The Quantity and Quality of Teachers: A Dynamic Trade-off *

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Abstract

We study the dynamics of the quantity and quality of teachers in the framework of dynamic general equilibrium OLG model. The quantity and quality are jointly set by a government agency wishing to maximize the quality of basic education per student while being bound by teachers' collective bargaining agreement which equalizes teacher pay. Our model features two stages of education: basic and advanced, the latter being required of teachers. The cost of hiring teachers is influenced by the outside opportunities that skilled individuals have in the production sector. We show that this factor strengthens in the process of endogenous growth and moreover that it pushes the optimal trade-off between quantity and quality of teachers in the direction of the former. Namely, the number of teachers hired will grow over time while their relative quality (but not the absolute human capital attainment) will fall. Furthermore, we show that this evolution of human capital accumulation is accompanied by increasing inequality within the group of college educated workers as well as between it and the unskilled.

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1. Introduction

Increasing focus on "individual based instruction" continues to be one of the main education policy priorities in the United States as a means to raising education quality. This is evidenced by the dynamics of student-teacher ratio which has fallen from 25.8 in 1960 to 15.9 in 2003 (*Digest for Education Statistics* 2004, table 63). Research, however, has shown that students' test scores have not risen despite increased individualized instruction. This discrepancy has compelled policy makers and researchers to question the factors affecting students' test scores and the role of the quality of teachers vs. their quantity (see Hanushek et al (2005)). This paper develops a theoretical framework for analyzing this quantity-quality trade-off and offers an explanation to the observed trend biased in favor of quantity.

Some of the changes in school inputs and students' test scores between 1955 and 2003 are displayed in Table 1. It shows that the decline in the student-teacher ratio was accompanied by declining relative teacher salaries while the overall K-12 public education expenditures have been increasing by roughly 1% of GDP per decade.

Table 1: Historical Data on Public Elementary and Secondary Schools From 1955:2003

Year	Enrollment[1,a]	Teachers[1,a]	Pupil / teacher ratio[1]	Expenditures to GDP[2,b]	Relative Teacher Salary[3,b,d]
1955	30,680	1,141	26.9	3.3[c]	
1960	36,281	1,408	25.8	3.6	43
1965	42,173	1,710	24.7	3.9	
1970	45,894	2,059	22.3	4.6	44
1975	44,819	2,198	20.4	4.6	
1980	40,877	2,184	18.7	4.0	41
1985	39,422	2,206	17.9	3.8	
1990	41,217	2,398	17.2	4.3	35
1995	44,840	2,598	17.3	4.3	
2000	47,204	2,941	16.0	4.5	36.5
2005	49,113	3,137	15.7	4.6	

Source:
1: Digest for Education Statistics 2004, Table 61
2: Digest for Education Statistics 2004, Table 25
3: Hanushek & Rivkin (2003)

Notes:
a: In thousands
b: In Percent
c: 1959 data
d: College educated females, age 20-29, earning less than average female teacher, age 20-29

Another significant effect from changing education policy is the decline in the aptitude of teachers relative to educated workers. Hoxby and Andrews (2004) estimate that in 1963 41% of all teachers were of the “middle” aptitude relative to their educated peers, with 17% above and 42% below the average; in comparison, in 2000, 28% of all teachers were of the “middle” aptitude with 5% above and 67% below average. Corcoran et al (2002) provide similar results. Interestingly, student test scores have remained roughly constant despite the substantial growth in public education outlays. In the literature, much attention has focused on explaining how these different inputs in K-12 public school system have affected students' test scores.

Many of the conflicting conclusions in the literature concerning the factors affecting student performance boil down to two general empirical strategies in the literature used to estimate the returns to quality and quantity of teachers. The first strategy attempts to estimate which teacher characteristics affect student achievement while partially controlling class size (see Aaronson (2007), Clotfelter (2007), Rivkin et al (2005), Goldhaber and Anthony (2007)). Class size is naturally constrained due to geographic and time proximities of the observations (teachers in the same state are under one mandated student-teacher ratio). The second empirical strategy aims to estimate how class size affects student achievement while attempting to control for teacher quality. Several studies who follow this strategy use data from policy experiments which resulted in random assignment of students to smaller and larger classes. Then controlling for teacher quality this data yields unbiased estimates of the effects of class size on student achievement (see Angrist and Lavy (1999), Krueger and Whitmore, 2001, Krueger (1999), Jepsen and Rivkin (2002)).

Using data from North Carolina, Clotfelter et al (2007) conclude that teacher experience, test scores and regular licensure all have greater positive effects on student achievement, whether compared to the effects of changes in class size or to the socioeconomic characteristics of students. Aaronson et al (2007) use the data on Chicago public high school students and teachers at the classroom level to estimate how teacher characteristics affect mathematics test scores. They find that replacing a teacher with another that is rated two standard deviations superior in quality can add 0.35 to 0.45 grade equivalents, or 30 to 40 percent of an average school year’s worth, to a student's math score performance. Goldhaber and Anthony (2007) also use the same North Carolina data to examine the effects of the National Board Certification process and find mixed evidence that improved observable teacher credentials have positive impact on student achievement. These results are similar to Rivkin et al (2005) who use the UTD Texas Schools

Project. Their results suggest that a ten student reduction in class size produces smaller benefits than a one standard deviation improvement in teacher quality.

On the other hand, Angrist and Lavy (1999) use Israel's class size maximum to estimate class size effects on student achievement. They find that reducing class size causes significant and substantial increase in test scores for fourth and fifth graders, although not for third graders. Krueger (1999) analyzes data from Tennessee Project STAR to estimate the effects of class size reductions on student performance on standardized tests. His results indicate that students' scores increase by four percentage points in the first year students attend smaller classes while in subsequent years the test scores grow by about one percentage point per year. Hanushek (1999) rebuts Krueger's findings citing important design and implementation issues from the STAR project that suggest the returns to class size reduction are biased upwards. Krueger and Whitmore (2001) follow up on students who participated in the Tennessee STAR experiment and find that they had, on average, ACT scores of .13 standard deviations higher.

Another approach is to use longitudinal data on declining class size. Card and Krueger (1992) find that a decrease in the pupil-teacher ratio from 30 to 25 is associated with a 0.4 percentage point increase in the rate of return to education. Hoxby (2000), however, estimates that there is no effect from decreased class size on student achievement. These opposing estimates are addressed by Jepson and Rivkin (2002). They argue that using mandated class size reduction programs as natural experiments for estimating the class size effect is problematic when these changes involve a trade-off between the quantity and quality of teachers, and that the same problem arises when time series data is used without the account for this endogenous trade-off. Specifically, their results indicate that California's class size reduction program came at a cost of hiring lower quality teachers to staff additional classrooms which offset the benefits of smaller classes. Similarly, Hoxby (1996) also finds that school inputs can increase without gains to student achievement due to teachers' unions reducing productivity enough to offset gains from lowered class sizes.

Thus, despite a significant attention in the literature, the questions about the determinants of education quality remain open. This underscores the need for a broader theoretical framework, which would capture the dynamic interaction between different inputs in education as it is influenced by labor market in the production economy. We note in this regard a branch of recent literature which has studied how outside job market opportunities have affected the quality of teachers. Flyer and Rosen (1997) report that the three-fold increase in direct costs of education per student is attributable to the growing market opportunities for women. Hanushek and Rivkin (1997) document the decline in the earnings of women teachers relative to women in other occupations and suggest that the expansion of alternative opportunities reduced teacher quality. Hanushek and Rivkin (2003) estimate that in 1955, 50% of all educated male workers earned less than male teachers, compared to 36% in 2000. Likewise, in 1955, 48% of all educated female workers earned less than female teachers compared to 29% in 2000. Similar analyses concerning the effect of the outside work opportunities on teacher quality are proposed in Goldhaber and Liu (2003), and Bacolad (2006). Lakdawalla (2006) demonstrates that a rising skill premium of educated workers due to faster technical change, coupled with low productivity growth of skilled teachers, has lead to lower teacher quality. The mechanism he highlights is the substitution of unskilled teachers for increasingly expensive skilled teachers.

In this paper we present a theoretical model which incorporates some of the factors of education quality discussed above, in a dynamic general equilibrium framework where government education policy decisions affect and are affected by individual education and employment decisions, whereas the dynamics of human capital accumulation and labor productivity has a feedback effect on both. In our model, the government agency wishes to maximize the quality of basic education per student and faces a trade-off between the quality and quantity of teachers to be hired. Furthermore, we assume that the agency is bound by teachers' collective bargaining agreement which equalizes teacher pay. It is, indeed, well documented that teachers' unions significantly contribute to the wage compression phenomenon. Unions provide tenure to teachers and tie salary primarily to experience rather than performance. Administrators wishing to hire higher quality teachers are forced by the unions to then provide matching raises to teachers across the board. [1]

In our model, a government education agency has two policy variables: teacher salary,

[1] It should be noted that unionization is not the sole explanation for the compression of teacher salaries. It is also due in part to the difficulty of measuring teacher productivity, especially in terms of educational value added given unobservable student characteristics. But even if such characteristics were observable, there still exists the challenge of determining criteria for performance based pay for teachers. For example, low ability students exhibit relatively low average gains in learning throughout the year, therefore an approach based on marginal improvement of students' performance would not fairly compensate teachers for working with lower ability children.

which is uniform according to the collective bargaining regime, and the number of teachers to be hired. The model features two stages of education: basic and advanced (college), the latter being required of teachers. College graduates can also take jobs in the skilled labor force of the production sector and get paid a competitive wage according to their human capital attainment. This opportunity cost implies that the level of teacher salary set by the government will determine the top quality (human capital level) of a teacher who can be hired at this salary. All college graduates whose human capital is below this level will be motivated to take a teaching job at the same salary. Therefore the number of teachers the government decides to hire along with the aforementioned top quality cut-off will determine the lowest human capital cut-off among teachers. Thus the total cost of hiring teachers is affected in our model by the outside opportunities available to skilled individuals in the production sector. We show, moreover, that in the process of endogenous growth this effect strengthens and that it pushes the optimal trade-off between quantity and quality of teachers in the direction of the former. Namely, in the face of rising (over time) cost of highly able skilled workers the government agency will find it optimal to opt for increasing the number of teachers hired while reducing the overall relative quality of the pool of teachers. (The absolute human capital attainment of teachers, however, does rise along with the overall human capital accumulation, while sliding toward the lower tail of the distribution of college educated population.) Furthermore, we show that this human capital dynamics is characterized by increasing inequality within the group of college educated workers as well as between it and the unskilled.

Thus this paper offers a theory explaining the trend in education policy in favor of lower student-teacher ratios (i.e., higher quantity of teachers) combined arguably with deteriorating teacher quality, despite growing per student schooling expenditures.

The paper is organized as follows. Section 2 develops a dynamic general equilibrium model with unionized public schools. Section 3 defines a competitive equilibrium. Section 4 provides main analytical results, while Section 5 illustrates them by means of a simple numerical example. Section 6 concludes. Appendix 1 contains the proofs of several auxiliary Lemmas. Appendix 2 contains a glossary of notation.

2. The Model

We develop a general equilibrium growth model of an economy populated by overlapping generations of individuals whose life consists of three periods: childhood, young adulthood, and old age. We identify a generation with the period when its members are young adults, thus the individuals born in period $t-1$ form a generation G_t. We assume that population size is constant in each generation G_t and that it forms a continuum on the interval $[0,1]$. Let $\mu(.)$ be the induced Lebesgue measure on the set of generation G_t individuals $[0,1]$, so that $\mu([0,1])=1$ for all t.

Children make no decisions of their own and receive basic (or first stage) education which is provided publicly. Young adults are endowed with a unit of time and face an option of devoting a fixed fraction n of it to acquiring higher education (which we will also refer to as college or second stage education); the balance of time not spent on education is inelastically devoted to work. Specifically, the individuals without college education will work for the full unit of time in the "unskilled" production workforce. Those with college education either work for the remaining fraction of time $1-n$ in the "skilled" production workforce or, if qualified by the government, can work as public school teachers. Individuals derive income from work. They spend part of it on consumption when young and invest the rest to use the returns to finance their consumption in retirement, the last period of life.

2.1. *Production*

The production sector of the economy consists of private perfectly competitive firms producing a homogeneous capital/consumption good by means of a constant returns technology which uses three factors of production - physical capital as well as unskilled and skilled human capital. The aggregate production function is given by

$$Y_t = DK_t^{\alpha}\left[H_t^u + \theta_t H_t^{sy}\right]^{1-\alpha}, \tag{1}$$

where $\alpha \in [0,1]$, $D>0$, while K_t, H_t^u , H_t^{sy} stand, respectively, for aggregate supply of physical capital, unskilled human capital, and skilled human capital employed in the production sector in period t. The coefficient θ_t characterizes the net productivity augmentation of skilled human capital (adjusted for the shorter employment duration dues to the time spent in college) which is imbedded technology. The sequence of $\{\theta_t\}_{t=o}^{\infty}$ characterizing the evolution of technological skill bias is assumed to be exogenously given.[2]

2.2. *Households*

All individuals ω of generation G_t, $t = 0,1,2,...$ have identical intertemporal preferences over consumption as young adults and retirees given by

$$\ln c_{t,t}(\omega) + \beta \ln c_{t,t+1}(\omega) \tag{2}$$

subject to the life-time budget constraint

$$c_{t,t}(\omega) + (1 + r_{t+1})^{-1} c_{t,t+1}(\omega) \leq (1 - \tau_t) I_t(\omega) \tag{3}$$

where r_{t+1} is the market interest rate, $I_t(\omega)$ is the individual's wage income which is derived from human capital, while τ_t is the uniform rate of labor income tax collected by the government. According to the production function (1) individuals working in the production economy receive the wage at competitive rates w_t and $\theta_t w_t$, respectively, per unit of their unskilled or skilled human capital, whichever applies. Thus the income of individual ω who receives only basic education and attains the level of unskilled human capital $h_t^u(\omega)$ will be

$$I_t^u(\omega) = w_t h_t^u(\omega) \tag{4}$$

The individual ω who obtains college education, attains the level of skilled human capital $h_t^s(\omega)$ and is employed in the production sector will receive income

$$I_t^{sy}(\omega) = \theta_t w_t h_t^s(\omega) \tag{5}$$

College educated individuals who become teachers will receive income I_t^h to be specified later.

2.3. *Human Capital Formation*

The human capital received by each child ω of generation $t+1$ at the first (basic) stage of his education is produced by combining children's random innate ability with public education, E_t, according to

$$h_{t+1}^u(\omega) = Ca(\omega)E_t \tag{6}$$

[2] See Appendix 2 for the glossary of notation.

where C is a positive constant, E_t is a uniform quality of public schooling received by each child in period t while $a(\omega)$ is the child's innate ability. We assume that innate ability is distributed independently and identically in each generation (the time indexation is thus omitted); specifically the distribution is uniform on the interval $[a, A]$. To simplify the exposition (but at no cost to the substance of the matter) we will later let $a = 0$.

We postulate that college education has a pre-requisite human capital threshold h^*. Rather than an *ad hoc* admission requirement (we assume that all individuals are free to choose to go to college but base this decision purely on income considerations) we view this threshold as a set of benchmark skills, such as adequate language and mathematical proficiency whose deficit would preclude any benefit from learning at an advanced stage.[3] Specifically, we postulate that if an individual ω of generation $t+1$ chooses to go to college, he will become a "skilled" agent with the level of human capital given by

$$h_{t+1}^{s}(\omega) = bh_{t+1}^{u}(\omega) + B[h_{t+1}^{u}(\omega) - h^{*}] \qquad (7)$$

where $b \in (0,1)$ and $B > 0$ are given constants. Thus according to the expression (7) the gains from college education depend on one's prior preparation, namely on the extent to which the individual's pre-college attainment exceeds the threshold h^*. The college education production function (7) also reflects a partial loss of pre-college human capital, according to the coefficient b, for the purposes of skilled human capital. While this loss is counteracted by the net productivity augmentation θ_t of skilled human capital according to the economy's production function (1), we impose a condition

$$b\theta_t < 1 \qquad (8)$$

which indeed implies that individuals whose pre-college human capital $h_{t+1}^{u}(\omega)$ is at or only slightly above the threshold h^* will not gain from attending college and therefore will not choose to do so.

[3] See Su (2004) for a similar approach to college eligibility. One can envision that this substantive threshold may evolve over time. For example, it now tends to incorporate computer literacy. While applicants are not tested on it for admission, their progress in many college specialties will critically depend on it. For the purposes of our analysis h^* is assumed fixed.

According to the expressions (6) and (7) human capital of each type, and therefore the corresponding income is a non-decreasing function of the innate ability. Therefore if a certain individual decides to attend college then all agents with higher ability will also do so. Thus in each period t there is an ability cut-off level a_t^* such that an individual ω in generation t will choose to attend college if and only if his ability $a(\omega)$ exceeds a_t^*. (Without loss of generality we'll make a convention that individuals with ability on the threshold do choose to go to college.)

Furthermore, we will later show that the college attendance ability cut-off level is given by the formula

$$a_t^* = \frac{1}{CE_{t-1}} \frac{\theta_t B h^*}{\theta_t (b+B) - 1} \tag{9}$$

which has a straightforward meaning: an individual will choose to attend college if and only if his resulting skilled human capital given by formula (7) adjusted for the net productivity augmentation θ_t will exceed his unskilled human capital derived from the first stage of education according to its production function (6).

2.4. *Quality of Basic Education*

We shall now introduce the *per student* basic education quality E_t, i.e. the public input in the basic education production function (6), as a function of the quality and quantity of teachers chosen by a government agency. Recall that only college educated individuals are eligible to be employed as teachers. Let Σ_t be the set of individuals ω in generation t employed as teachers. Let z_t be the total number of teachers. Since population size was normalized to 1 in all generations, z_t is also the fraction of teachers in the overall population in generation t, as well as the *student-teacher ratio* for generation $t+1$ students. We define the aggregate teacher quality as the aggregate human capital of teachers $q_t = \int_{\omega \in \Sigma_t} h_t^s(\omega) d\mu_t(\omega)$. Likewise, the average teacher quality is given by $z_t^{-1} \int_{\omega \in \Sigma_t} h_t^s(\omega) d\mu_t(\omega) = z_t^{-1} q_t$. We now define the quality of basic education as a Cobb-Douglas function of the quantity and aggregate quality of teachers:

$$E_t = z_t^{\gamma} q_t^{\nu} \tag{10}$$

Note that this formula corresponds to the one used by Tamura (2001) who assumed in particular that the role of personal instruction, i.e. that of teacher-student ratio, is more important for schooling effectiveness that the average quality of teachers, which in our formulation means that $\gamma \geq \nu$.

The special case of (10), when $\gamma = \nu = 1$, i.e.

$$E_t = z_t \int_{\omega \in \Sigma_t} h_t^s(\omega) d\mu_t(\omega) \tag{11}$$

has a particularly straightforward interpretation. Assume that all teachers are perfectly sorted across classes, each class of equal size z_t^{-1}, so that each student through his classes is exposed to a cross-section of teachers which perfectly represents their distribution of quality. Then the expression (11) which is equivalent to $E_t = \int_{\omega \in \Sigma_t} \frac{h_t^s(\omega)}{z_t^{-1}} d\mu_t(\omega)$ can be interpreted as *per student average teacher quality*.

2.5. *Government*

The government funds and administers public education at the basic level with the goal of maximizing its quality E_t, as defined above, subject to the budget constraint given by the revenue from a uniform labor income tax at a flat rate τ_t. To these ends in each period t, the government must set teachers' salary I_t^h and the number of teachers to be hired z_t . As discussed in the introduction, we postulate that the salary I_t^h received by all teachers in generational cohort t is uniform, according to a collective bargaining agreement. Since a college educated individual has an option to work in the production sector for a competitive wage as defined by the expression (6), the government's choice of teacher salary I_t^h will uniquely determine the highest level of human capital attainment $\overline{h}_t$ among individuals who will choose to become teachers. Indeed it should satisfy the equation[4]

[4] Since one's work career is summarily represented in our model by one time period, we do not model the wage dynamics over the course of a worker's or teacher's career as he accumulates seniority and experience. The appropriate understanding of the income variables in this framework is that they represent aggregates over the entire career, such as respective present values at the career's outset. While teachers' union collective bargaining agreements stipulate

$$\theta_t w_t \bar{h}_t = I_t^h \tag{12}$$

Thus all college graduates with human capital level $h_t^s(\omega)$ at or below $\bar{h}_t$ will be obviously motivated to accept employment as a teacher rather than work in the production sector. However, the government's goal to maximize the overall education quality for a set number of teachers z_t implies that the set Σ_t of teachers the government will hire consists of all individuals whose level of human capital attained in college falls into the interval: $h_t^s(\omega) \in [\underline{h}_t, \bar{h}_t]$, where the minimum teacher qualification threshold $\underline{h}_t$ is determined by the intended number of teachers, i.e. the measure[5]

$$z_t = \mu\left(\omega \mid \underline{h}_t \le h_t^s(\omega) \le \bar{h}_t\right) \tag{13}$$

where the top cut-off $\bar{h}_t$ is determined, according to (12), by the teacher salary I_t^h set by the government.

Recalling the production functions of basic and advanced education given, respectively, by the expressions (6) and (7), we define the cut-off innate ability levels $\underline{a}_t$ and $\bar{a}_t$ which characterize the teachers who possess, respectively, the cut-off levels of human capital $\underline{h}_t$ and $\bar{h}_t$ induced by the government policy choice. In other words,

$$\underline{a}_t = \frac{\underline{h}_t + Bh^*}{(b+B)CE_{t-1}} \quad \text{and} \quad \bar{a}_t = \frac{\bar{h}_t + Bh^*}{(b+B)CE_{t-1}} \tag{14}$$

For the government policy choice of I_t^h, z_t, to be feasible, the minimum teacher qualification threshold $\underline{h}_t$ defined by (13) obviously must belong the range of human capital levels attained by college graduates. In other words, the corresponding ability level $\underline{a}_t$ must exceed the college attendance cut-off level a_t^*.

Thus according to (10) the government's basic education quality optimization problem can be restated as

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wage differentials based on seniority, equation (12) should be understood as the comparison of respective aggregates over the course of the alternative careers in question.

[5] See Angrist and Guryan (2004) for details on complications of teacher screening.

$$\max_{z_t, \bar{h}_t} \; E_t$$

$$\text{subject to (13)} \qquad (15)$$

$$z_t \theta_t w_t \bar{h}_t = T_t \text{ and}$$

$$\underline{a}_t \geq a_t^*$$

where T_t - is the tax revenue collected by the government in period t.

Thanks to our assumption of the uniform distribution of innate ability on the interval $[a, A]$ and due to the linearity of basic and advanced education production functions (6) and (7) we can simplify expressions (10) and (13), respectively, as

$$E_t = \frac{z_t^{\gamma} \left[\bar{h}_t^2 - \underline{h}_t^2 \right]^{\nu}}{\left[2(A-a)(b+B)CE_{t-1} \right]^{\nu}} \qquad (16)$$

$$z_t = \frac{\bar{h}_t - \underline{h}_t}{(A-a)(b+B)CE_{t-1}} \qquad (17)$$

and therefore problem (15) to maximize the quality of basic education E_t subject to the government budget constraint can be restated as

$$\max_{z_t, \bar{h}_t} \; \frac{z_t^{\gamma} \left[\bar{h}_t^2 - \underline{h}_t^2 \right]^{\nu}}{\left[2(A-a)(b+B)CE_{t-1} \right]^{\nu}}$$

subject to (17),

$$z_t \theta_t w_t \bar{h}_t = T_t \text{ and}$$

$$\underline{a}_t \geq a_t^*$$

or equivalently, according to (17), as

$$\max_{z_t, \bar{h}_t} \; 2^{-\nu} \left(\bar{h}_t + \underline{h}_t \right)^{\nu} z_t^{\gamma+\nu}$$

$$\text{subject to (17),} \qquad (18)$$

$$z_t \theta_t w_t \bar{h}_t = T_t \text{ and}$$

$$\underline{a}_t \geq a_t^*$$

Note that the optimal minimum and maximum cut-off levels of teachers' human capital are related through the optimal choice of their number z_t according to the equation (17). The optimization in problem (18) thus expresses the trade-off between the quantity and quality of teachers to be hired. The quality of the top teacher $\overline{h}_t$ will not only determine his salary $I_t^h = \theta_t w_t \overline{h}_t$ due to his outside option as a skilled worker, but also set the identical salary for all other teachers according to the equal pay-based collective bargaining agreement. (Conversely, teacher salary I_t^h set by the government will uniquely determine the top teacher quality $\overline{h}_t$.) Hence the total teachers' wage bill $z_t \theta_t w_t \overline{h}_t$ in the government's budget constraint.

According to the relationships (14), the expression (17) is equivalent to

$$z_t = \frac{\overline{a}_t - \underline{a}_t}{A - a} \tag{19}$$

Therefore using relationships (14) to express $\overline{h}_t$ and $\underline{h}_t$ and then eliminate $\underline{a}_t$ according to formula (19), we can restate the government's education quality optimization problem (18) as

$$\begin{aligned} &\max_{z_t,\, \overline{a}_t} \; E_t = 2^{-\nu}\left[2\left(b+B\right)CE_{t-1}\overline{a}_t - 2Bh^* - z_t\left(A-a\right)\left(b+B\right)CE_{t-1}\right]^{\nu} z_t^{\gamma+\nu} \\ &\text{subject to } \; z_t\theta_t w_t\left[\left(b+B\right)CE_{t-1}\overline{a}_t - Bh^*\right] = T_t \; \text{ and} \\ &\overline{a}_t - z_t(A-a) \geq a_t^* \end{aligned} \tag{20}$$

3. General Equilibrium and Optimal Policy

We can now summarize the fundamental elements of the model and their relationships in a general equilibrium framework. We will first define the dynamic general equilibrium for a given government policy parameters and then explore the government's optimal determination of the quality of basic education.

Given the sequence of tax rates $\{\tau_t\}_{t=0}^{\infty}$ and the sequence of government education policy parameters $\{I_t^h, z_t\}_{t=0}^{\infty}$, i.e. teacher salaries and the numbers of teachers hired in each period, respectively, as well as the initial period $t=0$ aggregate supply of capital K_0, the distributions of the retirees' consumption levels $c_{-1,0}(\omega)$, and per students basic education quality E_{-1} provided to generation G_0 individuals as children, we define the dynamic general equilibrium as a collection of sequences of

(a) factor prices $\{(1+r_{t+1}), w_t,\ \theta_t w_t\}_{t=0}^{\infty}$ respectively of physical, unskilled and skilled human capital as inputs in production in period t;

(b) aggregate variables $\left\{Y_t, K_t, H_t^u, H_t^{sy}, T_t, E_t, a_t^*\right\}_{t=0}^{\infty}$, i.e., respectively, aggregate output, inputs of physical, unskilled and skilled human capital in production, government's tax revenue, the quality of basic education provided to each student in period t, as well as the endogenous innate ability cut-off for college attendance;

(c) distributions of individual consumption and education decisions $\left\{c_{t,t}(\omega), c_{t,t+1}(\omega), h_t^u(\omega), h_t^s(\omega)\right\}_{t=0}^{\infty}$, as well as employment decisions by college educated individuals

such that

(i) the factor prices are determined competitively, i.e. set equal to the marginal products of respective inputs:

$$1+r_{t+1} = \alpha D K_t^{\alpha-1}\left[H_t^u + \theta_t H_t^{sy}\right]^{1-\alpha},\ w_t = (1-\alpha) D K_t^{\alpha}\left[H_t^u + \theta_t H_t^{sy}\right]^{-\alpha}$$

(ii) each individual $\omega \in [0,1]$ in generation G_t makes a decision whether to go to college and if so whether to be employed as a teacher or in the production sector so as to maximize his income while taking as given their basic education quality E_{t-1}, production sector wage rates w_t and $\theta_t w_t$ (for unskilled and skilled labor, respectively), teacher salary I_t^h and the number of teachers z_t to be hired, whereas his human capital level $h_t^u(\omega)$ or $h_t^s(\omega)$ (depending on his college attendance decision) is determined according to the education production functions (6) and (7); (note that according to equation (12) and the collective bargaining agreement a teacher's salary will exceed production sector wage for all but the top quality teacher, so the government teacher employment limit z_t will bind;)

(iii) based on his income $I_t(\omega)$ determined according to (ii), each individual $\omega \in [0,1]$ makes his young- and old-age consumption decisions $c_{t,t}(\omega), c_{t,t+1}(\omega)$ by solving the optimization problem (2)-(3) while taking the rates of interest $1+r_{t+1}$ and tax τ_t as given;

(iv) the quality of basic education E_t provided to generation G_{t+1} individuals (as children) is determined by the expression (10) while the set of teachers Σ_t is defined by individual employment decisions according to (ii) while the number of teachers hired z_t is as given;

(v) the markets for goods, physical capital and both skilled and unskilled labor clears in each period:

$$Y_t = \int_{\omega\in[0,1]} c_{t,t}(\omega)d\mu_t(\omega) + \int_{\omega\in[0,1]} c_{t-1,t}(\omega)d\mu_{t-1}(\omega), \tag{21}$$

$$K_t = (1+r_{t+1})^{-1} \int_{\omega\in[0,1]} c_{t,t+1}(\omega)d\mu_t(\omega), \tag{22}$$

$$H_t^u = \int_{a\le a(\omega)\le a_t^*} h_t^u(\omega)d\mu_t(\omega), \tag{23}$$

$$H_t^{sy} = \int_{a_t^*\le a(\omega)\le A} h_t^s(\omega)d\mu_t(\omega) - \int_{\omega\in\Sigma_t} h_t^s(\omega)d\mu_t(\omega)\ , \tag{24}$$

where the ability cut-off for college attendance a_t^* is determined by individual college attendance decisions as defined in (ii);

(vi) the aggregate tax revenue is composed of labor income taxes collected from all categories of employees, i.e.

$$T_t = \tau_t\left(w_t H_t^u + \theta_t w_t H_t^{sy} + z_t I_t^h\right) \tag{25}$$

We can now define the government's <u>optimal education policy</u> in period t recursively, based on the above general equilibrium construct. Namely, the government chooses teacher salaries I_t^h and the numbers of teachers z_t for period t by solving the optimization problem (18) (or, equivalently, the problem (20)) where the top teacher quality $\overline{h}_t$ is determined by equation (12), while taking as given the economy's general equilibrium values of production sector wage rate w_t, aggregate tax revenue T_t and the distribution of skilled human capital attainment $h_t^s(\omega)$ by generation G_t individuals.

Noting the mutual dependence of the general equilibrium variables in period t and the government's optimal education policy we define the <u>Education-Economy recursive dynamic</u>

equilibrium (the recursive equilibrium for brevity) as a fixed point of this relationship, recursively determined for each period t.

Remark. Since we assumed that individuals make a decision whether to attend college solely on the basis of maximizing income, it is clear that the ability cut-off for college attendance a_t^* defined in part (ii) of the definition of the dynamic general equilibrium, should satisfy inequality

$$a_t^* \le \frac{1}{CE_{t-1}} \frac{\theta_t Bh^*}{\theta_t (b+B)-1} \tag{26}$$

Indeed, according to (6), (7) and (4), (5), an individual with ability exceeding the right hand side of (26) will certainly increase his income by going to college. In fact, we will show in the next section that in the recursive dynamic equilibrium inequality (26) is satisfied as equality, i.e. equality (9) is true.

4. Analysis of the Model

To reduce the unessential analytical complexity we will assume henceforth without any additional substantive loss of generality that parameter $a=0$, i.e. innate ability in each generation is distributed uniformly on the interval $[0, A]$. We begin by analyzing the government's optimal education policy problem equivalently stated as in (18) or (20).

We impose the following restrictions on the economy's parameters, where E_{-1} is an exogenously given per student basic education quality provided to generation G_0 individuals.

Assumption 1. $\left(\frac{\nu}{\gamma}(b+B)AC(1-\tau_t)\right)^{\frac{1}{2+\gamma/\nu}} \left(\left(\frac{\gamma}{2\nu+\gamma}\right)\left(\frac{\tau_t}{1-\tau_t}\right)\right)^{\frac{1+\gamma/2\nu}{2+\gamma/\nu}} \left(1-\frac{Bh^*}{(b+B)ACE_{t-1}}\right) > 1$ is true for any $t=0,1,\ldots$

Assumption 2. $\left(\frac{\nu(1-\tau_t)}{\gamma}-\frac{1}{2}\right)\left(\left(\frac{\gamma}{2\nu+\gamma}\right)\left(\frac{\tau_t}{1-\tau_t}\right)\right)^{1/2}\left(1-\frac{Bh^*}{(b+B)ACE_{t-1}}\right) > \frac{1}{\theta_t(b+B)}$ is true for any $t=0,1,\ldots$

The above assumptions require that education taxes τ_t not be too small while not exceeding $1-\frac{\gamma}{2\nu}$, which of course imposes a requirement that γ, the relative importance of the teacher-student ratio for schooling effectiveness, should not be substantially greater than ν, the relative importance of the teacher quality. The main thrust of the assumptions, however, concerns the parameters which characterize educational gains. Assumption 1 is satisfied if parameter C characterizing gains to basic education is sufficiently large. Assumption 2 will hold if $\left(b+B\right)$, a productivity characteristic of the college education production function, is large enough.

Based on these assumptions we will characterize the optimal solution of the education quality optimization problem in terms of the optimal number of teachers z_t for period t, the corresponding range of teachers' human capital, i.e. its maximum and minimum values $\overline{h}_t$, $\underline{h}_t$ induced by the policy and the corresponding innate ability levels $\overline{a}_t$, $\underline{a}_t$. In the process we will establish the following important facts (see Appendix 1 for the proofs):

Lemma 1 (Growth of Basic Education Quality). *The recursive equilibrium dynamics exhibits sustained growth of the quality of per student basic education. Specifically, there is a rate* $g>1$ *such that* $E_t > gE_{t-1}$ *is true for all* $t=0,1,...$

Lemma 2 (The Interiority Property). *In the recursive equilibrium, the ability of the least qualified teacher exceeds the college attendance cut-off ability in all time periods, i.e.* $\underline{a}_t > a_t^*$ *is true for* $t=0,1,...$*. Thereby the human capital of the least qualified teacher will not be the lowest among his contemporary college graduates.*

Lemma 3. *The ability cut-off for college attendance* a_t^* *satisfies equality (9), i.e.*

$$a_t^* = \frac{1}{CE_{t-1}}\frac{\theta_t Bh^*}{\theta_t\left(b+B\right)-1}$$

which means that an individual will choose to attend college if and only if his resulting skilled human capital given by formula (7) adjusted for the net productivity augmentation θ_t *will exceed* *his unskilled human capital derived from the first stage of education according to its production function (6).*

We now proceed to solving the optimization problem (20).

According to the teacher salary equation (12) and the tax revenue formula (25), the government budget constraint can be stated as

$$\left(1-\tau_t\right) z_t \theta_t \overline{h}_t = \tau_t \left(H_t^u + \theta_t H_t^{sy}\right) \tag{27}$$

Using the education production functions (6) and (7), and the assumption that innate ability is uniformly distributed on $[0, A]$ we can rewrite the general equilibrium relationships (23), (24) as

$$H_t^u = CE_{t-1} \int_0^{a_t^*} \frac{a}{A} da = \frac{(a_t^*)^2}{2A} CE_{t-1} \tag{28}$$

$$\begin{aligned} H_t^{sy} &= \int_{a_t^*}^{A} \left[\left(b+B\right) CE_{t-1} a - Bh^*\right] \frac{1}{A} da - \int_{\underline{a}_t}^{\overline{a}_t} \left[\left(b+B\right) CE_{t-1} a - Bh^*\right] \frac{1}{A} da = \\ &= \frac{\left(b+B\right) CE_{t-1}}{2A} \left[A^2 - (a_t^*)^2 - (\overline{a}_t)^2 + (\underline{a}_t)^2\right] - \frac{Bh^*}{A} \left[A - a_t^* - \overline{a}_t + \underline{a}_t\right] \end{aligned} \tag{29}$$

Therefore expressing $\overline{h}_t$ through $\overline{a}_t$ according to the relationship in (14) we can rewrite the budget constraint (27) as

$$\begin{aligned} &\left(1-\tau_t\right) \theta_t z_t \left(\left(b+B\right) CE_{t-1} \overline{a}_t - Bh^*\right) = \\ &\frac{\tau_t CE_{t-1}}{2A} \left[(a_t^*)^2 + \theta_t \left(b+B\right) \left(A^2 - (a_t^*)^2 - (\overline{a}_t)^2 + (\underline{a}_t)^2\right)\right] - \frac{\tau_t \theta_t Bh^*}{A} \left[A - a_t^* - \overline{a}_t + \underline{a}_t\right] \end{aligned} \tag{30}$$

We now eliminate variables a_t^* and $\underline{a}_t$ from (30) by substituting the value of a_t^* given by (9), based on the above Lemma 3, and using the expression $\underline{a}_t = \overline{a}_t - Az_t$, which follows from the relationship (19) since we set $a = 0$. This immediately turns expression (30) into a linear equation in terms of variable $\overline{a}_t$ which yields

$$\overline{a}_t = \frac{z_t \tau_t A}{2} + \frac{Bh^*}{\left(b+B\right) CE_{t-1}} + \frac{\tau_t A}{2 z_t} \left(1 - \frac{2Bh^*}{A\left(b+B\right) CE_{t-1}} + \frac{\theta_t \left(Bh^*\right)^2}{\left(\theta_t \left(b+B\right) - 1\right)\left(b+B\right)\left(ACE_{t-1}\right)^2}\right) \tag{31}$$

This expression incorporates the government budget constraint of problem (20). That problem's objective function, upon substituting the expression (31) for $\bar{a}_t$, becomes a function of a single variable z_t . We will solve for its unconstrained maximization and then refer to Lemma 2 which ensures that the only remaining constraint $\bar{a}_t - Az_t \geq a_t^*$ in the government optimization problem (20) is automatically fulfilled.

Thus we are looking at the unconstrained maximization of the following function:

$$F(z_t) = q_t^\nu z_t^\gamma = \left(\frac{\tau_t (b+B) ACE_{t-1}}{2} - \tau_t Bh^* + \frac{\tau_t \theta_t B^2 h^{*2}}{2\left(\theta_t (b+B) - 1\right) ACE_{t-1}} - \frac{(1-\tau_t)(b+B) ACE_{t-1} z_t^2}{2} \right)^\nu z_t^\gamma$$

Its first order necessary condition is given by the equation

$$\gamma z_t^{\gamma-1} q_t^\nu - \nu (1-\tau_t)(b+B) ACE_{t-1} z_t^{\gamma+1} q_t^{\nu-1} = 0 \tag{32}$$

yielding unique non-negative solution:

$$z_t = \left(\left(\frac{\gamma}{2\nu+\gamma} \right) \left(\frac{\tau_t}{1-\tau_t} \right) \right)^{1/2} \left(1 - \frac{2Bh^*}{(b+B) ACE_{t-1}} + \frac{\theta_t \left(Bh^*\right)^2}{\left(\theta_t (b+B) - 1\right)(b+B)\left(ACE_{t-1}\right)^2} \right)^{1/2} \tag{33}$$

It is straightforward to verify that this solution also satisfies the second order sufficient condition of the maximization problem. Substituting expression (33) back into formula (31) we obtain

$$\bar{a}_t = \frac{\tau \hat{a} z_t}{2} + \frac{Bh^*}{(b+B) CE_{t-1}} + \frac{\tau A}{2 z_t} \left(1 - \frac{2Bh^*}{(b+B) ACE_{t-1}} + \frac{\theta_t \left(Bh^*\right)^2}{\left(\theta_t (b+B) - 1\right)(b+B)\left(ACE_{t-1}\right)^2} \right)$$

which simplifies, by using equation (33) again, into

$$\bar{a}_t = \frac{\tau_t A z_t}{2} + \frac{Bh^*}{(b+B) CE_{t-1}} + \frac{(2\nu+\gamma)(1-\tau_t) A z_t}{2\gamma} = \frac{Bh^*}{(b+B) CE_{t-1}} + A z_t \left(\frac{\nu (1-\tau_t)}{\gamma} + \frac{1}{2} \right) \tag{34}$$

Recall that $\underline{a}_t = \bar{a}_t - Az_t$ according to (19) since we set $a = 0$. Applying this to (34) we obtain

$$\underline{a}_t = \frac{Bh^*}{(b+B) CE_{t-1}} + A z_t \left(\frac{\nu (1-\tau_t)}{\gamma} - \frac{1}{2} \right) \tag{35}$$

Observe that the education policy optimization as well as the individuals' and the production sector's general equilibrium reactions are determined recursively. Indeed, according to expressions (33) - (35), education quality E_{t-1} uniquely determines optimal education policy in period t, i.e. their number, as well as the range of their innate abilities and thereby due to (14) the range of their human capital attainment. This in turn will uniquely determine college attendance and employment decisions by generation t individuals, hence their incomes and their allocations. Government's education policy will also determine the current period's basic education quality E_t, so the recursion continues.

Consider now the effect of the level of previous period's education quality E_{t-1} on education decision variables in period t. By differentiating the expressions (9) and (33) we obtain:

$$\frac{\partial a_t^*}{\partial E_{t-1}} = \frac{-\theta_t B h^*}{\left(\theta_t (b+B)-1\right) C E_{t-1}^2} < 0 \tag{36}$$

$$\frac{\partial z_t}{\partial E_{t-1}} = \frac{1}{z_t}\left(\frac{\gamma}{2\nu+\gamma}\right)\left(\frac{\tau_t}{1-\tau_t}\right)\left(\frac{Bh^*}{(b+B)ACE_{t-1}^2}\left(1-\frac{a_t^*}{A}\right)\right) > 0 \tag{37}$$

According to (34) and (35), respectively, we can write

$$\begin{gathered}\frac{\partial \overline{a}_t}{\partial E_{t-1}} - \frac{Bh^*}{(b+B)CE_{t-1}^2} + \frac{A}{z_t}\left(\frac{2\nu(1-\tau_t)+\gamma}{2\gamma}\right)\left(\frac{\gamma}{2\nu+\gamma}\right)\left(\frac{\tau_t}{1-\tau_t}\right)\left(\frac{Bh^*}{(b+B)ACE_{t-1}^2}\left(1-\frac{a_t^*}{A}\right)\right) = \\ \frac{Bh^*}{(b+B)CE_{t-1}^2}\left(-1 + z_t^{-1}\left(\frac{2\nu(1-\tau_t)+\gamma}{2\gamma}\right)\left(\frac{\gamma}{2\nu+\gamma}\right)\left(\frac{\tau_t}{1-\tau_t}\right)\left(1-\frac{a_t^*}{A}\right)\right)\end{gathered} \tag{38}$$

$$\begin{gathered}\frac{\partial \underline{a}_t}{\partial E_{t-1}} = -\frac{Bh^*}{(b+B)CE_{t-1}^2} + Az_t^{-1}\left(\frac{2\nu(1-\tau_t)-\gamma}{2\gamma}\right)\left(\frac{\gamma}{2\nu+\gamma}\right)\left(\frac{\tau_t}{1-\tau_t}\right)\left(\frac{Bh^*}{(b+B)ACE_{t-1}^2}\left(1-\frac{a_t^*}{A}\right)\right) = \\ \frac{Bh^*}{(b+B)CE_{t-1}^2}\left(-1 + z_t^{-1}\left(\frac{2\nu(1-\tau_t)-\gamma}{2\gamma}\right)\left(\frac{\gamma}{2\nu+\gamma}\right)\left(\frac{\tau_t}{1-\tau_t}\right)\left(1-\frac{a_t^*}{A}\right)\right)\end{gathered} \tag{39}$$

Note that since $\dfrac{\theta_t(b+B)}{\theta_t(b+B)-1} > 1$, the following inequality is true

$$1-\frac{2Bh^*}{(b+B)ACE_{t-1}}+\frac{\theta_t\left(Bh^*\right)^2}{(b+B)\left(\theta_t(b+B)-1\right)\left(ACE_{t-1}\right)^2}>\left[1-\frac{Bh^*}{(b+B)ACE_{t-1}}\right]^2 \tag{40}$$

Therefore according to (33)

$$z_t>\left(\left(\frac{\gamma}{2\nu+\gamma}\right)\left(\frac{\tau_t}{1-\tau_t}\right)\right)^{1/2}\left(1-\frac{Bh^*}{(b+B)ACE_{t-1}}\right)>\left(\left(\frac{\gamma}{2\nu+\gamma}\right)\left(\frac{\tau_t}{1-\tau_t}\right)\right)^{1/2}\left(1-\frac{\theta_t Bh^*}{\left(\theta_t(b+B)-1\right)\left(ACE_{t-1}\right)}\right)=$$

$$=\left(\left(\frac{\gamma}{2\nu+\gamma}\right)\left(\frac{\tau_t}{1-\tau_t}\right)\right)^{1/2}\left(1-\frac{a_t^*}{A}\right)$$

Thus the expression (38) will be negative as long as

$$\left(\left(\frac{\gamma}{2\nu+\gamma}\right)\left(\frac{\tau_t}{1-\tau_t}\right)\right)^{1/2}>\left(\frac{\gamma}{2\nu+\gamma}\right)\left(\frac{\tau_t}{1-\tau_t}\right)\left(\frac{2\nu(1-\tau_t)+\gamma}{2\gamma}\right) \tag{41}$$

is true, which is certainly the case under the non-binding condition $\tau_t<\frac{4}{5+2\nu/\gamma}$ on the tax rate .[6] Comparing expressions (38) and (39) one can see that negativity of (38) implies the same for (39). Therefore we can conclude that

$$\frac{\partial\overline{a}_t}{\partial E_{t-1}}<0,\quad \frac{\partial\underline{a}_t}{\partial E_{t-1}}<0$$

Combining these facts with Lemma 1, which shows that education quality E_{t-1} does in fact grow over time, we obtain our central result.

Theorem 1 (Dynamics of the Quantity and Quality of Teachers). *The recursive dynamic equilibrium exhibits the following evolution of education policy variables:*

- *the quantity of teachers z_t will grow over time;*
- *the relative quality of teachers characterized by the range of their innate abilities falls: both the upper and the lower thresholds $\overline{a}_t$, $\underline{a}_t$ decrease over time;*
- *the college attendance ability-cut-off a_t^* also drops over time and (according to Lemma 2) remains consistently below the lower ability threshold for teachers $\underline{a}_t$.*

[6] See Appendix 1 for the proof.

Recall that according to relationships (14)

$$\underline{h}_t = \underline{a}_t \left(b+B\right) CE_{t-1} - Bh^* \quad \text{and} \quad \overline{h}_t = \overline{a}_t \left(b+B\right) CE_{t-1} - Bh^*$$

Therefore due to (34) and (35), respectively, as well as (33)

$$\begin{aligned} \overline{h}_t &= \left(b+B\right) ACE_{t-1} \left(\frac{\nu(1-\tau_t)}{\gamma} - \frac{1}{2} \right) z_t \\ &= \left(b+B\right) CA \left(\frac{\nu(1-\tau_t)}{\gamma} - \frac{1}{2} \right) \left(\left(\frac{\gamma}{2\nu+\gamma} \right) \left(\frac{\tau_t}{1-\tau_t} \right) \right)^{1/2} \left(E_{t-1}^2 - \frac{2Bh^* E_{t-1}}{\left(b+B\right) AC} + \frac{\theta_t \left(Bh^*\right)^2}{\left(\theta_t \left(b+B\right)-1\right)\left(b+B\right)\left(AC\right)^2} \right)^{1/2} \end{aligned} \quad (42)$$

$$\begin{aligned} \underline{h}_t &= \left(b+B\right) ACE_{t-1} \left(\frac{\nu\left(1-\tau_t\right)}{\gamma} + \frac{1}{2} \right) z_t \\ &= \left(b+B\right) AC \left(\frac{\nu\left(1-\tau_t\right)}{\gamma} + \frac{1}{2} \right) \left(\left(\frac{\gamma}{2\nu+\gamma} \right) \left(\frac{\tau_t}{1-\tau_t} \right) \right)^{1/2} \left(E_{t-1}^2 - \frac{2Bh^* E_{t-1}}{\left(b+B\right) AC} + \frac{\theta_t \left(Bh^*\right)^2}{\left(\theta_t \left(b+B\right)-1\right)\left(b+B\right)\left(AC\right)^2} \right)^{1/2} \end{aligned} \quad (43)$$

This leads to the following

Corollary. *As the relative quality of teachers falls over time in the recursive dynamic equilibrium (according to the Theorem), the absolute quality of teachers characterized by their human capital attainment grows: both the human capital of the top teacher and the least qualified one,* $\overline{h}_t$, $\underline{h}_t$ *, increase over time.*

Discussion. The intuition for the above results is based on the facts characterizing economic growth in our model. The growth of per student quality of education increases educational opportunity for an expanding group of students. Namely, college attendance becomes worthwhile for an ever broader group of students, while adding on students with relatively low ability. At the same time, the human capital attainment of higher ability students increases disproportionately relative to their less able peers due to non-linearity in the college education production function (7). In other words, economic growth drives the rise of income inequality within the group of college educated individuals. As a result, hiring high ability individuals as teachers becomes a relatively more expensive option, which pushes the quality-quantity trade-off in the education policy in favor of the latter. This argument is made explicit by the following result which characterizes the evolution of income inequality in our model.

Based on the income formulas (4)-(5) and the human capital accumulation formulas (6)-(7) and using the uniform distribution of abilities, as well as the formula (9) for the threshold ability between the groups, we can obtain mean incomes of unskilled individuals:

$$\overline{I}_t^u = \frac{I_t^u\left(a_t^*\right)}{2} = \frac{w_t \theta_t B h^*}{2\left(\theta_t\left(b+B\right)-1\right)}$$

and the mean income of the skilled (ignoring the distortion due to collective bargaining in the education sector):

$$\overline{I}_t^s = \frac{I_t^s\left(a_t^*\right)+I_t^s\left(A\right)}{2} = \frac{w_t\theta_t}{2}\left(\frac{\left(b+B\right)\theta_t B h^*}{\left(\theta_t\left(b+B\right)-1\right)} + A\left(b+B\right)E_{t-1} - 2Bh^*\right)$$

Thus the inequality between the groups is given by

$$\sigma_t^{s/u} = \frac{\overline{I}_t^s}{\overline{I}_t^u} = \frac{A\left(b+B\right)E_{t-1}\left(\theta_t\left(b+B\right)-1\right)}{Bh^*} + \left(2-\theta_t\left(b+B\right)\right)$$

which according to Lemma 1 increases as education quality rises over time.

The inequality within the skilled group (ignoring the distortion due to collective bargaining in the education sector) is characterized by

$$\sigma_t^s = \frac{I_t^s\left(A\right)}{I_t^s\left(a_t^*\right)} = \frac{\left(b+B\right)ACE_{t-1} - Bh^*}{\left(b+B\right)a_t^* CE_{t-1} - Bh^*} = \left(\theta_t\left(b+B\right)-1\right)\frac{\left(b+B\right)ACE_{t-1} - Bh^*}{Bh^*}$$

which also grows as education quality rises over time.

We summarize these results as

Theorem 2 (The Evolution of Income Inequality). *The recursive dynamic equilibrium dynamics exhibits growing inequality within the group skilled individuals, as well the increase in inequality between this group and the unskilled.*

These results are consistent with the trend of rising skill premium over the recent decades accompanied by rising dispersion of incomes of skilled workers. A common argument in the growth literature is that these phenomena can be explained by the skill biased nature of technological change (see Acemoglu (1998), (2000) and Galor & Moav (2000)).

6. Conclusion

Over the last forty years, education policy in the U.S. has changed significantly. We have developed a model which offers an insight into this evolution by relating it to the changes in the US economy characterized by rising skill premium and overall income inequality. Collective bargaining agreements imposed by teachers' unions have a significant effect on decisions concerning quantity-quality trade-offs in hiring teachers. Our model predicts that as incomes rise and become more dispersed, education policy-makers are forced to adjust relative teacher salaries and thereby quality standards. Education quality is optimized by lowering relative quality of teachers while increasing their numbers. This causes the higher ability college educated people to choose private sector employment which offers higher skill premium.

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Appendix 1

We will first prove Lemmas 1 and 2 under the hypothesis that Lemma 3 is correct. We will then prove that Lemma 3 is indeed correct in the recursive dynamic equilibrium, and thereby the imposition of the hypothesis will not have diminished the generality of (or create circularity problems with) the argument.

Proof of Lemma 1

Recall that according to (20) $E_t = \left(z_t (b+B) CE_{t-1}\bar{a}_t - z_t Bh^* - \frac{1}{2}(b+B) ACE_{t-1} z_t^2 \right)^{\nu} z_t^{\gamma}$.

Substituting

the expression for $\bar{a}_t$ given in (34), we obtain $E_t = \left(A(b+B) CE_{t-1} \left(\frac{\nu(1-\tau_t)}{\gamma} \right) \right)^{\nu} z_t^{2\nu+\gamma}$, or

according to (33)

$$E_t = \left(\frac{A(b+B) CE_{t-1} \nu (1-\tau_t)}{\gamma} \right)^{\nu} \left(\left(\frac{\gamma}{2\nu+\gamma} \right) \left(\frac{\tau_t}{1-\tau_t} \right) \left(1 - \frac{2Bh^*}{(b+B) ACE_{t-1}} + \frac{\theta_t (Bh^*)^2}{(\theta_t (b+B) - 1)(b+B)(ACE_{t-1})^2} \right) \right)^{\nu+\gamma/2}$$

Note that since $\frac{\theta_t (b+B)}{\theta_t (b+B) - 1} > 1$, the following inequality is true

$$1 - \frac{2Bh^*}{(b+B) ACE_{t-1}} + \frac{\theta_t (Bh^*)^2}{(b+B)(\theta_t (b+B) - 1)(ACE_{t-1})^2} > \left[1 - \frac{Bh^*}{(b+B) ACE_{t-1}} \right]^2 \tag{41}$$

Therefore we can write

$$E_t > \left(\frac{\nu}{\gamma}(b+B) ACE_{t-1}(1-\tau_t) \right)^{\nu} \left(\left(\frac{\gamma}{2\nu+\gamma} \right) \left(\frac{\tau_t}{1-\tau_t} \right) \right)^{\nu+\gamma/2} \left(1 - \frac{Bh^*}{(b+B) ACE_{t-1}} \right)^{2\nu+\gamma}$$

Thus, in order to prove the Lemma it is sufficient to show that for all $t = 0, 1, \ldots$

$$\left(\frac{\nu}{\gamma}(b+B) AC(1-\tau_t) \right)^{\frac{1}{2+\gamma/\nu}} \left(\left(\frac{\gamma}{2\nu+\gamma} \right) \left(\frac{\tau_t}{1-\tau_t} \right) \right)^{\frac{1+\gamma/2\nu}{2+\gamma/\nu}} \left(1 - \frac{Bh^*}{(b+B) ACE_{t-1}} \right) > 1$$

which is indeed true according to Assumption 1 and by the induction argument. ■

Proof of Lemma 2

Based on Lemma 3 we use expression (9) for a_t^* . Then according to (35) our task of proving the inequality $\underline{a}_t > a_t^*$. is equivalent to verifying the inequality

$$\frac{Bh^*}{(b+B)CE_{t-1}} + Az_t\left(\frac{\nu(1-\tau_t)}{\gamma} - \frac{1}{2}\right) > \frac{1}{CE_{t-1}}\frac{\theta_t Bh^*}{\theta_t(b+B)-1} \text{ or } Az_t\left(\frac{\nu(1-\tau_t)}{\gamma} - \frac{1}{2}\right) > \frac{1}{(b+B)CE_{t-1}}\frac{Bh^*}{\theta_t(b+B)-1}$$

Upon substituting the expression (33) for z_t, the last inequality becomes

$$A\left(\frac{\nu(1-\tau_t)}{\gamma} - \frac{1}{2}\right)\left(\left(\frac{\gamma}{2\nu+\gamma}\right)\left(\frac{\tau_t}{1-\tau_t}\right)\right)^{1/2}\left(1 - \frac{2Bh^*}{(b+B)ACE_{t-1}} + \frac{\theta_t\left(Bh^*\right)^2}{\left(\theta_t(b+B)-1\right)(b+B)\left(ACE_{t-1}\right)^2}\right)^{1/2} > \frac{1}{(b+B)CE_{t-1}}\frac{Bh^*}{\theta_t(b+B)-1} \quad (42)$$

Under Lemma 3 the right hand side in (42) is less than $\frac{A}{\theta_t(b+B)}$ since $a_t^* < A.$. Therefore according to (41) in order to prove inequality (42) it is by far sufficient to establish

$$\left(\frac{\nu(1-\tau_t)}{\gamma} - \frac{1}{2}\right)\left(\left(\frac{\gamma}{2\nu+\gamma}\right)\left(\frac{\tau_t}{1-\tau_t}\right)\right)^{1/2}\left(1 - \frac{Bh^*}{(b+B)ACE_{t-1}}\right) > \frac{1}{\theta_t(b+B)}$$

which is indeed true for all $t = 0,1,...$ according to Assumption 2 combined with Lemma 1. ■

Proof of Lemma 3

The above proofs were based on the hypothesis that Lemma 3 is correct, i.e. that the ability cut-off for college attendance a_t^* satisfies equality (9), i.e. we proved that if college attendance cut-off ability is $a_t^* = \frac{1}{CE_{t-1}}\frac{\theta_t Bh^*}{\theta_t(b+B)-1}$ then the optimal education policy requires that all teachers' ability strictly exceed this threshold.. This in turn means that the marginal college graduate will be employed in the production sector. As we explained after stating equality (9), if an individual with ability below attended college his skilled human capital given adjusted for the net productivity augmentation θ_t will be inferior to his unskilled human capital derived from the first stage of education, therefore a job in production sector's skilled labor force would not compel such individual to attend college. Thus the only way the violation of Lemma 3 could occur is if such individual had an opportunity to be hired as a teacher. Compare, however, optimization problem (20) where $a_t^* < \frac{1}{CE_{t-1}} \frac{\theta_t B h^*}{\theta_t(b+B)-1}$ versus the one where $a_t^* = \frac{1}{CE_{t-1}} \frac{\theta_t B h^*}{\theta_t(b+B)-1}$. One can easily see that the only difference would be a lower tax revenue T_t in the former case. Therefore such government policy would be inferior to the one where $a_t^* = \frac{1}{CE_{t-1}} \frac{\theta_t B h^*}{\theta_t(b+B)-1}$. Thus the latter indeed characterizes the recursive dynamic equilibrium optimum, i.e. Lemma 3 is correct. ■

Proof of inequality (41)

We rewrite the inequality by squaring its both sides:

$$\left(\left(\frac{\gamma}{2\nu+\gamma}\right)\left(\frac{\tau_t}{1-\tau_t}\right)\right) > \left(\frac{\gamma}{2\nu+\gamma}\right)^2 \left(\frac{\tau_t}{1-\tau_t}\right)^2 \left(\frac{2\nu(1-\tau_t)+\gamma}{2\gamma}\right)^2$$

which can be simplified: $(1-\tau_t)\left(2\frac{\nu}{\gamma}+1\right) > \tau_t\left(\frac{\nu}{\gamma}(1-\tau_t)+\frac{1}{2}\right)^2$. This last inequality is certainly true if $(1-\tau_t)\left(2\frac{\nu}{\gamma}+1\right) > \tau_t\left(\frac{\nu}{\gamma}+\frac{1}{2}\right)^2$ which reduces to $\tau_t < \frac{4}{5+2\nu/\gamma}$. ■

Appendix 2

Glossary of Mathematical Terms

β	Discount factor in individual intertemporal preferences
τ_t	Labor income tax rate
T_t	Total government revenue in period t
a	Lower bound on innate ability, from Section 4 on a =0 is assumed
A	Upper bound on innate ability
$a(\omega)$	The innate ability of individual ω
a_t^*	Ability cut-off level for attending college
$h_t^u(\omega)$	Unskilled individual's level of human capital
H_t^u	Aggregate unskilled human capital
$h_t^s(\omega)$	Skilled individual's level of human capital
H_t^s	Aggregate skilled human capital
C	Productivity coefficient of compulsory basic education
E_t	Public education quality
b	Coefficient of in-college depreciation of pre-college human capital.
B	Productivity coefficient of higher education
h^*	Human capital threshold for admission to college
D	TFP coefficient in the goods production sector
α	Physical capital income share in the goods production sector
θ_t	Productivity augmentation of skilled human capital (technological skill bias) in the production sector
H_t^u	Aggregate unskilled human capital in goods production
H_t^{sy}	Aggregate skilled human capital in goods production
γ	Returns to quantity of teachers
ν	Returns to quality of teachers
Σ_t	Set of individuals employed as teachers in period t
z_t	Number of teachers (the share of teachers in the working population) in period t
I_t^h	Teacher's salary
$\underline{a}_t$	The lowest ability level among teachers in period t
$\underline{h}_t$	The lowest human capital level among teachers in period t
$\overline{a}_t$	The highest ability level among teachers in period t
$\overline{h}_t$	The highest human capital level among teachers in period t