_h = 0$ and $b_h = e^{(1-\rho)\gamma h}$. ■

*Proof of cor. 3.* Let $V_c = \sum_\tau V_c^\tau$, where $V_c^{\tau_0}$ with $c = \nu - \beta$ is the utility aggregator defined as $\int_{-\infty}^\infty e^{-\beta t}\sum_\tau \mathbf{1}_{\tau_0} N_t^\tau\,(v_t^\tau(\pi) - \bar{u}_t^\tau)\,dt$. Applying cor.2 to each $V_c^\tau$, i.e., to the economy in which utilities of all types but $\tau$ are identically zero, one obtains the differential of $W$:

$$\sum_\tau \int_{-\infty}^{\infty} e^{(\nu-\beta+(1-\rho^\tau)\gamma)t}\langle q^\tau, \delta\pi(t)\rangle dt = \int_{-\infty}^{\infty} e^{(\nu-\beta+\gamma)t}\Big\langle \sum_\tau e^{-\rho^\tau\gamma t}q^\tau, \delta\pi(t)\Big\rangle dt$$

Visibly the criterion $q^\tau$ of the types with the smallest $\rho$ asymptotically gets all the weight. ■

*Proof of prop. 3.* Shift invariance of $A$ implies $\mathbb{T}$-invariance of the set of the induced (acceptable) allocations under an outcome function $\varsigma$, thus if $\varsigma(\pi) = (c,l)$ for some $\pi \in A$, then $\forall h\ \varsigma(\mathbb{S}_h\pi) = (e^{\gamma h}\mathbb{S}_h c, \mathbb{S}_h l)$, and $\mathbb{S}_h\pi \in A$. So the utility difference between the worst and best acceptable allocations for an agent of type $\tau$ is, by homogeneity of utilities, proportional to $e^{(1-\rho^\tau)\gamma x}$: this is the normalisation factor. Thus, again by homogeneity, $\mathbb{M}_A(v_x^\tau(\pi)) = w^\tau U^\tau(e^{-\gamma x}c_x^\tau, l_x^\tau)$. Hence, since $\varsigma\circ\mathbb{S}_h = \mathbb{T}_h\circ\varsigma$ and $e^{-\gamma x}\mathbb{T}_h c_x^\tau = \mathbb{S}_h(e^{-\gamma x}c_x^\tau)$, one gets $\mathbb{M}_A(v_x^\tau(\mathbb{S}_h\pi)) = w^\tau U^\tau(\mathbb{S}_h(e^{-\gamma x}c_x^\tau, l_x^\tau)) = \mathbb{S}_h\mathbb{M}_A(v_x^\tau(\pi))$. ■

## Appendix C. Initial Conditions

Even with the weakest initial condition, say $K_t$ bounded at $-\infty$, one should expect $K_t^i$ to converge to 0 at $-\infty$ if $\delta^i > 0$. But land and natural resources are the typical examples of goods with $\delta^i = 0$, so the natural value for $\delta$ in a general form of the initial condition and lemma 6 is 0.[29] The initial condition is thus a bit too stringent conceptually, requiring exponential convergence to the initial value 0 instead of just plain convergence. An additional reason to want just plain convergence there is that then the "initial condition" becomes equivalent to the initial condition for the integral formula of $K_t$ in terms of $I_t$: even with $\delta^i = 0$, $K_t^i = K_{-\infty}^i + \int_{-\infty}^t e^{\delta^i(s-t)} I^i(s)ds$ implies $K_t^i \to K_{-\infty}^i$ at $-\infty$.[30] We make a first attempt here to address this issue.

**Lemma 10.** *(i) Every $\beta \gg 0$ in $\mathbb{R}^m$ is the $\alpha^I$ of some linear functional $\alpha$ having a unique maximiser on $Y$.*

*(ii) For $\alpha \gg 0$, let $\psi_\alpha(x) = \mathsf{sup}\{\langle\alpha, I\rangle \mid \exists K \geq 0, \|K\| \leq x, (-(1,1,1,\ldots), -K, I, 0) \in Y\}$. Then $\int_1^\infty \frac{\psi_\alpha(x)}{x^2}dx$ is finite iff the same integral is finite replacing $\psi_\alpha(x)$ by $\mathsf{sup}\{\langle\alpha, I\rangle \mid \exists K \geq 0, \langle\alpha, K\rangle \leq x, (-\bar{L}, -K, I, 0) \in Y\}$.*

*Proof.* For point (i), let $E$ be the commodity space $\mathbb{R}^h \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n$, containing $Y$, with vectors typically denoted $(-L, -K, I, C)$. Let $F$ be the subspace where $L = C = 0$, and let $\beta'$ extend $\beta$ with arbitrary positive $K$-coordinates. Let $G = \{x \in F \mid \langle\beta', x\rangle = 0\}$. By the No-Rabbit assumption, $G \cap Y = \{0\}$. By irreversibility, there exists a linear functional $\gamma$ on $E$ having a unique maximizer on $Y$; so $Y' = \{y \in Y \mid \langle\gamma, y\rangle \leq -1\}$ and $G$ are disjoint closed convex sets, with disjoint asymptotic cones: their difference is a closed convex set disjoint from 0, hence they can be strictly separated: there is a linear functional $\alpha$ with $\langle\alpha, G\rangle > \langle\alpha, Y'\rangle$. $G$ being a subspace, this implies $\alpha$ vanishes on $G$, and has 0 as unique maximiser on $Y$. Thus some positive multiple of $\alpha$ coincides with $\beta'$ on $F$; in particular, $\alpha^I = \beta$.

For point (ii), observe first that the integrability condition on $\psi_\alpha(x)$ is equivalent to that on $\psi_\alpha(cx)$, for any $c > 0$. Now, $K \geq 0$ and $\alpha \gg 0$ imply that $\langle\alpha, K\rangle$ is a norm, so for any norm there exist $\underline{c} > 0$ and $\bar{c}$ such that $\underline{c}\|K\| \leq \langle\alpha, K\rangle \leq \bar{c}\|K\|$. The the independence from $c$ of the condition on $\psi_\alpha$ allows then to replace that inner product by $\|K\|$. Similarly, to replace $\bar{L}$ by a vector of 1's, first majorise and minorise it by a multiple of this vector. ∎

As the proof of lemma 6 shows, the "No-Rabbit" condition is equivalent to $\frac{\varphi(x)}{x} \to 0$, so the condition $\int_1^\infty \frac{\varphi(x)}{x^2}dx < \infty$ appears as a very slight strengthening. This justifies the following:

**Definition 11.** The "Strong No-Rabbit" condition on $Y$ is that $\int_1^\infty \frac{\varphi_\alpha(x)}{x^2}dx < \infty$ for some $\alpha \gg 0$.

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[29]Conceptually our "Initial Condition" is best thought of as a pair: on the one hand, a general form, say something like $K_t$ bounded at $-\infty$, provided one can prove from this convergence at $-\infty$, and on the other hand a specific assumption to ensure balanced growth, i.e., that the limit is 0.

[30]Independently of the natural requirement that for natural resources (e.g., mining —), $Y$ should force $I^i \leq 0$, and for land (raw acreage), $I^i = 0$.

**Proposition 4.** *The conclusions of lemma 6 remain true when weakening in the Initial Condition the exponentially fast convergence to plain convergence, provided the Strong No-Rabbit condition holds.*

*Proof.* Fix $\alpha \in A$ according to the Strong No-Rabbit condition, and, using lemma 10.i to find a corresponding $\alpha$ in the proof of lemma 6, follow that proof till where $\varphi$ is majorised by $A + \varepsilon x$, and let now $f(x) = Rx - \varphi(x)$, $\bar{K} = \inf\{k \mid f(k) > 0\}$, and fix a corresponding $\kappa_{t_0}$.

At that point, prove first that, for $t \le t_0$, $\kappa_t \ge y_t$, with $y_t$ the solution of $y'_t = \varphi(y_t) - Ry_t$ with prescribed value at $t_0$: reversing time, and translating $t_0$ to 0, we have, using $x_t$ for $\kappa_t$, $x'_t \ge f(x_t)$ a.e., $y'_t = f(y_t)$ a.e., $x_0 = y_0$, $f(x_0) > 0$, $f$ is increasing for $x \ge x_0$, and need to show that $x_t \ge y_t$ for $t > 0$. Translating $f$ and the functions $x$, $y$, we can even assume $x_0 = y_0 = 0$, $f(0) > 0$, so $f$ is positive and increasing on $\mathbb{R}_+$. So $H(x) = \int_0^x \frac{1}{f(y)} dy$ is well-defined, positive, $C^1$, concave and increasing on $\mathbb{R}_+$. Assuming the chain-rule for differentiation established for the composition $H \circ x$ of such an $H$ with a Denjoy primitive like $x_t$, we obtain $(H \circ x)'_t = H'(x_t)x'_t = \frac{x'_t}{F(x_t)} \ge 1$, and similarly $(H \circ y)'_t = 1$, hence, for $t \ge 0$, $H(x_t) \ge H(y_t)$ and so $x_t \ge y_t$ by strict monotonicity of $H$.

Thus indeed $\kappa_t \ge y_t$ for $t \le t_0$. Since further $\kappa_t$ and $y_t$ are decreasing and continuous on that range, they have continuous and decreasing inverse functions $t^\kappa$ and $t^y$ defined on $[\kappa_{t_0}, \infty[$ and values in $]-\infty, t_0]$, and there $t^\kappa \ge t^y$. Now $y'_t = -f(y_t)$ means $\frac{dy}{f(y)} = -dt$, hence, since $y_{t_0} = \kappa_{t_0}$, $t^y(x) = t_0 - \int_{\kappa_{t_0}}^x \frac{dz}{f(z)}$. So $t^\kappa(x) \ge t_0 - \int_{\kappa_{t_0}}^x \frac{dz}{f(z)}$.

But the "weak" initial condition is that $e^{Rt}\kappa(t) \xrightarrow[t \to -\infty]{} 0$, so, $e^{Rt^\kappa(x)}x \xrightarrow[x \to \infty]{} 0$, i.e., taking logarithms, $\ln(x) + Rt^\kappa(x) \xrightarrow[x \to \infty]{} -\infty$, and thus, by our bound on $t^\kappa$,[31] and replacing $\ln$ by its integral definition, neglecting constants, $\int_a^x \frac{dz}{z} - R\int_a^x \frac{dz}{f(z)} \xrightarrow[x \to \infty]{} -\infty$, with $a = \max\{1, \kappa_{t_0}\}$. Given the formula for $f$, this means $\int_a^x \frac{\varphi(z)}{z(Rz - \varphi(z))} \to \infty$, and hence, $\varphi(z)$ being negligible compared to $z$, and $R > 0$, that $\int_1^x \frac{\varphi(z)}{z^2} \to \infty$, contradicting the Strong No-Rabbit condition by lemma 10.ii. Thus indeed $\kappa_t \le \bar{K}\, \forall t$. The rest of the proof of lemma 6 remains as is. ■

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[31]Note that for $m = 1$ this bound is attained, so the Strong No-Rabbit condition is best possible: else, under the "weak" initial condition, there exist feasible paths with $\|k_t\|$ unbounded at $-\infty$, and for any fixed $t$ the set of feasible $K_t$ is unbounded.

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