Asymmetric Contests with Interdependent Valuations and Incomplete Information

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Abstract

I show that a unique equilibrium exists in a two-player all-pay auction with asymmetric independent discrete signal distributions and asymmetric interdependent valuations.

1 Introduction

This paper investigates a contest model in which two asymmetric contestants compete for a prize by expending resources. Each contestant has some private information that may affect both contestants' valuation for the prize, and contestants are asymmetric in that their private information may be drawn from different distributions and impact their valuations differently. For example, consider a research and development race in which the firm with the highest-quality product enjoys a dominant market position. Each firm may be informed about different attributes of the market, which together determine the value of winning. This value may differ across firms, because the profit associated with a

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dominant market position may depend on firm-specific characteristics such as production costs and marketing expertise.

Section 2 models the contest as an asymmetric all-pay auction with independent signals and interdependent valuations. Each player's signal is drawn from a finite set of ordered signals according to a probability distribution with full support. The sets of signals and corresponding probability distributions may differ across players. After observing his signal, each player decides how much to expend, and the player with the higher expenditure wins the prize. The value of the prize is a player-specific function of both players' signals. The only restriction is that this function increase in the player's own signal. This model includes complete information, private values, common values, and one informed and one uninformed player as special cases.

Section 3 contains the main result of the paper, which is a constructive characterization of the unique equilibrium. This characterization uses the finiteness of players' type spaces to employ insights from the analysis of complete-information models. Each player, for each of his types, chooses his expenditure from an interval, and higher types choose expenditures from higher intervals. This ordering of intervals means that, by proceeding from the top, the equilibrium can be constructed in a finite number of steps. In each step, one type of player 1 competes against one type of player 2. In the resulting interval of competition, players compete as in a complete-information all-pay auction with valuations that correspond to the competing types. Once one player has exhausted the probability mass associated with his lowest type, any remaining probability mass of the other player is expended as an atom at 0. This simple procedure is easy to implement as a computer program.[1]

As an application of the construction result, Section 3.1 enumerates the possible equilibrium orderings of players' intervals, which depend on players' valuations and probability distributions. Section 4 provides a complete characterization when each player has one or two types (excluding the case of complete information, which has been well studied by, for example, Hillman and Riley (1989)). Section 4 also shows that when the value of the prize is common to both players, their equilibrium strategies are identical from an ex–ante

[1]A Matlab implementation of this procedure is available on my website, http://faculty.wcas.northwestern.edu/~rsi665/.

perspective. Players' payoffs may differ, however, because each player may condition his expenditure on his private information, which may differ between the players. A closed form description of the equilibrium is available when, in addition to common values, only one player has private information. A closed-form solution is also available when players are symmetric.

Section 5 uses the equilibrium construction result to derive a candidate equilibrium when players' types are drawn from continuous distributions. The continuous distributions are approximated by increasingly finer discrete distributions, and the limit of the equilibria of the corresponding contests deliver a differential equation that identifies a candidate equilibrium. When players have private values, the candidate equilibrium coincides with that of Amann and Leininger (1996). They considered an asymmetric two-player all-pay auction with private values and continuous type distributions, and characterized the unique equilibrium candidate within the class of differentiable equilibria.[2] They did they consider discrete distributions or interdependent valuations.

Beyond Amann and Leininger's (1996) work, few papers have studied auction-like contests with incomplete information and asymmetries in players' valuations and information structure, even though these features arise in many real-world competitions with sunk investments. Lizzeri and Persico (2000) showed that with a binding reservation price, a unique equilibrium exists in a general two-player auction model with continuous type distributions. Their proof, however, is not constructive, and also relies on the continuity of the type distributions and on the reservation price excluding a positive measure of types. Parreiras and Rubinchik's (2010) characterized some equilibrium properties in an asymmetric all-pay auction with private values and more than two players.

[2] They did not prove that the candidate equilibrium is indeed an equilibrium, or that it is unique within the class of all equilibria. These lacunae can most likely be filled by the tools developed in Lizzeri and Persico (2000).

2 Model

There are two players and one prize. Each player $i = 1, 2$ observes a private signal $s_i \in S_i$, where S_i is a finite ordered set. The signals in S_i are distributed according to a commonly-known probability distribution F_i, where $F_i(s_i) > 0$ is the probability of signal s_i. The distributions F_1 and F_2 are statistically independent. Player i's valuation for the prize is $V_i : S_1 \times S_2 \rightarrow \mathbb{R}_{++}$, which is strictly increasing in player i's signal, but need not be increasing or monotonic in the other player's signal. After observing their signals, the players compete in an all-pay auction: they simultaneously chooses how much money to bid, forfeit their bid, and the player with the higher bid wins the prize (in case of a tie, any procedure can be used to allocate the prize between the players). Thus, player i's payoff if he observes signal s_i and players' bids are b_1 and b_2 is

$$u_i(s_i, b_1, b_2) = P_i(b_1, b_2) \sum_{s_{-i} \in S_{-i}} \left(F_{-i}(s_{-i}) V_i(s_i, s_{-i})\right) - b_i,$$

where $-i$ refers to player $3 - i$,

$$V_i(s_i, s_{-i}) = \begin{cases} V_1(s_1, s_2) & \text{if } i = 1 \\ V_2(s_1, s_2) & \text{if } i = 2 \end{cases},$$

and

$$P_i(b_1, b_2) = \begin{cases} 1 & \text{if } b_i > b_{-i}, \\ 0 & \text{if } b_i < b_{-i}, \\ \text{any value in } [0, 1] & \text{if } b_i = b_{-i}, \end{cases},$$

such that $P_1(b_1, b_2) + P_2(b_1, b_2) = 1$.

3 Equilibrium

Denote a mixed strategy of player i by $G_i : S_i \times \mathbb{R} \rightarrow [0, 1]$, where $G_i(s_i, x)$ is the probability that player i bids at most x when he observes s_i (so $G_i(s_i, \cdot)$ is a CDF for every signal s_i). Abusing notation, I will sometimes treat G_i as a function of one variable, $G_i(x) = \sum_{s_i \in S_i} F_i(s_i) G_i(s_i, x)$, so G_i is also a CDF. An equilibrium is a pair $\mathbf{G} = (G_1, G_2)$, such that given that player i plays the mixed strategy G_i, the CDF $G_{-i}(s_{-i}, \cdot)$ assigns measure

1 to player $-i$'s set of best responses when he observes signal s_{-i}, for every signal s_{-i}. I say that a player has an *atom* at x if the player bids x with positive probability when he observes one of his signals.

One difficulty in solving for equilibrium is that a player's valuation for the prize may depend on the other player's signal, which can be inferred (at least partially) from the other player's equilibrium bid. This does not happen in a private-value model. Of course, even with private values a player's probability of winning with a given bid depends on the other player's strategy. The additional complication here is that the player's valuation for the prize also depends on the other player's strategy (through the equilibrium mapping between the other player's signal and his strategy). The key to solving for equilibrium is to show that a simple structure governs this dependency.

The remainder of the section characterizes the unique equilibrium. I begin with a few preliminaries.

Lemma 1 *In any equilibrium* $\mathbf{G}$*, (i) there is no bid at which both players have an atom, (ii) there is no positive bid at which either player has an atom, (iii) if* $x > 0$ *is not a best response for player* i *for any signal, then no bid* $y \geq x$ *is a best response for either player for any signal, and (iv) both players have best responses at* 0 *or arbitrarily close to* 0.

Proof. For (i), suppose that player 1 chose x with positive probability when he observed s_1, and player 2 did the same when observing s_2. Because $F_1(s_1) > 0$, player 2 could do strictly better by choosing a bid slightly above x, so x cannot be a best response for player 2 when observing s_2, a contradiction. For (ii), suppose that player i chose $x > 0$ with positive probability when he observed s_i. By an argument similar to the one used to prove (i), the other player would not have best responses on some positive-length interval with upper bound x for any signal. But then player i could do strictly better by bidding slightly below x, so x cannot be a best response for player i when observing s_i, a contradiction. For (iii), note that (ii) proved that each player's CDF is continuous above 0 for any signal he observes. Therefore, if $x > 0$ is not a best response for player i at any signal, the same is true for all bids in a some maximal neighborhood of x. This implies that the other player also does not choose any bids in this neighborhood. But then, again by continuity, no player would have a best response at the top of this neighborhood. For (iv), suppose that 0 is not a best response for one of the players and that player does not have best responses arbitrarily close to 0. This means that the player does not have best responses in some interval with lower endpoint 0. By (iii), the player does not have any best-responses, so **G** is not an equilibrium. ■

Denote by $u_i(s_i, x)$ player i's expected payoff when he observes signal s_i and bids x and the other player uses strategy G_{-i}. Choose an equilibrium **G**, and denote by $BR_i(s_i)$ player i's best responses when he observes signal s_i and the other player uses strategy G_{-i}.

Lemma 2 *If $s_i' > s_i$, then for any x in $BR_i(s_i)$ and y in $BR_i(s_i')$, we have $y \geq x$.*

Proof. Choose x in $BR_i(s_i)$ and y in $BR_i(s_i')$. Suppose $x > y$. By part (ii) of Lemma 1, neither player has an atom at $x > 0$. And player $-i$ does not have an atom at y, otherwise y would not be in $BR_i(s_i')$. Therefore,

$$u_i(s_i, x) - u_i(s_i, y) = \sum_{s_{-i} \in S_{-i}} (F_{-i}(s_{-i}) V_i(s_i, s_{-i}) (G_{-i}(s_{-i}, x) - G_{-i}(s_{-i}, y))) - (x - y) \geq 0, \tag{1}$$

where the last inequality follows from $u_i(s_i, x) \geq u_i(s_i, y)$, because x is in $BR_i(s_i)$. This last inequality and $x - y > 0$ imply that $G_{-i}(s_{-i}, x) - G_{-i}(s_{-i}, y) > 0$ for at least one signal s_{-i}. This shows that the value of (1) strictly increases if s_i is replaced with s_i', because $V_i(s_i', s_{-i}) > V_i(s_i, s_{-i})$ for every signal s_{-i}. Therefore, $u_i(s_i', x) > u_i(s_i', y)$, which means that y is not in $BR_i(s_i')$, a contradiction. ■

The following corollary of Lemmas 1 and 2 describes the structure of players' best response sets in any equilibrium.

Corollary 1 *For every player i and every signal s_i, $BR_i(s_i)$ is an interval. For any two consecutive signals $s_i' > s_i$, the upper bound of $BR_i(s_i)$ is equal to the lower bound of $BR_i(s_i')$. Moreover,*

$$\sup \cup_{s_1 \in S_1} BR_1(s_1) = \sup \cup_{s_2 \in S_2} BR_2(s_2) \text{ and } \inf \cup_{s_1 \in S_1} BR_1(s_1) = \inf \cup_{s_2 \in S_2} BR_2(s_2) = 0. \tag{2}$$

Proof. By part (iii) of Lemma 1 and Lemma 2, $BR_i(s_i)$ is an interval. By part (iii) of Lemma 1, $BR_i(s_i) \cap BR_i(s_i')$ is not empty, and Lemma 2 shows that this intersection can include only the boundaries of the best-response sets. Parts (iii) and (iv) of Lemma 1 imply (2). ■

A structure consistent with Corollary 1 is depicted in Figure 1, where T denotes the common upper bounds of players' best response sets.[3]

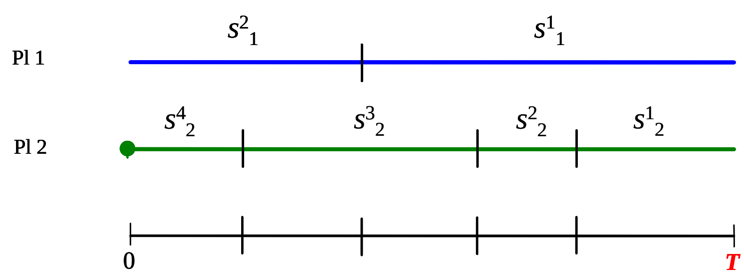

Figure 1: A possible equilibrium structure when player 1 has two signals, player 2 has four signals, and player 2 has an atom at 0

This structure shows that the equilibrium can be found by using an iterative procedure. To see this, let s_i^k be the k^{th} signal in S_i when signals are ordered from highest to lowest. Consider the coarsest partition of $[0,T]$ that includes both partitions of $[0,T]$ into players' best response sets (henceforth: the joint partition). In Figure 1, the joint partition is depicted on the bottom line. Consider two bids $x < y$ in the top interval of this partition, and let

$$U_i(s_1', s_2') = E[V_i(s_1', s_2) | s_2 < s_2']$$

be player i's expected valuation for the prize conditional on him observing s_1' and player 2 observing a signal below s_2'. Because both x and y are best responses for player 1, we have

$$\begin{aligned} & (1 - F_2(s_2^1)) U(s_1^1, s_2^1) + F_2(s_2^1) G_2(s_2^1, y) V_1(s_1^1, s_2^1) - y \qquad (3) \\ = \; & (1 - F_2(s_2^1)) U(s_1^1, s_2^1) + F_2(s_2^1) G_2(s_2^1, x) V_1(s_1^1, s_2^1) - x, \end{aligned}$$

[3]Note that because $T > 0$ (at most one player has an atom at 0) and players' strategies are continuous above 0 (part (ii) of Lemma 1), T is a best response for both players' highest types.

which can be rewritten as

$$\frac{G_2\left(s_2^1, y\right) - G_2\left(s_2^1, x\right)}{y - x} = \frac{1}{F_2\left(s_2^1\right) V_1\left(s_1^1, s_2^1\right)}.$$

Taking $y - x$ to 0 shows that in the top interval $G_2\left(s_2^1, \cdot\right)$ is differentiable with density

$$g_2\left(s_2^1, x\right) = \frac{1}{F_2\left(s_2^1\right) V_1\left(s_1^1, s_2^1\right)}.^4$$

Similarly, in the top interval $G_1\left(s_1^1, \cdot\right)$ is differentiable with density

$$g_1\left(s_1^1, x\right) = \frac{1}{F_1\left(s_1^1\right) V_2\left(s_1^1, s_2^1\right)}.$$

Once we identified the densities of players' strategies in the top interval of the joint partition, we can find the length of this interval. For this, note that because $BR_i\left(s_i\right)$ is an interval, the top interval of the joint partition ends when one of the two players exhausts the probability mass associated with his highest signal. Therefore, the length of the top interval is

$$\min\left\{F_2\left(s_2^1\right) V_1\left(s_1^1, s_2^1\right), F_1\left(s_1^1\right) V_2\left(s_1^1, s_2^1\right)\right\}, \tag{4}$$

with the player whose density determines the length of the interval exhausting the probability mass associated with his highest signal. Players' densities in the next interval are calculated in a similar fashion, with the player(s) who has exhausted the probability mass associated with his highest signal "moving" to his second highest signal. This process is iterated, calculating the length of each interval and players' densities in each interval. Suppose we are at the k^{th} interval of the joint partition, after player 1 has exhausted the probability mass associated with his k_1 highest signals and player 2 has exhausted the probability mass associated with his k_2 highest signals. The equivalent of Equation 3 is then

$$\left(1 - \sum_{j=0}^{k_2} F_2\left(s_2^{j+1}\right)\right) U_1\left(s_1^{k_1+1}, s_2^{k_2+1}\right) + F_2\left(s_2^{k_2+1}\right) G_2\left(s_2^{k_2+1}, y\right) V_1\left(s_1^{k_1+1}, s_2^{k_2+1}\right) - y \tag{5}$$

$$= \left(1 - \sum_{j=0}^{k_2} F_2\left(s_2^{j+1}\right)\right) U\left(s_1^{k_1+1}, s_2^{k_2+1}\right) + F_2\left(s_2^{k_2+1}\right) G_2\left(s_2^{k_2+1}, x\right) V_1\left(s_1^{k_1+1}, s_2^{k_2+1}\right) - x,$$

4 Compare this expression with $\frac{1}{V_1}$, the expression for player 2's density in a complete-information all-pay auction.

which leads to densities

$$g_2\left(s_2^{k_2+1},x\right)=\frac{1}{F_2\left(s_2^{k_2+1}\right)V_1\left(s_1^{k_1+1},s_2^{k_2+1}\right)}\text{ and }g_1\left(s_1^{k_1+1},x\right)=\frac{1}{F_1\left(s_1^{k_1+1}\right)V_2\left(s_1^{k_1+1},s_2^{k_2+1}\right)}. \tag{6}$$

When computing the length of this interval, the probability mass associated with the signals $s_1^{k_1+1}$ and $s_2^{k_2+1}$ expended on higher intervals must be taken into account (at most one of these signals will have probability mass expended on higher intervals, by definition of the joint partition).

When one of the players has exhausted the probability mass associated with his lowest signal, the remaining mass of the other player must be an atom, and this atom must be at bid 0 (part (ii) of Lemma 1). This atom may include the mass associated with several signals. If both players exhaust their probability mass simultaneously, then the point of exhaustion is also 0 (part (iv) of Lemma 1). By going from 0 upwards, the equilibrium can be constructed from players' densities on each interval. The following result shows that the resulting pair of strategies is the unique equilibrium.

Proposition 1 *There is a unique equilibrium, constructed by the procedure above. In this equilibrium, each player's strategy is continuous above* 0 *and piecewise uniform. At most one player has an atom* 0*.*

Proof. Because the procedure described above relies on necessary conditions for equilibrium, the pair of strategies resulting from the procedure is the unique candidate for an equilibrium. To show that it is indeed an equilibrium, it suffices to show that every type of each player chooses best responses with probability 1. Suppose that player 1 observes s_1^k, and denote by l_k and t_k the upper and lower bounds of the interval on which $g_1\left(s_1^k,\cdot\right)>0$ (as identified by the procedure above). By construction, player 2's strategy is continuous at all positive bids and player 1 obtains the same payoff at every bid $(l_k,t_k]$. Moreover, if player 2 does not have an atom at l_k, then player 1 obtains the same payoff at l_k as well. If player 2 does have an atom at l_k, then $l_k=0$ is not a profitable deviation for player 1. To complete the proof, therefore, it suffices to show that player 1 does not have profitable deviations lower than l_k or higher than t_k. To this end, denote by $0=l_K,t_K,\ldots,l_{k+1},t_{k+1},l_k,t_k,\ldots,l_2,t_2,l_1,t_1=T$ the partition of $[0,T]$ identified by the procedure that corresponds to player 1's types. That is, $g_1\left(s_i^j, x\right) > 0$ for every $j \leq K$ and x in (l_j, t_j). Suppose that player 1 has a profitable deviation below l_k when he observes s_i^k, and let $[l_i, t_i]$ be the highest interval below $[l_k, t_k]$ that contains a profitable deviation y. Because $t_i = l_{i-1}$, $y < t_i$. By construction, player 1 obtains the same payoff at y and t_i when he observes s_1^i.[5] Therefore, because $s_1^k > s_1^i$, player 1 obtains strictly more at $t_i = l_{i-1}$ than at y when he observes s_i^k (this follows from (1) and the argument that follows it in the proof of Lemma 2). If $i-1 = k$, this shows that y is not a profitable deviation. If $i-1 < k$, then $[l_{i-1}, t_{i-1}]$ contains a profitable deviation, contradicting the definition of $[l_i, t_i]$. This shows that there are no profitable deviations below $[l_k, t_k]$. A similar argument shows that there are no profitable deviations above t_k, by considering the lowest interval above $[l_k, t_k]$ that contains a hypothesized profitable deviation, and noting that bids above T are strictly dominated by slightly lower bids. Therefore, player 1 does not have profitable deviations. The same argument shows that player 2 chooses best responses with probability 1 as well.

■

3.1 Equilibrium Ordering

The procedure for constructing the equilibrium shows that players' types exhaust their equilibrium masses in an order that depends on their valuation functions and probability distributions. That is, if player 1 has n_1 types and player 2 has n_2 types, then the equilibrium induces an ordering $(s^1, \ldots, s^{n_1+n_2})$ of the elements in $S_1 \cup S_2$, such that the probability mass associated with s^j is expended on an interval of bids whose lower bound is (weakly) lower than those of the intervals of bids that correspond to types $s^1, \ldots, s^{j-1}$. And if the last type in the ordering, $s^{n_1+n_2}$, is a type of player i, then the lower bound of the interval of bids of the last type of player $-i$ in the ordering is 0. In any such ordering, and for any pair of types of a player, the higher type appears before the lower type. Thus, the number of equilibrium ordering of players' types that can be generated by varying players' valuation functions and probability distributions is at most $(n_1 + n_2)!/\left(n_1!n_2!\right)$: this is the number of orderings of $n_1 + n_2$ elements, n_1 of which are identical and the other

[5] If $y = 0$ and player 2 has an atom at 0, then choose a slightly higher y as the profitable deviation.

n_2 of which are identical. Conversely, it is easy to see that each ordering of n_1 identical elements and n_2 identical elements corresponds to an equilibrium ordering of players' types for some valuation functions and probability distributions.

4 Special Cases

4.1 Example 1 - Two Types for Player 1, One Type for Player 2

The $(2+1)!/2!1! = 6/2 = 3$ possible equilibrium orderings are (i) (s_1^1, s_1^2, s_2^1), (ii) (s_1^1, s_2^1, s_1^2), and (iii) (s_2^1, s_1^1, s_1^2).

In (i), player 1 exhausts the mass associated with both his types before player 2 exhausts the mass associated with his single type. Therefore, starting from the top, player 1's type s_1^1 exhausts his mass before player 2's type s_2^1, so we must have

$$g_1\left(s_1^1,\cdot\right) = \frac{1}{F_1\left(s_1^1\right)V_2\left(s_1^1,s_2^1\right)} > \frac{1}{V_1\left(s_1^1,s_2^1\right)} = g_2\left(s_2^1,\cdot\right),$$

or

$$V_1\left(s_1^1,s_2^1\right) > F_1\left(s_1^1\right)V_2\left(s_1^1,s_2^1\right). \tag{7}$$

The length of the top interval is therefore

$$\frac{1}{g_1\left(s_1^1,\cdot\right)} = F_1\left(s_1^1\right)V_2\left(s_1^1,s_2^1\right).$$

In the second interval, player 1's type s_1^2 exhausts his mass before player 2's type s_2^1 exhausts his remaining mass of

$$1-\frac{F_1\left(s_1^1\right)V_2\left(s_1^1,s_2^1\right)}{V_1\left(s_1^1,s_2^1\right)}.$$

Together with (6) this implies that

$$\left(1-\frac{F_1\left(s_1^1\right)V_2\left(s_1^1,s_2^1\right)}{V_1\left(s_1^1,s_2^1\right)}\right)\underbrace{V_1\left(s_1^2,s_2^1\right)}_{\text{The reciprocal of player 2's density}} \geq \underbrace{F_1\left(s_1^2\right)V_2\left(s_1^2,s_2^1\right)}_{\text{The reciprocal of player 1's density}}. \tag{8}$$

Fixing player 1's probability distribution and player 2's valuation function, (7) and (8) are satisfied when $V_1\left(s_1^1,s_2^1\right)$ and $V_1\left(s_1^2,s_2^1\right)$ are large enough.

In (ii), player 2 exhausts the mass associated with his single type after player 1 exhausts the mass associated with his high type, so (7) holds, but before player 1 exhausts the mass associated with his low type, so the reverse of (8) holds. Fixing player 1's probability distribution and player 2's valuation function, this happens when $V_1(s_1^1, s_2^1)$ is large enough and $V_1(s_1^2, s_2^1)$ is small enough.

In (iii), player 2 exhausts the mass associated with his single type before player 1 exhausts the mass associated with his high type. Therefore, the reverse of (7) holds, and the length of the top (and only) interval is $V_1(s_1^1, s_2^1)$. Fixing player 1's probability distribution and player 2's valuation function, this happens when $V_1(s_1^1, s_2^1)$ is small enough.

The three possible equilibrium configurations are illustrated in Figure 2.

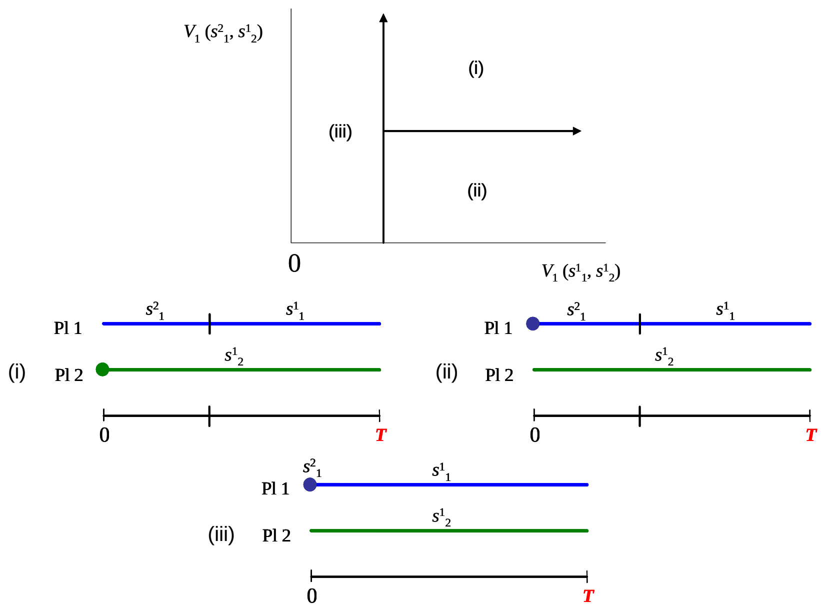

Figure 2: The possible equilibrium configurations in Example 1 as a function of $V_1(s_1^1, s_2^1)$ and $V_1(s_1^2, s_2^1)$, fixing player 1's probability distribution and player 2's valuation function

4.2 Example 2 - Two Types for Each Player

The $(2+2)!/(2!2!) = 24/4 = 6$ possible equilibrium orderings are (i) $(s_1^1, s_1^2, s_2^1, s_2^2)$, (ii) $(s_1^1, s_2^1, s_1^2, s_2^2)$, (iii) $(s_1^1, s_2^1, s_2^2, s_1^2)$, (iv) $(s_2^1, s_2^2, s_1^1, s_1^2)$, (v) $(s_2^1, s_1^1, s_2^2, s_1^2)$, and (vi) $(s_2^1, s_1^1, s_1^2, s_2^2)$.

In (i), player 1 exhausts the mass associated with both his types before player 2 exhausts the mass associated with his high type. Therefore, starting from the top, player 1's type

s_1^1 exhausts his mass before player 2's type s_2^1, so we must have

$$F_2\left(s_2^1\right) V_1\left(s_1^1, s_2^1\right) > F_1\left(s_1^1\right) V_2\left(s_1^1, s_2^1\right), \tag{9}$$

and the length of the top interval is $F_1\left(s_1^1\right) V_2\left(s_1^1, s_2^1\right)$. In the second interval, player 1's type s_1^2 exhausts his mass before player 2's type s_2^1 exhausts his remaining mass of

$$1 - \frac{F_1\left(s_1^1\right) V_2\left(s_1^1, s_2^1\right)}{F_2\left(s_2^1\right) V_1\left(s_1^1, s_2^1\right)}.$$

This implies that

$$\left(1 - \frac{F_1\left(s_1^1\right) V_2\left(s_1^1, s_2^1\right)}{F_2\left(s_2^1\right) V_1\left(s_1^1, s_2^1\right)}\right) F_2\left(s_2^1\right) V_1\left(s_1^2, s_2^1\right) \geq F_1\left(s_1^2\right) V_2\left(s_1^2, s_2^1\right). \tag{10}$$

Fixing players' probability distributions and player 2's valuation function, (9) and (10) are satisfied if $V_1\left(s_1^1, s_2^1\right)$ and $V_1\left(s_1^2, s_2^1\right)$ are large enough.

In (ii), player 2 exhausts the mass associated with his high type after player 1 exhausts the mass associated with his high type, so (9) holds, but before player 1 exhausts the mass associated with his low type, so the strict reverse of (10) holds. The length of the second interval of the joint partition is therefore

$$\left(1 - \frac{F_1\left(s_1^1\right) V_2\left(s_1^1, s_2^1\right)}{F_2\left(s_2^1\right) V_1\left(s_1^1, s_2^1\right)}\right) F_2\left(s_2^1\right) V_1\left(s_1^2, s_2^1\right),$$

and player 1 exhausts

$$\left(1 - \frac{F_1\left(s_1^1\right) V_2\left(s_1^1, s_2^1\right)}{F_2\left(s_2^1\right) V_1\left(s_1^1, s_2^1\right)}\right) \frac{F_2\left(s_2^1\right) V_1\left(s_1^2, s_2^1\right)}{F_1\left(s_1^2\right) V_2\left(s_1^2, s_2^1\right)} < 1 \tag{11}$$

of the mass associated with his low type in the second interval. In the third interval, player 1 exhausts the remaining mass associated with his low type before player 2 exhausts the mass associated with his low type. This implies that

$$\left(1 - \left(1 - \frac{F_1\left(s_1^1\right) V_2\left(s_1^1, s_2^1\right)}{F_2\left(s_2^1\right) V_1\left(s_1^1, s_2^1\right)}\right) \frac{F_2\left(s_2^1\right) V_1\left(s_1^2, s_2^1\right)}{F_1\left(s_1^2\right) V_2\left(s_1^2, s_2^1\right)}\right) F_1\left(s_1^2\right) V_2\left(s_1^2, s_2^2\right) < F_2\left(s_2^2\right) V_1\left(s_1^2, s_2^2\right). \tag{12}$$

Fixing players' probability distributions and player 2's valuation function, (9) and the strict reverse of (10) are satisfied when $V_1\left(s_1^1, s_2^1\right)$ is large enough and $V_1\left(s_1^2, s_2^1\right)$ is small enough. And (11) and (12) are satisfied when $V_1\left(s_1^2, s_2^1\right)$ is small enough and $V_1\left(s_1^2, s_2^2\right)$ is large enough.

In (iii), players 1 and 2 behave as in (ii) in the first two intervals, but in the third interval player 2 exhausts the remaining mass associated with his low type before player 1 exhausts the remaining mass associated with his low type. Therefore, (9), the strict reverse of (10), (11), and the reverse of (12) hold. Fixing players' probability distributions and player 2's valuation function, this happens when $V_1(s_1^1, s_2^1)$ is large enough and $V_1(s_1^2, s_2^1)$ and $V_1(s_1^2, s_2^2)$ are small enough.

The orderings (iv), (v), and (vi) are the symmetric counterparts of (i), (ii), and (iii). That is, (iv), (v), and (vi) are obtained from (i), (ii), and (iii) by switching the indices of players 1 and 2. The equilibrium configurations that correspond to (i), (ii), and (iii) are illustrated in Figure 3.

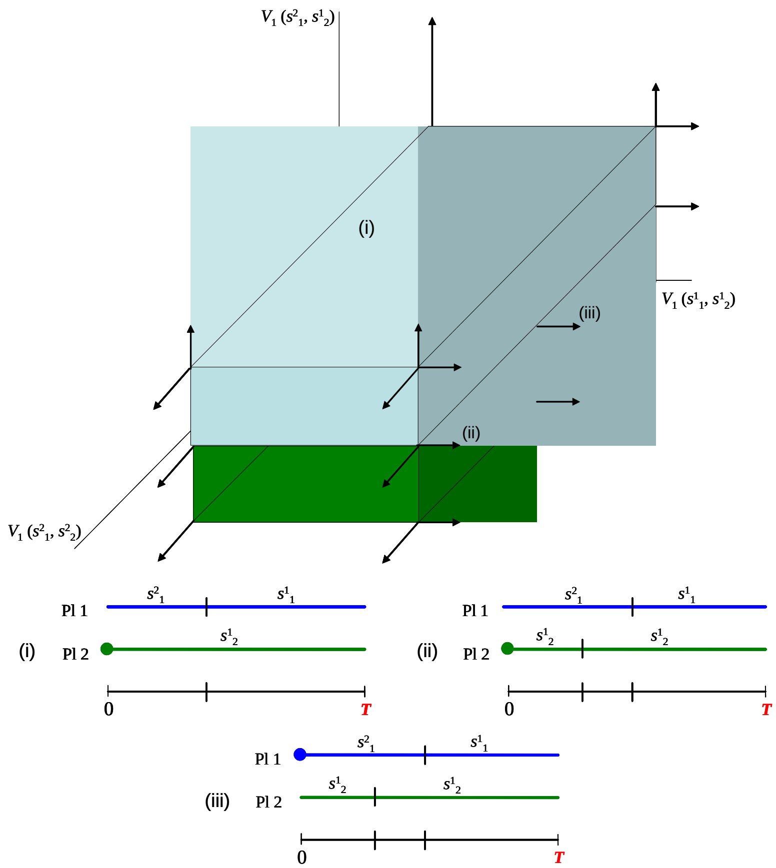

Figure 3: The equilibrium configurations in Example 2 that correspond to orderings (i), (ii), and (iii), as a function of $V_1(s_1^1, s_2^1)$, $V_1(s_1^2, s_2^1)$, and $V_1(s_1^2, s_2^2)$, fixing player 1's probability distribution and player 2's valuation function

4.3 Common Values

Suppose that the value of the prize is common to both players, and denote this common value function by $V = V_1 = V_2$. In equilibrium, the unconditional distribution of players' bids is the same, regardless of the information structure and the function V. To see why, note that for almost any x in $(0, T]$ we have

$$G_i(x) = \sum_{s_i \in S_i} F_i(s_i) G_i(s_i, x) \Rightarrow g_i(x) = F_i(s_i(x)) g_i(s_i(x), x),$$

where g_i is the density of G_i, and $s_i(x)$ is the signal of player i for which x is a best response. In conjunction with (6), this means that for almost any x in $(0, T]$ we have

$$g_1(x) = g_2(x) = \frac{1}{V(s_1(x), s_2(x))}.$$

In particular, both players exhaust the same unconditional probability mass on $(0, T]$. And since at most one player has an atom at 0, this mass must be 1, so no player has an atom at 0. Therefore, the lowest type of each player has a payoff of 0. However, other types' payoffs, and therefore the ex-ante expected payoffs, may differ between the players.

4.4 Common Values, One Informed Player

Suppose that the prize is of common value, that player 1 knows the common value, and that player 2 only knows it's distribution. This means that player 2 has only one type. Without loss of generality, let player 1's type equal the common value, so $s_1^i = V_1(s_1^i, s_2^1) = V_2(s_1^i, s_2^1)$. In this case, the equilibrium can be described in closed form. The number of intervals in the joint partition equals the number of player 1's signals, n, and no player has an atom at 0. When player 1 observes signal s_1, he chooses a bid from an interval of length $F_1(s_1) s_1$ according to a uniform distribution with density $1/F_1(s_1) s_1$. On the same interval, player 2 chooses a bid according to a uniform distribution with density $1/s_1$. Denote by s_1^j player 1's j^{th} signal when his signals are ordered from high to low, and let

$$l\left(s_1^j\right) = \begin{cases} \sum_{k=j+1}^{n} F_1\left(s_1^k\right) s_1^k & j < n \\ 0 & j = n \end{cases}.$$

The equilibrium densities are

$$g_1\left(s_1^j, x\right) = \begin{cases} \frac{1}{F_1\left(s_1^j\right)s_1^j} & \text{if } x \in \left[l\left(s_1^j\right), l\left(s_1^j\right) + F_1\left(s_1^j\right)s_1^j\right] \\ 0 & \text{otherwise} \end{cases}$$

and

$$g_2\left(s_2^1, x\right) = \begin{cases} \frac{1}{s_1^j} & \text{if } x \in \left[l\left(s_1^j\right), l\left(s_1^j\right) + F_1\left(s_1^j\right)s_1^j\right] \\ 0 & \text{otherwise} \end{cases}.$$

Player 2's payoff is 0 (no player has an atom at 0). Because $G_1 = G_2$ (as explained in Section 4.3), both players win the prize with the same probability. Player 1's payoff is positive, however, because he places higher bids, and therefore wins more often, when the prize is more valuable.

4.5 Symmetric Players

When the two players are symmetric, so $S = S_1 = S_2$, $F = F_1 = F_2$, and $V = V_1 = V_2$, the equilibrium, which is symmetric, can be described in closed form. The number of intervals in the joint partition equals the number of signals, n, and no player has an atom at 0. When a player observes signal s, he chooses a bid from an interval of length $F(s)V(s,s)$ according to a uniform distribution with density $1/F(s)V(s,s)$. Denote by s^j the j^{th} signal when signals are ordered from high to low, and let

$$l\left(s^j\right) = \begin{cases} \sum_{k=j+1}^{n} F\left(s^k\right)V\left(s^k, s^k\right) & j < n \\ 0 & j = n \end{cases}.$$

The equilibrium density $g = g_1 = g_2$ is

$$g\left(s^j, x\right) = \begin{cases} \frac{1}{F(s^j)V(s^j,s^j)} & \text{if } x \in \left[l\left(s^j\right), l\left(s^j\right) + F\left(s^j\right)V\left(s^j, s^j\right)\right] \\ 0 & \text{otherwise} \end{cases}.$$

The equilibrium is efficient, because higher types choose bids from higher intervals and the equilibrium is symmetric.

5 Approximating Continuous Type Distributions

5.1 Private Values and Uniform Distributions

To approximate private values drawn independently from the uniform distribution on $[0,1]$, suppose that for some $n > 1$ each player's valuation is independently drawn from the set $\mathcal{S}^n = \{j/n\}_{j=1}^n$ according to a uniform probability distribution. For every n, the equilibrium is symmetric, and the joint partition is comprised of n intervals. The density of each player's strategy in the j^{th} interval is

$$\frac{1}{\frac{1}{n}\frac{j}{n}} = \frac{n^2}{j},$$

so the length of the j^{th} interval is j/n^2. Type j/n chooses bids from the interval

$$\left[\sum_{k=1}^{j-1}\frac{k}{n^2}, \sum_{k=1}^{j}\frac{k}{n^2}\right] = \left[\frac{j(j-1)}{2n^2}, \frac{j(j+1)}{2n^2}\right] = \left[\frac{j^2-j}{2n^2}, \frac{j^2+j}{2n^2}\right].$$

For any $x \in [0,1]$, consider a sequence $\{j_n/n\}_{n=1}^\infty$ with j_n/n in $\mathcal{S}^n$ and $j_n/n \to x$. We have that

$$\frac{j_n^2-j_n}{2n^2}, \frac{j_n^2+j_n}{2n^2} \to \frac{x^2}{2},$$

which is the equilibrium bid of type x in the limiting all-pay auction (see, for example, Krishna (2002) and Amann and Leininger (1996)).

5.2 Private values

Using an approximation similar to the one in Section 5.1, we can heuristically derive a candidate equilibrium for the all-pay auction with independent private values and a continuum of signals, which was analyzed by Amann and Leininger (1996). Consider distributions on $[0,1]$, H_1 for player 1 and H_2 for player 2, with corresponding positive continuous densities h_1 and h_2. For any $n > 1$, consider a sequence of n values in $(0,1]$, $\alpha^n < \alpha^{n-1} < \ldots < \alpha^1 = 1$ such that $\alpha^j - \alpha^{j+1} < \frac{1}{n}$ for any $j \in 1,\ldots,n-1$. To approximate H_1 with a finite distribution, let player 1's valuation be α^j with probability $\int_{\alpha^{j+1}}^{\alpha^j} h_1(x)\,dx$. For player 2, consider a sequence of n values in $(0,1]$, $k(\alpha^n) < k(\alpha^{n-1}) < \ldots < k(\alpha^1) = 1$, and approximate player 2's distribution by letting player 2's valuation be $k(\alpha^j)$ with probability $\int_{k(\alpha^{j+1})}^{k(\alpha^j)} h_2(x)\,dx$. Given $\alpha^1,\ldots,\alpha^n$, choose the sequence $k(\alpha^1),\ldots,k(\alpha^n)$ such that in equilibrium type α^j of player 1 and type $k(\alpha^j)$ of player 2 choose bids from the same interval. This is useful because then each type of each player chooses bids using one density (whenever one player exhausts the mass associated with one of his types so does the other player). To do this, it suffices to choose $k(\alpha^j)$ such that the length of the intervals from which type α^j of player 1 and type $k(\alpha^j)$ of player 2 choose bids are the same. for this, it suffices that the densities according to which the players choose their bids are the same. From (6), these densities are

$$\frac{1}{k(\alpha^j)\int_{\alpha^{j+1}}^{\alpha^j} h_1(x)\,dx}$$

for player 1 and

$$\frac{1}{\alpha^j\int_{k(\alpha^{j+1})}^{k(\alpha^j)} h_2(x)\,dx}$$

for player 2. Equating these densities, we obtain

$$k\left(\alpha^j\right)\int_{\alpha^{j+1}}^{\alpha^j} h_1(x)\,dx = \alpha^j\int_{k(\alpha^{j+1})}^{k\left(\alpha^j\right)} h_2(x)\,dx. \tag{13}$$

The equality (13) shows how to uniquely identify $k(\alpha^1),\ldots,k(\alpha^n)$ by proceeding inductively from 1 to n: set $k(\alpha^1)=\alpha^1=1$ and, given α^j, α^{j+1}, and $k(\alpha^j)$, use (13) to solve for $k(\alpha^{j+1})$. Now, (13) implies that

$$\frac{\int_{\alpha^{j+1}}^{\alpha^j} h_1(x)\,dx}{\int_{k(\alpha^{j+1})}^{k(\alpha^j)} h_2(x)\,dx} = \frac{\alpha^j}{k(\alpha^j)},$$

so

$$\frac{\frac{\int_{\alpha^{j+1}}^{\alpha^j} h_1(x)dx}{\alpha^j-\alpha^{j+1}}}{\frac{\int_{k(\alpha^{j+1})}^{k(\alpha^j)} h_2(x)dx}{k(\alpha^j)-k(\alpha^{j+1})}} = \frac{\alpha^j\left(k(\alpha^j)-k(\alpha^{j+1})\right)}{k(\alpha^j)\left(\alpha^j-\alpha^{j+1}\right)}. \tag{14}$$

As n grows large, fixing α^j as an element in the sequence of player 1's types, (14) heuristically "becomes"

$$\frac{h_1(\alpha^j)}{h_2(k(\alpha^j))} = \frac{\alpha^j}{k(\alpha^j)}k'\left(\alpha^j\right),$$

which is precisely (1) in Amann and Leininger's (1996). As in Amann and Leininger (1996), in the limit, $k(\alpha^j)$ is the type of player 2 that submits a bid equal to that of type α^j of player 1.

We can say more about the limiting bid of type α^j of player 1. For simplicity, assume that for any n player 1's lowest type does not have an atom at 0. Then, the length of the bidding interval of type α^j of player 1 is $k(\alpha^j)\int_{\alpha^{j+1}}^{\alpha^j} h_1(x)\,dx$, so the lower endpoint of the bidding interval of type α^j is

$$\sum_{l=n}^{j+1} k\left(\alpha^l\right)\int_{\alpha^{l+1}}^{\alpha^l} h_1(x)\,dx. \tag{15}$$

As n grows large, fixing α^j as a point in the sequence of player 1's types, by definition of the Lebesgue integral (15) heuristically "becomes"

$$\int_0^{\alpha^j} k(x)\,h_1(x)\,dx,$$

which is precisely the formula in Amann and Leininger (1996) that describes the bid of type a^j of player 1 (on page 8, following (2)).

In particular, if $H_1 = H_2$, then $k(\alpha^i) = \alpha^i$ (directly from (13)), so

$$\int_0^{\alpha^j} k(x)\,h_1(x)\,dx = \int_0^{\alpha^j} xh_2(x)\,dx = E\left[x|x<\alpha^j\right]H_2\left(\alpha^j\right).$$

5.3 Interdependent Valuations

Using the same approximation heuristic as in Section 5.2, a differential equation that describes a candidate equilibrium can also be derived for the case of continuous type distributions with interdependent valuations (which were not considered by Amann and Leininger (1996)).

The condition that each type of each player bids using one density is

$$V_1\left(\alpha^j, k\left(\alpha^j\right)\right)\int_{k(\alpha^{j+1})}^{k\left(\alpha^j\right)} h_2(x)\,dx = V_2\left(\alpha^j, k\left(\alpha^j\right)\right)\int_{\alpha^{j+1}}^{\alpha^j} h_1(x)\,dx \Rightarrow$$

$$\frac{\frac{\int_{k(\alpha^{j+1})}^{k\left(\alpha^j\right)} h_2(x)dx}{k(\alpha^j)-k(\alpha^{j+1})}}{\frac{\int_{\alpha^{j+1}}^{\alpha^j} h_1(x)dx}{\alpha^j-\alpha^{j+1}}} = \frac{\alpha^j-\alpha^{j+1}}{k(\alpha^j)-k(\alpha^{j+1})}\frac{V_2(\alpha^j,k(\alpha^j))}{V_1(\alpha^j,k(\alpha^j))} \Rightarrow$$

$$\frac{h_2(x)}{h_1(x)} = \frac{1}{k'(x)}\frac{V_2(x,k(x))}{V_1(x,k(x))},$$

where $k(x)$ is the type of player 2 that bids the same as type x of player 1 in the candidate equilibrium. And the lower bound of player 1's bidding interval when his type is α^j is

$$\sum_{l=n}^{j+1} V_2\left(\alpha^l, k\left(\alpha^l\right)\right) \int_{\alpha^{l+1}}^{\alpha^l} h_1(x)\, dx \to \int_0^x V_2(y, k(y))\, h_1(y)\, dy,$$

where the last expression is the player 1's bid when his type is x in the candidate equilibrium.

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