

# Bayesian Decision Theory and the Representation of Beliefs

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## Abstract

This paper shows that Savage's (1954) definition of subjective probabilities relies on an arbitrary normalization, and that this normalization is not refutable in the framework of the revealed preference methodology. In contrast, in the framework of Karni (2006), choice-based subjective probabilities that measure Bayesian decision makers' prior and posterior beliefs contingent on his actions are identifiable. A principal-agent

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example shows that ascribing to the agent subjective probabilities that do not represent his beliefs, may result in contracts that are individually rational and incentive incompatible.

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**JEL classification numbers:** D80, D81, D82

## 1 Introduction

What is the meaning of subjective probabilities? The literature offers two distinct answers to this question that I shall refer to as the definitional and measurement interpretations.

The *definitional* interpretation maintains that subjective probability defines the decision maker's *degree of belief* that certain events obtain. According to this interpretation, subjective probability emerges as a consequence of the choice of a functional form to represent a decision maker's preference relation and, in itself, have no empirical meaning (that is, has no refutable implications in the context of revealed preference methodology).

The *measurement* interpretation maintains that subjective probabilities quantify the attribute of events, or propositions, relative to a decision maker, namely, the *degree of belief* that a decision maker holds that the events obtain, or that the propositions are true. This interpretation is as old as the notion of probability itself.<sup>1</sup> A modern day expression of this

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<sup>1</sup>“... the probability emerging in the time of Pascal is essentially dual. It has to do with both the stable frequencies and with degrees of belief.” Hecking (1975) p. 10.

view is found in the following quote: “When measuring some attribute of a class of objects or events, we associate numbers with the objects in such a way that the properties of the attribute are faithfully represented by the numerical properties.” (Krantz et. al. (1971), p. 1). The attribute of the events measured by subjective probability is their subjective likelihood.

The notion of subjective probability that has dominated the discussion in economics and decision theory during the half-century following the publication of Savage’s (1954) *The Foundations of Statistics*, is of the definitional type. However, being a tautology, Savage’s notion of subjective probability is inadequate foundation of Bayesian statistical theory.

In this paper I invoke the analytical framework advanced in Karni (2006) to develop a theory of subjective probability consistent with the measurement interpretation. I claim that this notion of probability is has the attributes of a subjective Bayesian prior and, consequently, constitutes a foundations of Bayesian statistics.

Whether the subjective probability measures the decision maker’s beliefs is not a purely philosophical issue - it has potential economic consequences. In the appendix I analyze a moral hazard problem illustrating the potential pitfalls of ascribing the agent an incorrect prior. In that example the principal knows the agent’s prior preference relation and uses Bayes’ rule to update the agent’s preferences in the light of new information. Yet, even though the agent is Bayesian, because the principal ascribes to him an incorrect prior, the principal fails to appreciate the effects of new information and, as a result, designs an

incentive incompatible contract.

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With few exceptions, modeling decision making under uncertainty (that is, choice among courses of action whose exact consequences are not determined solely by the decision maker's choice) invoke the analytical framework of Savage (1954). This framework consists of a set of states,  $S$ ; a set of consequences,  $C$ ; and the set,  $F$ , of all the mappings from the set of states to the set of consequences. Elements of  $F$ , referred to as *acts*, correspond to conceivable courses of action; a decision maker is characterized by a preference relation on  $F$ . States resolve the uncertainty in the sense that once the (unique) true state becomes known, the unique consequence implied by each and every act becomes known. Subsets of  $S$  are *events*. An event is said to obtain if the true state belongs to it.

In Savage's subjective expected utility theory, the structure of a (prior) preference relation,  $\succsim$ , on  $F$ , depicted axiomatically, allows its representation by an expected utility functional,

$$\int_S u(f(s)) d\pi(s), \tag{1}$$

where  $u$  is a real-valued (utility) function defined on the consequences;  $\pi$  is a finitely additive, nonatomic probability measure on  $S$ ; and  $f \in F$ . Moreover, the utility function  $u$  is unique up to positive linear transformation, and, given  $u$ , the subjective probability measure  $\pi$  is unique. Presumably, the interest in this representation stems from the notion that the

decision maker evaluates alternative acts by assessing, separately, the merit and likelihood of the possible consequences and then integrates these assessments. In other words, the merit of the consequences and the decision maker’s degree of belief regarding their likelihood are meaningful cognitive phenomena that may be quantified, respectively, by a utility function and a subjective probability measure. These numerical values are combined, using the functional in equation (1), to obtain a numerical valuation of the acts. In this interpretation, a decision maker’s beliefs is a relation on the set of events whose meaning is “at least as likely to obtain as.”

Clearly, any probability measure on  $S$  defines beliefs.<sup>2</sup> In Savage’s (1954) theory the decision maker’s (prior) beliefs are defined by the probability measure  $\pi$ . However, the uniqueness of  $\pi$  and, consequently, of Savage’s beliefs, is predicated on the convention that constant acts are constant utility acts (that is, the utility function is state-independent). *This convention, however, is not implied by the axioms and, more important, is not testable within Savage’s analytical framework.*<sup>3</sup> Yet without it, it is impossible to disentangle the probabilities and marginal utilities. Consequently, there are infinitely many prior probability measures consistent with a decision maker’s prior preferences. In other words, even if a decision maker’s beliefs constitute a quantifiable psychological phenomenon and his choice behavior is consistent with the axiomatic structure of expected utility theory, the proposition that the subjective probabilities ascribed to him by Savage’s model represent the decision maker’s

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<sup>2</sup>Let  $\succeq_{\mathcal{B}}$  be a binary relation on the set of events defined by  $E \succeq_{\mathcal{B}} E'$  if and only if  $\pi(E) \geq \pi(E')$ , for all events  $E$  and  $E'$ , then  $\succeq_{\mathcal{B}}$  is beliefs.

<sup>3</sup>For more details regarding the first part of this assertion, see Karni (1996, 2006).

beliefs is untestable. In what follows I use the term *identifiable* subjective probabilities to refer to a probability measure, consistent with the decision maker's choice behavior, whose uniqueness is not predicated on a particular choice of a utility function. The argument above implies that Savage's subjective probability measure is unidentifiable.

A decision maker who, upon receiving new information, displays a change of preferences reflecting his modified beliefs is said to be Bayesian if the representation of his prior preferences involves an identifiable prior probability measure on the set  $S$ ; the representations of his posterior preference relations involve well-defined posterior probability measures on  $S$ ; and the representations of the posterior preference relations are obtained from the prior preference relation by updating the prior probabilities according to Bayes rule. Next I show that Savage's model fails to yield an identifiable prior and is not a model of Bayesian decision making.<sup>4</sup>

To prove this assertion, let  $\gamma$  be a strictly positive, bounded, real-valued function on  $S$ , and let  $E(\gamma) = \int_S \gamma(s) d\pi(s)$ . Then the prior preference relation, depicted by the representation (1) is also represented by

$$\int_S \hat{u}(f(s), s) d\hat{\pi}(s), \tag{2}$$

where  $\hat{u}(\cdot, s) = u(\cdot) / \gamma(s)$  and  $\hat{\pi}(s) = \pi(s) \gamma(s) / E(\gamma)$ , for all  $s \in S$ .<sup>5</sup>

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<sup>4</sup>The same criticism applies to all subjective expected utility models that invoke Savage's analytical framework, such as the model of Anscombe and Aumann (1963).

<sup>5</sup>This point has been recognized by Drèze (1987); Schervish, Seidenfeldt, and Kadane (1990); Karni (1996), (2003); Karni and Schmeidler (1993); and Nau (1995).

Consider next an experiment whose possible outcomes are pertinent to the decision maker's assessment of the likely realization of events. Let  $X$  be the set of observations, and denote by  $q(x | s)$  the conditional probability of the observation  $x$  if the true state is  $s$  (e.g.,  $s$  may be thought of as the parameters underlying the distribution of  $X$ ) and  $\{q(\cdot | s) | s \in S\}$  is a family of likelihood functions. Then, invoking representation (1), the induced posterior preference relations  $\{\succsim^x | x \in X\}$  of a Bayesian decision maker are defined as follows: for all  $x \in X$ , and  $f, f' \in F$ ,

$$f \succsim^x f' \Leftrightarrow \int_S u(f(s)) d\pi(s | x) \geq \int_S u(f'(s)) d\pi(s | x), \quad (3)$$

where, for each  $x \in X$ ,

$$\pi(\cdot | x) = \frac{q(x | \cdot) \pi(\cdot)}{\int_S q(x | s') d\pi(s')} \quad (4)$$

is the posterior probability measure obtained by the application of Bayes' theorem.

Invoking representation (2), define the induced posterior preference relations,  $\{\hat{\succsim}^x | x \in X\}$ , of the same Bayesian decision maker whose prior probability measure is  $\hat{\pi}(\cdot)$  as follows: for all  $x \in X$ , and  $f, f' \in F$ ,

$$f \hat{\succsim}^x f' \Leftrightarrow \int_S \hat{u}(f(s), s) d\hat{\pi}(s | x) \geq \int_S \hat{u}(f'(s), s) d\hat{\pi}(s | x), \quad (5)$$

where

$$\hat{\pi}(\cdot | x) = \frac{q(x | \cdot) \hat{\pi}(\cdot)}{\int_S q(x | s') d\hat{\pi}(s')}. \quad (6)$$

However,

$$\int_S \hat{u}(f(s), s) d\hat{\pi}(s | x) = \left[ \frac{\int_S q(x | s') d\pi(s')}{\int_S q(x | s') d\hat{\pi}(s')} \right] \int_S u(f(s)) d\pi(s | x). \quad (7)$$

Hence for all  $f, f' \in F$  and  $x \in X$ ,

$$\int_S \hat{u}(f(s), s) d\hat{\pi}(s | x) \geq \int_S \hat{u}(f'(s), s) d\hat{\pi}(s | x) \quad (8)$$

if and only if

$$\int_S u(f(s)) d\pi(s | x) \geq \int_S u(f'(s)) d\pi(s | x). \quad (9)$$

Thus  $\succsim^x = \hat{\succsim}^x$  for all  $x \in X$ . *The fact that a decision maker updates his preferences using Bayes' rule does not imply that either his prior or his posterior beliefs, as defined by the representing probabilities, are unique.*<sup>6</sup>

The fact that the subjective probabilities in Savage's theory (and in other theories that invoke Savage's analytical framework) are not identifiable means that the definitional notion of probability does constitute a behavioral foundations of Bayesian statistics.

From the viewpoint of economic and decision theories, the popularity of Savage's notion of subjective probabilities is due to the elegance of the representation it affords, both in terms of its mathematical formulation and the linguistic ease of describing its ingredients.<sup>7</sup>

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<sup>6</sup>Note that if the prior is nonunique it then follows from Bayes rule that the posterior probabilities are not unique.

<sup>7</sup>Quite often in the literature, the term subjective probability is used interchangeably with the term

However, to attain the separation of subjective probabilities from utilities it is necessary to assume that the preference relation exhibit state independence, which entails substantial restrictions on generality and range of applications of the model. For instance, Anscombe and Aumann (1963) impose state-independence to decompose the terms of a separately additive representation into a product of utility and probability.<sup>8</sup> The imposition of substantive restrictions to attain mathematical elegance is not good scholarship. A more meaningful notion of subjective probability, one that is a measurement of subjective beliefs when these beliefs have structure that allows their representation by probability measure, is developed in this paper.

Section 2 reviews the analytical framework and states the main result of Karni (2006). Section 3 presents a Bayesian decision theory and shows that the prior and posterior subjective probabilities are identifiable. Concluding remarks appear in section 4.

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beliefs. Yet, as just demonstrated, this usage is hardly justified if not outright misleading. It does, however, serve the purpose of lending the theory with intuitive meaning that readers apparently find compelling.

<sup>8</sup>The analogous axioms in Savage's model are P3 and P4. Like Savage, to deliver that uniqueness of the probabilities, Anscombe and Aumann (1963) invoke the convention that the utility function is state independent.

## 2 The Theory

### 2.1 The analytical framework

Let  $\Theta$  be a finite set of *effects* and  $A$  a set of *actions*. Actions represent initiatives that decision makers may take to influence the likely realization of effects. A *bet*,  $b$ , is a mapping from  $\Theta$  into  $\mathbb{R}$ , the set of real numbers. Bets have the interpretation of monetary payoffs contingent on effects. Let  $B := \mathbb{R}^{\Theta}$  denote the set of all bets, and assume that it is endowed with the  $\mathbb{R}^{|\Theta|}$  topology. Denote by  $(b_{-\theta}r)$  the bet obtained from  $b \in B$  by replacing the  $\theta$  coordinate of  $b$  (that is,  $b(\theta)$ ) with  $r$ . In this framework, effects are analogous to states, in the sense that they resolve the uncertainty associated with bets. However, unlike states, decision makers believe that they may influence the likely realization of effects by their actions.

Decision makers are supposed to, simultaneously, choose actions and place bets on the effects. For example, a decision maker may adopt an exercise and diet regimen to reduce the risk of heart attack and, at the same time, take out health insurance and life insurance policies. The diet and exercise regimens correspond to actions, the states of health are effects, and the financial terms of an insurance policy constitute a bet. Formally, the *choice set*,  $\mathbb{C}$ , consists of all action-bet pairs (that is,  $\mathbb{C} = A \times B$ ). A choice of an action  $a$  and a bet  $b$  results in an effect-payoff pair, or *consequence*,  $(\theta, b(\theta))$ . Let  $C$  denote the set of all consequences ( $C = \Theta \times \mathbb{R}$ ).

A decision maker is characterized by prior preference relation,  $\succsim$ , on  $\mathbb{C}$ . The strict pref-

erence relation,  $\succ$ , and the indifference relation,  $\sim$ , are the asymmetric and symmetric parts of  $\succsim$ , respectively. For each  $a \in A$ , the preference relation  $\succsim$  on  $\mathbb{C}$  induces a conditional preference relation on  $B$  defined as follows: for all  $b, b' \in B$ ,  $b \succsim_a b'$  if and only if  $(a, b) \succsim (a, b')$ .

An effect,  $\theta$ , is said to be *nonnull given the action  $a$*  if  $(a, (b_{-\theta}r)) \succ (a, (b_{-\theta}r'))$ , for some  $b \in B$  and  $r, r' \in \mathbb{R}$ ; it is *null given the action  $a$*  otherwise. I assume that every effect is nonnull for some action  $a$ . Given a preference relation,  $\succsim$ , denote by  $\Theta(a; \succsim)$  the subset of effects that are nonnull given  $a$  according to  $\succsim$ . To simplify the notation, when there is no risk of confusion, I write  $\Theta(a)$  instead of  $\Theta(a; \succsim)$ .

Two effects,  $\theta$  and  $\theta'$ , are said to be *elementarily linked* if there are actions  $a, a' \in A$  such that  $\theta, \theta' \in \Theta(a) \cap \Theta(a')$ . Two effects are said to be *linked* if there exists a sequence of effects  $\theta = \theta_0, \dots, \theta_n = \theta'$  such that  $\theta_j$  and  $\theta_{j+1}$  are elementarily linked,  $j = 0, \dots, n - 1$ . The set of effects,  $\Theta$ , is linked if every pair of its elements is linked.

Actions may affect the decision maker's well-being directly. For example, adopting a diet regimen may require the decision maker to avoid eating food he likes. With this in mind, I refer to a bet as constant valuation if, once accepted, it leaves the decision maker indifferent among actions whose direct utility costs are just compensated for by the improved chances of winning better outcomes they afford.<sup>9</sup> To simplify the exposition, I assume that the set  $A$  consists of those actions whose induced variations in the likely realization of alternative

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<sup>9</sup>Because of the possible direct impact of the actions on the decision maker's well-being, constant valuation bets are different from Drèze's (1987) notion of "omnipotent" acts and the idea of constant valuation acts in Karni (1993, 2006) and Skiadas (1997).

effects and associated direct utility implications determine a unique set of constant valuation bets.<sup>10</sup> Formally, let  $I(a; b) = \{b' \in B \mid (a, b') \sim (a, b)\}$ . A bet  $\bar{b}$  is said to be a *constant-valuation bet* if  $(a, \bar{b}) \sim (a', \bar{b})$  for all  $a, a' \in A$ , and  $\bigcap_{a \in A} I(a; \bar{b}) = \{\bar{b}\}$ .

Let  $B^{cv}$  denote the set of all constant valuation bets. If  $b^{**}$  and  $b^*$  are constant valuation bets satisfying  $(a', b^{**}) \succ (a', b^*)$  for some  $a' \in A$ , then, by transitivity of  $\succ$ ,  $(a, b^{**}) \succ (a, b^*)$  for all  $a \in A$ . Since transitivity will be assumed, I write  $b^{**} \succ b^*$  instead of  $(a, b^{**}) \succ (a, b^*)$ .

The following richness assumption is maintained throughout:

- (A.0) *The set  $\Theta$  is linked, there exist constant-valuation bets  $b^{**}$  and  $b^*$  such that  $b^{**} \succ b^*$  and, for every  $(a, b) \in \mathbb{C}$ , there is  $\bar{b} \in B^{cv}$  satisfying  $(a, b) \sim \bar{b}$ .*

## 2.2 Axioms and representation

The first two axioms are standard.

- (A.1) **(Weak order)**  $\succ$  on  $\mathbb{C}$  is a complete and transitive binary relation.

- (A.2) **(Continuity)** For all  $(a, b) \in \mathbb{C}$  the sets  $\{(a, b') \in \mathbb{C} \mid (a, b') \succ (a, b)\}$  and  $\{(a, b') \in \mathbb{C} \mid (a, b) \succ (a, b')\}$  are closed.

The third axiom requires that the “intensity of preferences” for monetary payoffs contingent on any given effect be independent of the action that resulted in that effect.

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<sup>10</sup>The model may be extended to include additional actions (see in Karni 2006).

(A.3) (**Action-independent betting preferences**) For all  $a, a' \in A, b, b', b'', b''' \in B, \theta \in \Theta(a) \cap \Theta(a')$ , and  $r, r', r'', r''' \in \mathbb{R}$ , if  $(a, b_{-\theta}r) \succcurlyeq (a, b'_{-\theta}r')$ ,  $(a, b'_{-\theta}r'') \succcurlyeq (a, b_{-\theta}r''')$ , and  $(a', b''_{-\theta}r') \succcurlyeq (a', b'''_{-\theta}r)$  then  $(a', b''_{-\theta}r'') \succcurlyeq (a', b'''_{-\theta}r''')$ .

To grasp the meaning of action-independent betting preferences, think of the preferences  $(a, b_{-\theta}r) \succcurlyeq (a, b'_{-\theta}r')$  and  $(a, b'_{-\theta}r'') \succcurlyeq (a, b_{-\theta}r''')$  as indicating that, given an action  $a$  and an effect  $\theta$ , the intensity of the preferences of  $r''$  over  $r'''$  is sufficiently larger than that of  $r$  over  $r'$  as to reverse the preference ordering of the effect-contingent payoffs  $b_{-\theta}$  and  $b'_{-\theta}$ . The axiom requires that these intensities not be contradicted when the action is  $a'$  instead of  $a$  and bets are  $b''$  and  $b'''$ .<sup>11</sup> It implies that the curvature of the utility function is independent of the action. In a world in which lotteries are consequences this would imply that the decision maker's risk attitudes are action independent.

Karni (2006) shows that a preference relation on  $\mathbb{C}$  satisfying (A.0), has the structure described by axioms (A.1)–(A.3) if and only if there exist a continuous utility function,  $u$ , on the set of consequences, a family of action-dependent probability measures,  $\{\pi(\cdot; a) \mid a \in A\}$ , on the set of effects, and a family of continuous strictly increasing functions  $\{f_a : \mathbb{R} \rightarrow \mathbb{R} \mid a \in A\}$  such that, for all  $(a, b), (a', b') \in \mathbb{C}$ ,

$$(a, b) \succcurlyeq (a', b') \Leftrightarrow f_a \left( \sum_{\theta \in \Theta} u(b(\theta); \theta) \pi(\theta; a) \right) \geq f_{a'} \left( \sum_{\theta \in \Theta} u(b'(\theta); \theta) \pi(\theta; a') \right). \quad (10)$$

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<sup>11</sup>Action-independent betting preferences invokes Wakker's (1987) idea of cardinal consistency. A more elaborate discussion of this axiom is provided in Karni (2006).

Furthermore,  $u$  is unique;  $\{f_a\}_{a \in A}$  are unique up to common, strictly increasing, transformation; and, for each  $a \in A$ , the probability measure  $\pi(\cdot; a)$  is unique, satisfying  $\pi(\theta; a) = 0$  if and only if  $\theta$  is null given  $a$ .

The fact that the actions do not appear as argument in  $u$  is a reflection of action-independent risk attitudes, implied by (A.3). As in Savage's (1954) model, there is no guarantee that the prior probabilities correspond to the decision maker's prior beliefs. To see this, fix a non-constant function  $\gamma : \Theta \rightarrow \mathbb{R}_{++}$  and let  $\Gamma(a) = \sum_{\theta \in \Theta} \gamma(\theta) \pi(\theta; a)$ ,  $\hat{u}(b(\theta); \theta) = u(b(\theta); \theta) / \gamma(\theta)$ ,  $\hat{\pi}(\theta; a) = \gamma(\theta) \pi(\theta; a) / \Gamma(a)$ , and  $\hat{f}_a = f_a \circ \Gamma(a)$ . Then, by (10), the prior preference relation  $\succsim$  is represented by

$$f_a \left( \Gamma(a) \sum_{\theta \in \Theta} \frac{u(b(\theta); \theta) \gamma(\theta) \pi(\theta; a)}{\gamma(\theta) \Gamma(a)} \right) = \hat{f}_a \left( \sum_{\theta \in \Theta} \hat{u}(b(\theta); \theta) \hat{\pi}(\theta; a) \right). \quad (11)$$

But  $\hat{\pi}(\theta; a) \neq \pi(\theta; a)$  for some  $a$  and  $\theta$ . Therefore the prior subjective probabilities are not unique.

### 3 Bayesian Preferences and Subjective Probabilities

#### 3.1 Bayesian decision makers

A Bayesian decision maker is characterized by a prior preference relation, a set of information-dependent posterior preferences, and the condition that the posterior preferences are obtained from the prior preference solely by updating his subjective probabilities using Bayes' rule

(that is, leaving intact the other functions that figure in the representation). To model the behavior of Bayesian decision makers it is necessary to formally incorporate into the analytical framework new information.

Let  $X$  be a finite set of *observations* and consider the case in which an observation may be obtained prior to taking action. For each  $\theta \in \Theta$ ,  $q(\cdot | \theta)$  is a likelihood function on  $X$  (that is,  $q(\cdot | \theta) \geq 0$  and  $\sum_{x \in X} q(x | \theta) = 1$ ).<sup>12</sup> Elements of  $X$  are *observations* that might influence the decision maker's beliefs regarding the likelihoods of the alternative effects (experimental results, expert opinions, etc.). For every  $x \in X$ , let  $\succsim^x$  be a (posterior) preference relation on  $\mathbb{C}$  characterizing the decision maker's choice behavior after he has been informed of the observation  $x$ .

Consider a decision maker who, before choosing an action-bet pair, receives information pertinent to his assessment of the likelihood of alternative effects, conditional on his choice of action. Suppose that the decision maker updates his prior beliefs by the application of Bayes' rule. Formally,

**Definition:** A decision maker whose prior preference relation,  $\succsim$  on  $\mathbb{C}$ , is represented by

$f_a \left( \sum_{\theta \in \Theta} u(b(\theta); \theta) \pi(\theta; a) \right)$  is *Bayesian* if his posterior preference relations,  $\{\succsim^x\}_{x \in X}$ , are represented by  $f_a \left( \sum_{\theta \in \Theta} u(b(\theta); \theta) \pi(\theta | x, a) \right)$ , where, for all  $x \in X$ ,  $\theta \in \Theta$  and

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<sup>12</sup>This assume that the likelihood functions are independent of  $a$  (see the example in the appendix). I revisit this assumption in the concluding remarks.

$a \in A$ ,

$$\pi(\theta \mid x, a) = \frac{q(x \mid \theta) \pi(\theta; a)}{\sum_{\theta' \in \Theta} q(x \mid \theta') \pi(\theta'; a)}. \quad (12)$$

The hybrid model presented here entails subjective assessment of likelihoods of effects and objective (in the sense of relative frequencies) assessment of likelihoods of observations conditional on the effects. This is a reasonable assumption in many situations. For instance, if the effects are the parameters underlying the binomial distribution and the observations corresponds to realizations of the process then, conditional on the parameter value, the distribution of the observation is objective, while the prior on the parameter space is subjective. Similarly, the empirical distribution of the level of PSA that shows in a blood test depends on whether the person has of prostate cancer.

### 3.2 The uniqueness of the priors

The main result asserts that if a decision maker is Bayesian, then the prior, action-dependent, subjective probabilities representing his beliefs are identifiable. It requires, however, that the set of observations and associated likelihood functions be rich in the following sense: for every nonconstant function  $\gamma : \Theta \rightarrow \mathbb{R}_{++}$  let  $\Gamma_\gamma(a) := \sum_{\theta \in \Theta} \gamma(\theta) \pi(\theta \mid a)$  and  $\Gamma_\gamma(a, x) := \sum_{\theta \in \Theta} \gamma(\theta) \pi(\theta \mid a, x)$ , where  $\pi(\theta \mid a)$  and  $\pi(\theta \mid a, x)$  denote, respectively, the the prior probability of  $\theta$  conditional on  $a$  and the posterior probability of  $\theta$  conditional on  $a$  and  $x$ , then,

(R) For every nonconstant function  $\gamma : \Theta \rightarrow \mathbb{R}_{++}$  there are  $a, a' \in A$  and  $x \in X$  such

that

$$\frac{\Gamma_\gamma(a', x)}{\Gamma_\gamma(a, x)} = \frac{\Gamma_\gamma(a')}{\Gamma_\gamma(a)}. \quad (13)$$

Note that the richness condition R is weak, requiring that, if  $\gamma$  is nonconstant function then the ratio of the means of  $\gamma$  conditional on, at least one pair of actions, is not the same under all possible observations.

**Theorem 1** *Suppose that there are at least two effects, R holds, and the decision maker's prior preference relation,  $\succsim$  on  $\mathbb{C}$ , satisfies (A.0)–(A.3). If the decision maker is Bayesian, then the representation of his prior preferences admits a unique set of action-dependent probability measures.*

*Proof.* Consider a Bayesian decision maker whose prior preference relation  $\succsim$  on  $\mathbb{C}$  satisfies (A.0)–(A.3). Then, by Karni (2006),  $\succsim$  has the representation (10). Denote by  $\succsim^x$  the decision maker's posterior preferences on  $\mathbb{C}$  conditional on the observation  $x \in X$ . By definition, if the representation of  $\succsim$  on  $\mathbb{C}$  is given by

$$(a, b) \rightarrow f_a \left( \sum_{\theta \in \Theta} u(b(\theta); \theta) \pi(\theta; a) \right), \quad (14)$$

then for every  $x \in X$ , the posterior preference relation is represented by

$$(a, b) \mapsto f_a \left( \sum_{\theta \in \Theta} u(b(\theta); \theta) \pi(\theta | x, a) \right), \quad (15)$$

where  $\pi(\theta | x, a)$  is given in (12).

Given a nonconstant function  $\gamma : \Theta \rightarrow \mathbb{R}_{++}$  let  $\Gamma_\gamma(a) = \sum_{\theta \in \Theta} \gamma(\theta) \pi(\theta; a)$ .<sup>13</sup> Then the prior preferences  $\succsim$  on  $\mathbb{C}$  is represented by

$$(a, b) \rightarrow \hat{f}_a \left( \sum_{\theta \in \Theta} \hat{u}(b(\theta); \theta) \hat{\pi}(\theta; a) \right), \quad (16)$$

where  $\hat{u}(b(\theta); \theta) = u(b(\theta); \theta) / \gamma(\theta)$ ,  $\hat{\pi}(\theta; a) = \gamma(\theta) \pi(\theta; a) / \Gamma_\gamma(a)$ , and  $\hat{f}_a = f_a \circ \Gamma_\gamma(a)$ .

Moreover, by definition, for every  $x \in X$ , the corresponding posterior preference relation is represented by

$$(a, b) \mapsto \hat{f}_a \left( \sum_{\theta \in \Theta} \hat{u}(b(\theta); \theta) \hat{\pi}(\theta | x, a) \right), \quad (17)$$

where  $\hat{\pi}(\theta | x, a) = q(x | \theta) \hat{\pi}(\theta; a) / \sum_{\theta' \in \Theta} q(x | \theta') \hat{\pi}(\theta'; a)$ . By definition,

$$\hat{\pi}(\theta | x, a) = \frac{q(x | \theta) \gamma(\theta) \pi(\theta; a)}{\sum_{\theta' \in \Theta} q(x | \theta') \gamma(\theta') \pi(\theta'; a)}. \quad (18)$$

Recall that  $\Gamma_\gamma(a, x) = \sum_{\theta \in \Theta} \gamma(\theta) \pi(\theta | a, x)$ . Hence, using (18),

$$\sum_{\theta \in \Theta} \hat{u}(b(\theta), \theta) \hat{\pi}(\theta | x, a) = \frac{1}{\Gamma_\gamma(a, x)} \sum_{\theta \in \Theta} u(b(\theta); \theta) \pi(\theta | x, a), \quad (19)$$

and, by definition,

$$\hat{f}_a \left( \sum_{\theta \in \Theta} \hat{u}(b(\theta), \theta) \hat{\pi}(\theta | x, a) \right) = f_a \left( \frac{\Gamma_\gamma(a)}{\Gamma_\gamma(a, x)} \sum_{\theta \in \Theta} u(b(\theta); \theta) \pi(\theta | x, a) \right). \quad (20)$$

Next I show that there exists an observation  $x \in X$  such that  $\succsim^x \neq \hat{\succsim}^x$ .

**Lemma 2** *For every constant-valuation bet,  $\bar{b}$ , there exist constant-utility bets,  $\hat{b}_{\bar{b}}, \hat{b}'_{\bar{b}} \in B$ , such that, for all  $\theta \in \Theta$ ,  $u(\hat{b}_{\bar{b}}(\theta), \theta) = \bar{u}$  and  $u(\hat{b}'_{\bar{b}}(\theta), \theta) = \bar{u}'$  and, for all  $a, a' \in A$ ,  $f_a(\bar{u}) = f_{a'}(\bar{u}')$ .*

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<sup>13</sup>Since  $\gamma$  is not constant,  $\Gamma_\gamma(a) \neq 1$  for some  $a \in A$ .

*Proof of Lemma:* Fix  $a, a' \in A$ . Take a constant-valuation bet  $\bar{b}$ , rearrange the effects so that  $u(\bar{b}(1); 1) \geq u(\bar{b}(2); 2) \geq \dots \geq u(\bar{b}(n); n)$ . If all the weak inequalities are equalities then  $\bar{b}$  is a constant-utility bet and, by definition,  $f_a(\bar{u}) = f_{a'}(\bar{u}')$ .

Suppose that some of the inequalities are strict. Holding  $\sum_{\theta \in \Theta} u(b(\theta); \theta) \pi(\theta | a)$  constant, for all  $\bar{b}(i) \in \arg \max\{u(\bar{b}(\theta), \theta) | \theta \in \Theta\}$ , decrease the value of the payoff keeping  $u(\bar{b}(i), i) = u(\bar{b}(j), j)$  for all  $\bar{b}(i), \bar{b}(j) \in \arg \max\{u(\bar{b}(\theta), \theta) | \theta \in \Theta\}$  and, for all  $\bar{b}(i) \in \arg \min\{u(\bar{b}(\theta), \theta) | \theta \in \Theta\}$ , increase the value of the payoff keeping  $u(\bar{b}(i), i) = u(\bar{b}(j), j)$  for all  $\bar{b}(i), \bar{b}(j) \in \arg \min\{u(\bar{b}(\theta), \theta) | \theta \in \Theta\}$ . This is possible by continuity. Continue this process, until the payoff of the argmax set is equal to the second highest payoff or the payoff of the argmin set is equal to the second lowest payoff in  $\{\bar{b}(\theta) | \theta \in \Theta\}$ , whichever happens first. Repeat the process with the new argmin and/or argmax sets. After a finite number of repetitions the payoffs,  $\hat{b}_{\bar{b}}(\theta)$ , will be such that the utilities are the same. By construction  $(a, \hat{b}_{\bar{b}}) \sim (a, \bar{b})$ . Repeating the process with  $(a', \bar{b})$  to obtain  $\hat{b}'_{\bar{b}}$  such that  $(a', \hat{b}'_{\bar{b}}) \sim (a', \bar{b})$ . By definition  $(a', \bar{b}) \sim (a, \bar{b})$ . Hence, by transitivity,  $(a', \hat{b}'_{\bar{b}}) \sim (a, \hat{b}_{\bar{b}})$ , which implies  $f_a(\bar{u}) = f_{a'}(\bar{u}')$ .  $\square$

Given the function  $\gamma$ , the representation (15) implies that  $(a, \hat{b}_{\bar{b}}) \sim^x (a', \hat{b}'_{\bar{b}})$ , for all  $x \in X$ . But

$$\hat{f}_a \left( \sum_{\theta \in \Theta} \hat{u}(\hat{b}_{\bar{b}}(\theta), \theta) \hat{\pi}(\theta | x, a) \right) = f_a \left( \frac{\Gamma_{\gamma}(a)}{\Gamma_{\gamma}(a, x)} \bar{u} \right)$$

and

$$\hat{f}_{a'} \left( \sum_{\theta \in \Theta} \hat{u}(\hat{b}'_{\bar{b}}(\theta), \theta) \hat{\pi}(\theta | x, a') \right) = f_{a'} \left( \frac{\Gamma_{\gamma}(a')}{\Gamma_{\gamma}(a', x)} \bar{u}' \right).$$

Because  $f_a$  is monotonic increasing and continuous, the inverse function  $f_a^{-1}$  exist for all  $a \in A$ . Thus  $f_a(\bar{u}) = f_{a'}(\bar{u}')$  implies that  $\bar{u} = f_a^{-1} \circ f_{a'}(\bar{u}')$ , and, for all  $\bar{u}$ ,

$$\frac{\Gamma_\gamma(a)}{\Gamma_\gamma(a, x)} \bar{u} = f_a^{-1} \circ f_{a'} \left( \frac{\Gamma_\gamma(a')}{\Gamma_\gamma(a', x)} \bar{u}' \right).$$

Hence, for all  $x \in X$  and  $\bar{u}'$

$$\frac{\Gamma_\gamma(a)}{\Gamma_\gamma(a, x)} f_a^{-1} \circ f_{a'}(\bar{u}') = f_a^{-1} \circ f_{a'} \left( \frac{\Gamma_\gamma(a')}{\Gamma_\gamma(a', x)} \bar{u}' \right).$$

If  $f_a^{-1} \circ f_{a'}$  is linear then  $\frac{\Gamma_\gamma(a)}{\Gamma_\gamma(a, x)} = \frac{\Gamma_\gamma(a')}{\Gamma_\gamma(a', x)}$  for all  $x \in X$ . If  $f_a^{-1} \circ f_{a'}$  is nonlinear then  $\frac{\Gamma_\gamma(a)}{\Gamma_\gamma(a, x)} = \frac{\Gamma_\gamma(a')}{\Gamma_\gamma(a', x)} = 1$  for all  $x \in X$ . Either case contradicts R. Thus  $(a, \hat{b}_b) \approx^{x'} (a', \hat{b}_b)$ . But  $f_a(\bar{u}) = f_{a'}(\bar{u}')$  implies that  $(a, \hat{b}_b) \sim^x (a', \hat{b}_b')$ , for all  $x \in X$ . Hence  $\succ^{x'} \neq \hat{\succ}^{x'}$ .

Therefore, in either case,  $\succ^x$  and  $\hat{\succ}^x$  cannot both be true for all  $x \in X$ . Consequently, there is a unique representation of the prior preference relations of a Bayesian decision maker that induces the correct posterior representations. The set of action-dependent probability measures in this representation is unique. ■

The proof of theorem 1 is a demonstration that updating the prior preference relation by applying Bayes' rule to a misspecified prior must, for some observations, induce posterior preference relations that disagree with the decision maker's actual posterior preferences. To develop an intuitive understanding of this result, consider the case of additive representation, namely,  $f_a(\sum_{\theta \in \Theta} u(b(\theta); \theta) \pi(\theta; a)) = \sum_{\theta \in \Theta} u(b(\theta); \theta) \pi(\theta; a) + v(a)$  or, equivalently,  $\hat{f}_a(\sum_{\theta \in \Theta} \hat{u}(b(\theta); \theta) \hat{\pi}(\theta; a)) = \Gamma_\gamma(a) \sum_{\theta \in \Theta} \hat{u}(b(\theta); \theta) \hat{\pi}(\theta; a) + v(a)$ , where  $\hat{u}(b(\theta); \theta) = u(b(\theta); \theta) / \gamma(\theta)$ ,  $\hat{\pi}(\theta; a) = \gamma(\theta) \pi(\theta; a) / \Gamma_\gamma(a)$ , and  $\Gamma_\gamma(a) = \sum_{\theta \in \Theta} \gamma(\theta) \pi(\theta; a)$ .

For every given  $a$  and *any* observation,  $x$ , the representation of the posterior preferences of a Bayesian decision maker is obtained by the application of Bayes' rule to the probability measure that figures in the prior representation. Hence the representations of the posterior preferences are:  $\sum_{\theta \in \Theta} u(b(\theta); \theta) \pi(\theta | x, a) + v(a)$  or  $\Gamma_\gamma(a) \sum_{\theta \in \Theta} \hat{u}(b(\theta); \theta) \hat{\pi}(\theta | x, a) + v(a)$ , where  $\pi(\theta | x, a)$  is given in (12) and  $\hat{\pi}(\theta | x, a)$  is obtained from  $\hat{\pi}(\theta; a)$  by Bayes' formula. But, by definition,

$$\Gamma_\gamma(a) \sum_{\theta \in \Theta} \hat{u}(b(\theta); \theta) \hat{\pi}(\theta | x, a) + v(a) = \frac{\Gamma_\gamma(a)}{\Gamma_\gamma(a, x)} \sum_{\theta \in \Theta} u(b(\theta); \theta) \pi(\theta | x, a) + v(a).$$

Since  $\Gamma_\gamma(a, x) > 0$ ,  $\sum_{\theta \in \Theta} u(b(\theta); \theta) \pi(\theta | x, a) / \Gamma_\gamma(a, x)$  is a positive increasing transformation of  $\sum_{\theta \in \Theta} u(b(\theta); \theta) \pi(\theta | x, a)$ . Hence, as in the case of Savage's theory, for a given  $a$ , the induced posterior preferences are necessarily the same, regardless of representation of the prior preferences.

To obtain the uniqueness of the prior, it is necessary to consider the posterior preferences on bets *conditional on distinct actions*. Under the richness assumption the normalization associated with distinct actions makes for diverse multiplicative coefficients which imply that, for some observations,

$$\frac{\Gamma_\gamma(a)}{\Gamma_\gamma(a, x)} \sum_{\theta \in \Theta} u(b(\theta); \theta) \pi(\theta | x, a) + v(a) \geq \frac{\Gamma_\gamma(a')}{\Gamma_\gamma(a', x)} \sum_{\theta \in \Theta} u(b'(\theta); \theta) \pi(\theta | x, a') + v(a')$$

and

$$\sum_{\theta \in \Theta} u(b(\theta); \theta) \pi(\theta | x, a) + v(a) < \sum_{\theta \in \Theta} u(b'(\theta); \theta) \pi(\theta | x, a') + v(a').$$

At least one of these representations of the posterior preferences cannot be true. The same argument applies to the more general, nonadditive, representation (10).

## 4 Concluding Remarks

The main issue addressed by this work is the existence of identifiable prior representing the statistician's beliefs. As demonstrated in the introduction, subjective expected utility theories invoking Savage's analytical framework do not yield identifiable priors. In the theory presented here, the ability of a decision maker to influence the likely realization of effects by his choice of action, his manifested willingness to bet on the effects, and the changes in his betting behavior when new information becomes available yield sufficient additional data to identify the subjective probability representing the decision maker's beliefs.

The assumption that the observation may be obtained prior to taking an action and that likelihood functions are independent of  $a$  is adequate in many situations. One example is provided in the appendix. Another instance, of medical decision making, concerns a mutation in breast cancer gene BRCA1 or BRCA2. If a woman has this mutation she is at significantly increased risk of developing breast and ovarian cancers. Preventive (prophylactic) oophorectomy, or the surgical removal of the ovaries, can be performed to reduce both risks. A woman may undergo genetic testing to determine whether she has this genetic mutation and, as a result, is predisposed to developing breast cancer and ovarian cancer and use this information to decide whether or not to undergo prophylactic oophorectomy. In this case, the probability of the existence of the mutation conditional on developing breast or ovarian cancer is independent of the action. Yet, undergoing surgery affects the probability of developing breast or ovarian cancer. However, the aforementioned assumption is

not necessary for the main result. To see this, consider the case in which, for every  $\theta \in \Theta$  and  $a \in A$ , the likelihood function is  $q(\cdot | \theta, a)$ . Let  $\bar{a}$  denote the default action (that is, the status quo). Having observed  $x$ , the decision maker can calculate the posterior probabilities on  $\Theta$  as follows:  $\pi(\theta | x, a) = q(x | \theta, a) \pi(\theta | a) / \sum_{\theta' \in \Theta} q(x | \theta', a) \pi(\theta' | a)$  and the posterior preferences,  $\succsim^x$  of a Bayesian decision maker are represented by

$$(a, b) \rightarrow f_a \left( \sum_{\theta \in \Theta} u(b(\theta); \theta) \pi(\theta | x, a) \right). \quad (21)$$

The decision maker's must choose between the status quo and switching to some other action in  $A$ . In this case the functions  $f_a, a \in A - \{\bar{a}\}$ , have the interpretation of the extra disutility associated with switching from  $\bar{a}$  to  $a$ . However, the main result holds and the proof is the same.

## References

- [1] Anscombe, F., and R. Aumann. 1963. A definition of subjective probability, *Annals of Mathematical Statistics* 34, 199-205.
- [2] Drèze, J. H. 1987. Decision Theory with Moral Hazard and State-Dependent Preferences, in Drèze, J. H. *Essays on Economic Decisions Under Uncertainty*. (Cambridge University Press, Cambridge).
- [3] Grossman, S. J., and O. D. Hart. 1983. An Analysis of the Principal-Agent Problem, *Econometrica* 51, 7-45.
- [4] Hacking, I. 1975. *The Emergence of Probability*. (Cambridge University Press, Cambridge).
- [5] Karni, E. 1993. A Definition of Subjective Probabilities with State-Dependent Preferences, *Econometrica* 61, 187-198.
- [6] Karni, E. 1996. Probabilities and Beliefs. *Journal of Risk and Uncertainty* 13, 249-262.
- [7] Karni, E. 2003. On the Representation of Beliefs by Probabilities. *Journal of Risk and Uncertainty*, 26, 17-38.
- [8] Karni, E. 2006. Subjective Expected Utility Theory without States of the World. *Journal of Mathematical Economics* 42, 325 - 342.
- [9] Karni, E. 2007. A Foundations of Bayesian Theory. *Journal of Economic Theory*, 132, 167-188.

- [10] Karni, E. and D. Schmeidler. 1993. On the uniqueness of subjective probabilities. *Economic Theory* 3, 267-277.
- [11] Krantz, D. E., R. D. Luce, P. Suppes, and A. Tversky. 1971. *Foundations of Measurement*. (Academic Press, New York and London).
- [12] Nau, Robert F. 1995. Coherent Decision Analysis with Inseparable Probabilities and Utilities, *Journal of Risk and Uncertainty* 10, 71-91.
- [13] Ramsey, Frank P. 1931. Truth and Probability, in *The Foundations of Mathematics and Other Logical Essays*. London: K. Paul, Trench, Truber and Co.
- [14] Savage, L. J. 1954. *The Foundations of Statistics*. (John Wiley and Sons, New York).
- [15] Schervish, M., J. T. Seidenfeldt and J. B. Kadane. 1990. State-Dependent Utilities. *Journal of American Statistical Association* 85, 840-847.
- [16] Skiadas, C. 1997. Conditioning and Aggregation of Preferences. *Econometrica* 65, 347-367.
- [17] Wakker, P. P. 1987. Subjective Probabilities for State-Dependent Continuous Utility. *Mathematical Social Sciences* 14, 289-298.

## APPENDIX

### A MORAL HAZARD PROBLEM

The following moral hazard problem is formulated using the parametrized distribution approach. As usual, the principal-agent relationship is governed by a contract specifying the monetary payoff to the agent contingent on the outcome. However, if new pertinent information becomes available before the terms of the original contract are fulfilled, the contract is subject to renegotiation. Assume throughout that new information is a signal observed by both parties to the contract.

Let the agent be a risk-averse, Bayesian, expected utility-maximizing decision maker. Suppose that the principal is not only aware of this but that he also knows the agent's prior preferences. However, the principal *does not know* the agent's preferences conditional on every possible signal. Hence, if new information arrives which warrants renegotiating the terms of the original contract, the principal must predict the agent's preferences conditional on the new information. In view of the fact that the agent is Bayesian, the principal ascribes to the agent posterior preferences which are obtained from the prior preferences solely by updating the prior probabilities he ascribed to the agent, using Bayes rule.

Even if it is state-independent, the agent's prior preference relation admits infinitely many representations, involving distinct probabilities and state-dependent utility functions (see Karni [1996]). Among these, presumably, is a unique probability measure that is the true

representation of the agent's beliefs. I intend to show that, even if the principal knows the agent's prior preferences but ascribes to him the wrong probabilities (and consequently the wrong utilities), it is possible that once new information becomes available, the two parties to the original contract may agree to replace it with a contract that induces the agent to select an action that is not in the principal's best interest.

For the sake of concreteness I consider the following example. At the end of a regatta, monetary prizes will be awarded to the boats that finish first and second. Suppose that there are three entrants,  $x, y$ , and  $z$ , and the prizes satisfy  $m_1 > m_2 > m_3 = 0$ , where  $m_i$  denotes the prize awarded to the boat in place  $i$ . According to the rules, if one of the boats withdraws from the race, it is automatically assigned third place; if two boats withdraw, the race is cancelled and no prizes are awarded. The relevant set of outcomes,  $\Theta$ , consists of the six permutations of the order in which the boats cross the finish line.<sup>14</sup>

The boat owners recruit skippers and put them in charge of training their crews. Training entails costly effort on the part of skippers and crews and cannot be monitor by the owners. To motivate the sailors, an owner (principal) offers his skipper (agent) an incentive contract.

In what follows I describe the initial incentive contract and examine the implications of renegotiating the contract of one of the remaining competitors - say, the owner of  $x$  - if one of the other entrants announces its withdrawal from the race. To simplify the analysis,

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<sup>14</sup>The outcomes of the race correspond to effects. As this example illustrates, the distinguishing characteristic of effects is that, unlike states, their likely realization may be influenced by the decision maker.

I assume that the set of actions is a doubleton, with  $a'$  denoting training vigorously and  $a$  training lightly. Let  $c : \{a, a'\} \rightarrow \mathbb{R}$  be a function depicting the disutility to the agent associated with the actions, and assume that  $c(a') > c(a) = 0$ . Let  $E_i$  denote the event that  $x$  finishes in position  $i$  (for example,  $E_1 = \{(x, y, z), (x, z, y)\}$ ).

Recall that the skipper is assumed to be a risk-averse, Bayesian, expected utility - maximizer, and that *the boat owner knows the skipper's prior preferences, and that the skipper is a Bayesian*. Suppose further that by ascribing to the skipper his own probabilities regarding the likely outcome, conditional on the level of training and outcome independent utility function, the owner in fact obtains a correct representation of the skipper's prior preferences. Under these assumptions, if the principal wants to induce the agent to train vigorously, his problem may be stated as follows:

PROGRAM 1: Choose a contract  $\mathbf{w} := (w(E_1), w(E_2), w(E_3)) \in \mathbb{R}^3$  so as to maximize

$$\sum_{i=1}^3 u(m_i - w(E_i)) \pi(E_i; a') \quad (22)$$

subject to the individual rationality constraint

$$\sum_{i=1}^3 v(w(E_i)) \pi(E_i; a') - c(a') \geq \bar{v} \quad (23)$$

and the incentive compatibility constraint

$$\sum_{i=1}^3 v(w(E_i)) \pi(E_i; a') - c(a') \geq \sum_{i=1}^3 v(w(E_i)) \pi(E_i; a) - c(a), \quad (24)$$

where  $u$  and  $\{\pi(E_i, j) \mid i \in \{1, 2, 3\}, j \in \{a, a'\}\}$  denote the utility function and the conditional (on the actions) subjective probabilities of the principal, and  $v$  and  $c$  denote,

respectively, the utility of money and the disutility associated with the actions ascribed by the principal to the agent. In this case,  $\pi(\cdot, \cdot)$  also denotes the conditional probabilities ascribed by the principal to the agent. Thus, the common prior assumption, often invoked in the parametrized distribution formulation of principal-agent problems, is maintained.

By assumption, *the principal employs a representation of the agent's prior preferences consistent with the agent's choice behavior.* This depiction of the principal's problem agrees with the parametrized distribution formulation in the literature on agency theory.

#### REPRESENTATION OF THE AGENT'S PREFERENCES

It is conceivable that, independently of his monetary reward, the agent cares about winning but that the principal is unaware of this.<sup>15</sup> Moreover, the principal cannot detect the agent's sentiments about winning from his choice behavior. Suppose that the agent's true utility function is given by  $V(w, i, j) = \Gamma(j) v(w) / \gamma(i)$ , where  $0 < \gamma(1) < \gamma(2) < \gamma(3)$ , and  $\Gamma(j) = \sum_{i=1}^3 \gamma(i) \pi(E_i; j)$ ,  $j \in \{a, a'\}$ . Note that the presence of a nonadditive component,  $\Gamma(j)$ , in the agent's utility of action is also undetectable by observing his choice behavior.<sup>16</sup> Consistency of the agent's prior preferences with their representation in program 1 requires that his beliefs be represented by the probabilities

$$\pi^A(E_i; j) = \frac{\gamma(i) \pi(E_i; j)}{\Gamma(j)}, \quad j \in \{a, a'\}, \quad i = 1, 2, 3. \quad (25)$$

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<sup>15</sup>This is, of course, a simplifying assumption. In general, it is sufficient that the principal not know *how much* the agent cares about winning.

<sup>16</sup>See Grossman and Hart (1983) for an example of principal-agent analysis in which the utility of action has a multiplicative factor, as above.

Hence the agent's true objective function is given by:

$$\Gamma(j) \sum_{i=1}^3 \gamma^{-1}(i) v(w(E_i)) \pi^A(E_i; j) - c(j), \quad j \in \{a, a'\}. \quad (26)$$

Program 1 may then be restated as follows:

PROGRAM 1': Choose a contract,  $\mathbf{w} \in \mathbb{R}^3$ , so as to maximize

$$\sum_{i=1}^3 u(m_i - w(E_i)) \pi(E_i; a') \quad (27)$$

subject to the individual rationality constraint

$$\sum_{i=1}^3 V(w(E_i), i, a') \pi^A(E_i; a') - c(a') \geq \bar{v} \quad (28)$$

and the incentive compatibility constraint

$$\sum_{i=1}^3 V(w(E_i), i, a') \pi^A(E_i; a') - c(a') \geq \sum_{i=1}^3 V(w(E_i), i, a) \pi^A(E_i; a). \quad (29)$$

Clearly, the solution,  $\mathbf{w}^*$ , to the two programs is the same. Thus insofar as the principal is concerned, as long as the contract  $\mathbf{w}^*$  is in effect, the misspecification of the agent's utilities and probabilities is of no consequence. More generally, this discussion shows that, from a substantive point of view, when possible, ascribing to the agent the principal's own beliefs is immaterial, provided that the utility functions are adjusted so that the agent's preferences are accurately represented.

#### NEW INFORMATION AND RECONTRACTING

Suppose that after having signed a contract, the owner and skipper of boat  $x$  learn that one of the entrants – say,  $z$  – decides to withdraw from the race. This decision eliminates

the possibility that one of the two remaining boats finishes third. Equivalently, the event consisting of all the outcomes in which  $z$  finishes first or second becomes null. Let  $E_3(z)$  denote the event consisting of outcomes in which  $z$  finishes third. Since both the principal and the agent are Bayesian, the principal's posterior probabilities are  $\pi(E_i; j | E_3(z)) = \pi(E_3(z); j | E_i) \pi(E_i; j) / \pi(E_3(z); j)$ , and  $\pi(E_3; j | E_3(z)) = 0$ ,  $j \in \{a, a'\}$ . Moreover, being aware that the agent is Bayesian, the principal updates the agent's probabilities in the same way. However – and here's the rub – the agent's *true* posterior probabilities are  $\pi^A(E_3; j | E_3(z)) = \pi^A(E_3(z); j | E_i) \pi^A(E_i; j) / \pi^A(E_3(z); j)$ , and  $\pi^A(E_3; j | E_3(z)) = 0$ ,  $j \in \{a, a'\}$ . To simplify the notation, henceforth I denote the posteriors  $\pi(E_i; j | E_3(z))$  and  $\pi^A(E_i; j | E_3(z))$  by  $\hat{\pi}(E_i, j)$  and  $\hat{\pi}^A(E_i, j)$ , respectively.

Consider next the issue of recontracting. The withdrawal of  $z$  from the race raises two questions. First, are there perceived benefits to recontracting? Second, if the original contract is replaced, is it necessarily true that the replacement contract induces the agent to implement the action desired by the principal?

In principle, the original and the replacement contracts may be negotiated simultaneously or sequentially. In other words, given that the announcement that one of the competitors has pulled out of the race is made public before the beginning of training, the contract governing the principal-agent relationship in this event may be negotiated before or after the announcement. However, except for the cost of contracting, there is no difference between the simultaneous and sequential negotiations. If the contracts are negotiated simultaneously, the reservation utility of the agent must be the same under both contracts. If the contracts

are negotiated sequentially, the first contract can be written to include a clause that makes it null and void if one of the competitors withdraws from the race. The second contract is negotiated under the same individual rationality and incentive compatibility constraints and is, therefore, no different from the contract covering the same contingency that was negotiated simultaneously. In the presence of contracting costs, the sequential negotiation is preferable on two counts. First, if none of the competitors pulls out of the race, there is no need for a second contract, which saves the contracting cost. Second, if one of the competitors does pull out, it is better to incur the cost of recontracting later than earlier.

In what follows, I pursue the scenario of sequential contracting and analyze the case in which the principal wants the agent to implement the rigorous training program. From the principal's point of view, the subsequent contract,  $\mathbf{w}^{**} = (w^{**}(E_1), w^{**}(E_2))$ , is the solution to the following program:

PROGRAM 2: Choose  $\mathbf{w} \in \mathbb{R}^2$  so as to maximize

$$\sum_{i=1}^2 u(m_i - w(E_i)) \hat{\pi}(E_i; a') \quad (30)$$

subject to the individual rationality constraint

$$\sum_{i=1}^2 v(w(E_i)) \hat{\pi}(E_i; a') - c(a') \geq \bar{v} \quad (31)$$

and the incentive compatibility constraint

$$\sum_{i=1}^2 v(w(E_i)) \hat{\pi}(E_i; a') - c(a') \geq \sum_{i=1}^2 v(w(E_i)) \hat{\pi}(E_i; a) - c(a). \quad (32)$$

Recalling that  $c(a) = 0$ , it is easily verifiable that the solution,  $\mathbf{w}^{**}$ , to program 2, is given by the solution to the following equations:

$$v(w(E_1)) = \bar{v} + \frac{[1 - \hat{\pi}(E_1; a)] c(a')}{\hat{\pi}(E_1; a') - \hat{\pi}(E_1; a)} \text{ and } v(w(E_2)) = \bar{v} - \frac{\hat{\pi}(E_1; a) c(a')}{\hat{\pi}(E_1; a') - \hat{\pi}(E_1; a)}. \quad (33)$$

Consider next the problem as viewed by the agent. The agent should agree to  $\mathbf{w}^{**}$  if, under the corresponding best action, it meets his reservation utility. Formally,  $\mathbf{w}^{**}$  is acceptable to the agent if

$$\max_{j \in \{a, a'\}} \sum_{i=1}^2 V(w^{**}(E_i), i, j) \hat{\pi}^A(E_i; j) - c(j) \geq \bar{v}. \quad (34)$$

The second question - namely, whether it is necessarily true that  $\mathbf{w}^{**}$  induces the agent to implement the action desired by the principal - may be restated as follows: Does the incentive compatibility constraint

$$\sum_{i=1}^2 V(w^{**}(E_i), i, a') \hat{\pi}^A(E_i; a') - c(a') \geq \sum_{i=1}^2 V(w^{**}(E_i), i, a) \hat{\pi}^A(E_i; a) \quad (35)$$

hold?

I am interested in showing that it is possible that inequality (35) fails to hold. In this instance, if  $z$  pulls out of the race, the original contract becomes null and void and the two parties sign a new contract,  $\mathbf{w}^{**}$ . However, contrary to the expectations and the interests of the principal, under the new contract the agent implements the light training program. In other words, *because he failed to ascribe to the agent the correct prior probabilities and utilities, the principal misrepresents the agent's posterior preference relation and, as a result,*

*fails to provide the agent with the incentives that would have induced him to act in the principal's best interest.*

The numerical example below demonstrate this phenomenon. In this example the agent believes that his choice of the more costly action shifts relatively large probability mass from the worst outcome, to the best outcome, but little from the intermediate outcome to the best outcome. Hence the agent readily responds to incentives when  $E_3$  is feasible. However, once this possibility has been eliminated, the agent's responsiveness diminishes. Because he ascribed to the agent the wrong probabilities, the principal fails to appreciate this lack of responsiveness and offers the agent the "wrong" replacement contract.

#### A NUMERICAL EXAMPLE

In order to simplify the calculations, assume that the principal is risk neutral. Let the agent's utility function be  $V(w, i, j) = \Gamma(j) \sqrt{w}/\gamma(i)$ ,  $i \in \{1, 2, 3\}$ ,  $j \in \{a, a'\}$ . Suppose that the principal perceives the problem facing him as program 1 above, with the prior probabilities as follows:

$$\begin{array}{rcccccc}
 \text{Outcomes} & \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \begin{pmatrix} x \\ z \\ y \end{pmatrix} & \begin{pmatrix} y \\ x \\ z \end{pmatrix} & \begin{pmatrix} z \\ x \\ y \end{pmatrix} & \begin{pmatrix} y \\ z \\ x \end{pmatrix} & \begin{pmatrix} z \\ y \\ x \end{pmatrix} \\
 \pi(\cdot; a) & \frac{1.5}{12} & \frac{1.5}{12} & \frac{2}{12} & \frac{2}{12} & \frac{2.5}{12} & \frac{2.5}{12} \\
 \pi(\cdot; a') & \frac{4}{12} & \frac{4}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12}
 \end{array} \tag{36}$$

where, in describing the outcomes, the boats are ranked, from top to bottom, according to their positions. Recall that the principal is the owner of boat  $x$ . Hence the principal's prior probabilities of the relevant events are:

$$\begin{aligned}\pi(E_1; a) &= \frac{3}{12} & \pi(E_2; a) &= \frac{4}{12} & \pi(E_3; a) &= \frac{5}{12} \\ \pi(E_1; a') &= \frac{8}{12} & \pi(E_2; a') &= \frac{2}{12} & \pi(E_3; a') &= \frac{2}{12}\end{aligned}\tag{37}$$

Maintain the assumption that  $c(a) = 0$ , and suppose that the agent's disutility associated with the rigorous training is  $c(a') = 0.1$  and his prior reservation utility level is  $\bar{v} = 11.8333$ . Consider the contract

$$\mathbf{w}^* = (w^*(E_1) = 144.26, w^*(E_2) = 137.82, w^*(E_3) = 139.65).\tag{38}$$

Insofar as the principal is concerned, this contract satisfies the agent's perceived individual rationality constraint

$$\frac{8}{12}\sqrt{144.26} + \frac{2}{12}\sqrt{137.82} + \frac{2}{12}\sqrt{139.65} - 0.1 = 11.8333\tag{39}$$

and the incentive compatibility constraint<sup>17</sup>

$$11.8333 = \frac{3}{12}\sqrt{144.26} + \frac{4}{12}\sqrt{137.82} + \frac{5}{12}\sqrt{139.65}.\tag{40}$$

Moreover, the principal would rather the agent implement  $a'$  under the contract  $\mathbf{w}^*$  than  $a$

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<sup>17</sup>The fact that in this example the incentive compatibility constraint is binding does not affect the generality of the conclusion below.

under the fixed-pay contract  $\bar{\mathbf{w}} = 11.8333^2$ . To see this, note that the value of the principal's objective function under  $(\mathbf{w}^*, a')$  is

$$\frac{8}{12}(1000 - 144.26) + \frac{2}{12}(500 - 137.82) - \frac{2}{12}139.65 = 607.58, \quad (41)$$

far exceeding the value of his objective function under the fixed contract and  $a$ , which is

$$\frac{3}{12}1000 + \frac{4}{12}500 - 11.8333^2 = 276.65. \quad (42)$$

Next consider the situation as seen by the agent. Suppose that the agent's true prior beliefs are represented by the action dependent probabilities as follows:

$$\begin{array}{l} \text{Outcomes} \\ \pi^A(\cdot; a) \\ \pi^A(\cdot; a') \end{array} \begin{array}{cccccc} \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \begin{pmatrix} x \\ z \\ y \end{pmatrix} & \begin{pmatrix} y \\ x \\ z \end{pmatrix} & \begin{pmatrix} z \\ x \\ y \end{pmatrix} & \begin{pmatrix} y \\ z \\ x \end{pmatrix} & \begin{pmatrix} z \\ y \\ x \end{pmatrix} \\ \frac{32,847}{1,000,000} & \frac{32,847}{1,000,000} & \frac{175,180}{1,000,000} & \frac{175,180}{1,000,000} & \frac{291,970}{1,000,000} & \frac{291,970}{1,000,000} \\ \frac{15}{100} & \frac{15}{100} & \frac{15}{100} & \frac{15}{100} & \frac{20}{100} & \frac{20}{100} \end{array} \quad (43)$$

This means that there are positive numbers  $\gamma(i)$ ,  $i = 1, 2, 3$ , satisfying  $\gamma(1) < \gamma(2) < \gamma(3)$ , and  $\Gamma(j) = \sum_{i=1}^3 \gamma(i) \pi(E_i; j)$ ,  $j \in \{a, a'\}$ , such that  $\pi^A(E_i; j) = \gamma(i) \pi(E_i; j) / \Gamma(j)$ . The solution to  $\gamma(i)$ ,  $i = 1, 2, 3$ , and  $\Gamma(j)$ ,  $j \in \{a, a'\}$ , is  $\gamma(1) = 6.0336$ ,  $\gamma(2) = 24.134$ ,  $\gamma(3) = 32.179$ ,  $\Gamma(a') = 13.408$ , and  $\Gamma(a) = 22.961$ .<sup>18</sup>

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<sup>18</sup>The calculation of  $\{\gamma(i) \mid i = 1, 2, 3\}$  and  $\Gamma(j)$ ,  $j \in (a, a')$  is based on the following equations:

Hence his probabilities of the relevant events are<sup>19</sup>

$$\begin{aligned}\pi^A(E_1; a) &= 0.065694 & \pi^A(E_2; a) &= 0.35036 & \pi^A(E_3; a) &= 0.58394 \\ \pi^A(E_1; a') &= 0.3 & \pi^A(E_2; a') &= 0.3 & \pi^A(E_3; a') &= 0.4\end{aligned}\tag{44}$$

Because  $\mathbf{w}^*$  in (38) is the solution of program 1, it satisfies the individual rationality and incentive compatibility constraints. Thus the agent accepts the contract  $\mathbf{w}^*$  and, under its terms, implements the rigorous training program, as expected by the principal.

$$\begin{aligned}\frac{\frac{8}{12}\gamma(1)}{\frac{8}{12}\gamma(1)+\frac{2}{12}\gamma(2)+\frac{2}{12}\gamma(3)} &= 0.3 \\ \frac{\frac{2}{12}\gamma(2)}{\frac{8}{12}\gamma(1)+\frac{2}{12}\gamma(2)+\frac{2}{12}\gamma(3)} &= 0.3 \\ \frac{\frac{2}{12}\gamma(3)}{\frac{8}{12}\gamma(1)+\frac{2}{12}\gamma(2)+\frac{2}{12}\gamma(3)} &= 0.4\end{aligned}$$

the solution to which is

$$\gamma(1) = 6.0336, \gamma(2) = 24.134, \gamma(3) = 32.179$$

Thus

$$\Gamma(a) = \frac{3}{12}6.0336 + \frac{4}{12}24.134 + \frac{5}{12}32.179 = 22.961$$

and

$$\Gamma(a') = \frac{8}{12}6.0336 + \frac{2}{12}24.134 + \frac{2}{12}32.179 = 13.408.$$

<sup>19</sup>The conditional probabilities  $\{\pi^A(E_i; a) \mid i = 1, 2, 3\}$  are the solution to the equations

$$\begin{aligned}\frac{\frac{3}{12}6.0336}{\frac{3}{12}6.0336+\frac{4}{12}24.134+\frac{5}{12}32.179} &= \pi^A(E_1; a) \\ \frac{\frac{4}{12}24.134}{\frac{3}{12}6.0336+\frac{4}{12}24.134+\frac{5}{12}32.179} &= \pi^A(E_2; a) \\ \frac{\frac{5}{12}32.179}{\frac{3}{12}6.0336+\frac{4}{12}24.134+\frac{5}{12}32.179} &= \pi^A(E_3; a)\end{aligned}$$

Consider next the effect of  $z$  pulling out of the race.<sup>20</sup> The principal's posterior probabilities, which are also the posterior probabilities he ascribes to the agent, are obtained by the application of Bayes' rule. They are given by:

$$\begin{aligned}\hat{\pi}(E_1; a) &= \frac{15}{35} & \hat{\pi}(E_2; a) &= \frac{20}{35} \\ \hat{\pi}(E_1; a') &= \frac{4}{5} & \hat{\pi}(E_2; a') &= \frac{1}{5}\end{aligned}\tag{45}$$

The principal considers offering the agent the contract,  $\mathbf{w}^{**} = (w^{**}(E_1), w^{**}(E_2))$ , that would motivate him to implement  $a'$ . This contract is obtained by the solution to the equations (33) and is given by  $w^{**}(E_1) = 143.69$ , and  $w^{**}(E_2) = 137.31$ .<sup>21</sup>

The principal's payoff under  $(\mathbf{w}^{**}, a')$ , is<sup>22</sup>

$$\frac{4}{5}(1000 - 143.69) + \frac{1}{5}(500 - 137.31) = 757.6\tag{46}$$

His payoff under  $(\bar{\mathbf{w}}, a)$ , is

$$\frac{15}{35}1000 + \frac{20}{35}500 - 11.8333^2 = 574.26.\tag{47}$$

Clearly, the principal is better off with  $(\mathbf{w}^{**}, a')$  than with  $(\bar{\mathbf{w}}, a)$ . That is, the principal is better off offering to replace contract  $\mathbf{w}^*$  by contract  $\mathbf{w}^{**}$ , expecting the agent to implement the rigorous training program.

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<sup>20</sup>Recall that, by the rules of the regatta, pulling out means that  $z$  is assigned third place.

<sup>21</sup>Specifically,  $\mathbf{w}^{**}$  is given by the equations

$$\sqrt{w^{**}(E_1)} = 11.8333 + \frac{20 \times 0.1}{13}, \text{ and } \sqrt{w^{**}(E_2)} = 11.8333 - \frac{15 \times 0.1}{13}.$$

<sup>22</sup>The principal believes that if the agent accepts  $\mathbf{w}^{**}$ , it will induce him to select the action  $a'$ .

Next I show that the agent accepts  $\mathbf{w}^{**}$  and, contrary to the anticipations of the principal, chooses action  $a$ . Note that the agent's posterior beliefs are represented by

$$\begin{aligned}\hat{\pi}^A(E_1; a) &= \frac{32,847}{32,847+175,180} = 0.1579 & \hat{\pi}^A(E_2; a) &= \frac{175,180}{32,847+175,180} = 0.8421 \\ \hat{\pi}^A(E_1; a') &= 0.5 & \hat{\pi}^A(E_2; a') &= 0.5\end{aligned}\tag{48}$$

If the agent accepts  $\mathbf{w}^{**}$  and implements  $a$ , his posterior expected utility is

$$22.961 \left( \frac{0.1579\sqrt{143.69}}{6.0336} + \frac{0.8421\sqrt{137.31}}{24.134} \right) = 16.591,\tag{49}$$

which exceeds his reservation utility. Thus the agent accepts  $\mathbf{w}^{**}$ . The agent's payoff under  $(\mathbf{w}^{**}, a')$  is

$$13.408 \left( \frac{0.5\sqrt{143.69}}{6.0336} + \frac{0.5\sqrt{137.31}}{24.134} \right) - 0.1 = 16.474.\tag{50}$$

Comparing the agent's payoffs corresponding to  $\mathbf{w}^{**}$  under the two actions, it is clear that  $\mathbf{w}^{**}$  is not incentive compatible. Hence *against the principal's best interest (and wishes), the agent implements the light training program.*

The explanation for this result is that the boat owner erroneously ascribes to the skipper action-dependent probabilities that exaggerate the impact of implementing the rigorous training program  $a'$ . In other words, given that  $z$  dropped out of the race, the owner perceives that the skipper believes that implementing  $a'$  reduces the probability of ending in second

place from 57 percent to 20 percent, whereas, in fact, the skipper believes that by implementing  $a'$  he reduces the probability of loss from approximately 84 percent to 50 percent. Thus, the estimated effect of training according to the owner is more than twice that of the skipper. As a result, the boat owner underestimates the exposure to risk necessary to produce the incentives that would induce the skipper to implement  $a'$ . The owner designs a contract that, once accepted, reduces the skipper's exposure to risk to the point at which the cost of rigorous training is no longer justified (that is, the contract is not incentive compatible).

This example illustrates the incentive incompatibility of the contracts that may result from the principal's ascribing to the agent incorrect subjective probabilities. It is possible to produce example in which the replacement contract is incentive compatible and individually rational but "overcompensates" the agent by offering him more than the minimum acceptable payoff.<sup>23</sup>

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<sup>23</sup>If the contract is not individually rational, the principal readily detects that something is wrong as the agent rejects a contract that the principal expect him to accept.