No Externalities: A Characterization of Efficiency and Incentive Compatibility with Public Goods*

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Abstract

We show that efficient anonymous incentive compatible (dominant strategy) mechanisms for public goods eliminate externalities, i.e., each individual is unable to influence the welfare of anyone else. The characterization applies whether preferences are quasilinear or ordinal and public goods are costless or costly. The characterization is used to derive existence and non-existence results for models with a finite number of individuals and to explain existence results in the continuum. With public goods, elimination of externalities implies that individuals can have no effect at all. Hence, such mechanisms provide only weak incentives for truth-telling. Comparisons with the no externality characterization for private goods are also included.

Keywords: Public goods, private goods, no externalities, incentive compatibility

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1 Introduction

We characterize efficient anonymous incentive compatible mechanisms for public goods as the elimination of externalities — the inability of any individual to effect the welfare of anyone else. In one direction, this characterization is immediate: *an efficient mechanism that exhibits no externalities satisfies incentive compatibility*: Suppose an individual enjoys a certain utility level, say $\bar{u}$, from reporting his preferences truthfully. If the individual were to misrepresent his preferences, the no externalities condition says that others' utilities would not change. Therefore, if the misrepresentation led to a utility level greater than $\bar{u}$, i.e., was not incentive compatible, that would contradict efficiency. Hence, the goal is to show that the roles of no externalities and incentive compatibility in the above statement can be reversed.

The italicized statement applies whether goods are public or private. And, it is demonstrated in [9] that the no externalities characterization holds for private goods. Here, we show that the same principle applies to public goods, while also pointing out the different implications of no externalities in the two environments.[1]

The Vickrey-Clarke-Groves Characterization: Our focus on no externalities resembles the conclusions of [16], [2], and [5], called VCG mechanisms. Such schemes need not satisfy anonymity, but that requirement can be added without changing its essential features. More important is the fact that its conclusions are based on the hypothesis that preferences are quasilinear.

Quasilinearity allows the allocation of resources to be separated into "non-money" commodities and the "money" commodity. This, in turn, permits a *qualified* notion of efficiency with respect to non-money allocations that is well-defined even if the allocation of the money commodity does not satisfy budget balance, i.e., is not feasible or efficient with respect to the money commodity. The main result of VCG schemes is that to achieve qualified efficiency and incentive compatibility, individuals should internalize the externalities they create. With quasilinearity this implies that the money payment associated with the individual's non-money allocation should, up to a constant, equal its social cost as measured by its consequences on the welfare of others. In principle, this is the same as no externalities. But, in practice it can differ because the payment made by the individual causing the externalities is not

[1]The externality described here is a consequence of a change in the allocation of resources caused by a change in one person's preferences. It does not include those situations in which individuals can report information that changes either the feasible set of allocations or the utility of a given allocation, e.g., as in [7].

necessarily received by those on whom the externality is imposed. An implication of Theorem 2, below, is that an anonymous VCG scheme satisfies our definition of no externalities if and only if it satisfies budget balance.

Valid Characterization without Quasilinearity: When attention shifts to models with ordinal preferences, an appealing notion of qualified efficiency is no longer available. Then, the existence of fully efficient incentive compatible mechanisms is problematic. And without existence, characterization is suspect.

The point of departure in our approach is a resolution of this dilemma: to obtain a valid characterization of efficiency and incentive compatibility in environments where existence cannot be taken for granted. By insisting on a characterization of efficiency and incentive compatibility with budget balance in quasilinear models — where existence is questionable, we are placing it on a similar footing to ordinal models — where qualified notions of efficiency with incentive compatibility are not available. This repositioning allows us to demonstrate that the no externalities characterization is equally applicable to both.

The following desiderata on the domain of the mechanism are invoked to provide a valid characterization. Calling a domain $\mathcal{D}$,

(1) there should be examples of $\mathcal{D}$ on which the characterization is non-empty,

(2) the characterization that emerges from $\mathcal{D}$ should be unique,

Condition (1) ensures that the characterization is not vacuous. Condition (2) precludes non-representative characterizations arising from idiosyncratic domains. In addition, the characterization should help to understand (non-)existence.

(3) Where existence is problematic, the characterization should show how and why $\mathcal{D}$ can be constructed to demonstrate impossibility.

(4) Conversely, the characterization should point to those environments where efficiency and incentive compatibility co-exist.

The hypothesis, below, that $\mathcal{D}$ is *comprehensive around some population* $\mathbf{u}$ yields the no externalities characterization that satisfies (1)–(4).

The Relation between Public and Private Goods: With private goods, the no externalities characterization in [9] has a canonical interpretation. Efficient anonymous incentive compatible mechanisms for an exchange environment exist if and only if a change in any individual's preferences leads to (i) no change in prices and to (ii) no change in the wealth of any individual. In other words, the elimination of externalities is achieved if and only if each individual is a perfect competitor. Observe that "no externalities" does not simply refer to a property of utility functions — that each person's utility depends on his own consumption. Rather, the definition is based on what the mechanism allows the individual to do, or not to do, to others' utilities. Similarly, "perfect competitor" does not refer to the individual as a price-taker, but to the fact that the mechanism does not permit the individual to create, as it were, any pecuniary externalities.

The distinguishing feature of a public good is that it applies to everyone. In a broad sense, however, private goods may interpreted as "public" by regarding the entire allocation as available to all. The difference, of course, is that by private goods we mean that each individual's utility is only sensitive to the part of the overall allocation specifically assigned to the person. Thus, public goods can be seen as an extension of private in which one person's utility also depends, in an unrestricted way, on others' shares of the overall allocation. Not surprisingly, the no externalities condition is more demanding when there are public goods. It implies that the following trivial sufficient condition is also necessary: *To ensure that individuals create no externalities, each must be unable to have any effect at all.* Therefore, elimination of externalities has different consequences for public compared to private goods. The information an individual reveals about his preferences will change his allocation of private goods. This provides a strong incentive to report truthfully. In contrast, the no-externalities-implies-no-change result for public goods evidently implies weak incentives to reveal.

The differences between no externalities for private versus public goods is further confirmation of the well-known reversal of roles between prices and quantities that characterizes the relation between private and public goods [13]. For private goods, quantities are personalized while prices are common to all individuals, whereas the situation is just the opposite with public goods. With private goods, prices are the "public goods" and the no externalities characterization implies that no one should be able to change them. Similarly, the no externalities characterization with public goods is that no one should be able to change what is common to all, i.e., quantities.

From Characterization to (Non-)Existence: There is a tension between efficiency and no externalities, hence between efficiency and incentive compatibility. While domains $\mathcal{D}$ will be found on which these two conditions can co-exist, the care that must be taken in their construction immediately implies that they are exceptional, i.e., non-generic. Impossibility of efficient incentive compatible mechanisms have been demonstrated in, for example, [17]. An advantage of the characterization of possibility is that it provides an explanation and simple demonstration of impossibility.

The tension between efficiency and no externalities can be reduced as the population grows if the ability of any one individual to influence the efficient public good decision decreases as the number of individuals increases. To illustrate, if a mechanism chooses an efficient allocation while requiring each person to pay his per capita share of the cost, the no-externalities-as-no-change condition will be attained in the limiting continuum economy provided that the efficient allocation of public goods various continuously with the distribution of agents' characteristics.[2] Existence results for the continuum in [6], [4], and [8], along with asymptotic results for large economies in [15], [3], [12], and [10] can therefore be seen as following from the characterization for the finite model.

Public Goods and Social Choice: If the choice set is one dimensional and preferences are single-peaked, Moulin [11] showed that selecting the most preferred alternative of the median voter is anonymous, efficient and incentive compatible. If preferences are convex, they are single-peaked along any one dimension. Nevertheless, Zhou [18] showed that when the dimension of the set of alternatives is greater than one, restricting to continuous and convex preferences implies that the only incentive compatible mechanisms are dictatorial. Barbera and Jackson [1] characterize incentive compatible mechanisms that are not necessarily efficient or anonymous.

The special case in which the set of public goods alternatives may be one-dimensional is not highlighted in the conclusions, below. We explain why. With quasilinearity, the median voter's most preferred alternative need not be efficient and, in addition, utility functions for the public good are not assumed to be concave, so preferences are not single-peaked. In the ordinal model, preferences are assumed to be convex, but by construction choices cannot be one dimensional because they consist of combinations of public and private goods. In a setting close to the one considered here, Serizawa [14] characterizes anonymous incentive compatible mechanisms with one private good that individuals supply to produce one public good. When the cost function is convex,

[2] This continuity condition is generic in the model studied here.

he shows that the hypotheses of the model restrict the range of the mechanism to induce single-peaked preferences. But single-peaked preferences are not consistent with efficiency, even in the case of one public good and one private good.

A statement of the problem that applies to the quasilinear and ordinal versions of the model is given in Section 2. Results for the quasilinear case are presented in Section 3 without assuming convexity, either of preferences or costs. Conclusions for costly public goods are derived as extensions of the costless version. Section 4 gives the results for the ordinal model with convex preferences when there are no costs. The same method of extending the conclusions to the costly version applies *provided* that costs are convex and preferences can be represented by concave functions. Section 5 is devoted to the implications of these findings for pricing and for the comparison with private goods. Proofs are in the Appendix.

2 Preliminaries

2.1 Definitions

Society is composed of n consumers. Person $i = 1, \dots, n$ has utility $u_i(a, b)$ from a vector of public goods $a \in A$, where A is a compact convex subset of $\Re^k$, and from a private good $b \in \Re$. It is assumed throughout that u_i is continuous, increasing in b, but not necessarily differentiable or increasing in a. The cost of producing public goods is measured in units of the private good and is given by the function $\varphi : A \to \Re_+$. Society is endowed with sufficient units of the private good to produce any quantity of the public good. In the special case of costless public goods $\varphi \equiv 0$.

Denote by $\mathcal{F}$ the set of allocations $(a, b_1, \dots, b_n)$ of the public and private goods that are feasible and by $\mathcal{U}$ the set of all utility functions for any individual i. (Further details on $\mathcal{F}$ and $\mathcal{U}$ will be given, below.)

A mechanism for a subset $\mathcal{D} \subseteq \mathcal{U}^n$ is a function $f : \mathcal{D} \to \mathcal{F}$. Its interpretation is the following. Given the vector of utility functions $\mathbf{u} = (u_1, \dots, u_n) \in \mathcal{D}$, the mechanism suggests producing the vector $a = f_0(\mathbf{u}) \in A$ of public goods, and to give individual i, $i = 1, \dots, n$, $b_i = f_i(\mathbf{u})$ units of the private good. For $\mathbf{u} = (u_1, \dots, u_n)$ and $u' \in \mathcal{U}$, let $(\mathbf{u}_{-i}, u'_i) = (u_1, \dots, u'_i, \dots, u_n)$.

The main features of the mechanism are:

Efficiency For every $\mathbf{u} \in \mathcal{D}$, there is no allocation $(a, b_1, \dots, b_n) \in \mathcal{F}$ such that for all i, $u_i(a, b_i) \geqslant u_i(f_0(\mathbf{u}), f_i(\mathbf{u}))$; and for some j, $u_j(a, b_j) > u_j(f_0(\mathbf{u}), f_j(\mathbf{u}))$.

Incentive Compatibility For all i, $\mathbf{u} \in \mathcal{D}$, u'_i such that $(\mathbf{u}_{-i}, u'_i) \in \mathcal{D}$,

$$u_i(f_0(\mathbf{u}), f_i(\mathbf{u})) \geqslant u_i(f_0(\mathbf{u}_{-i}, u'_i), f_i(\mathbf{u}_{-i}, u'_i))$$

We shall assume that individuals are only identifiable by their characteristics, not their 'names.' Hence, two individuals with the same utilities are indistinguishable and will therefore receive the same allocation of the private, as well as the public, good.

Anonymity If $u_i = u_j$, then $f_i(\mathbf{u}) = f_j(\mathbf{u})$.

Restrictions to be derived from the above properties (Theorems 2 and 5, below) of a mechanism are:

No Externalities For all i, $\mathbf{u} \in \mathcal{D}$, u'_i such that $(\mathbf{u}_{-i}, u'_i) \in \mathcal{D}$, and $j \neq i$,

$$u_j(f_0(\mathbf{u}_{-i}, u'_i), f_j(\mathbf{u}_{-i}, u'_i)) = u_j(f_0(\mathbf{u}), f_j(\mathbf{u}))$$

No Change For all i, $\mathbf{u} \in \mathcal{D}$, u'_i such that $(\mathbf{u}_{-i}, u'_i) \in \mathcal{D}$,

$$f_0(\mathbf{u}_{-i}, u'_i) = f_0(\mathbf{u}) \text{ and for all } j \text{ (including } i), f_j(\mathbf{u}_{-i}, u'_i) = f_j(\mathbf{u})$$

The no externalities condition says that the welfare of any individual $j \neq i$ is invariant to the reported utility of i. The no change condition says that the outcome of the mechanism is itself invariant to the reported utility of any individual. Below, it will be shown that for efficient and anonymous mechanisms, the following are equivalent: incentive compatibility, no externalities, and no change.

2.2 Ordinal and quasilinear models

The set of feasible allocations $\mathcal{F}_k$ will depend on the model $k \in \{\text{ORD}, \text{QL}\}$ where

$$\mathcal{F}_k = \{(a, b_1, \ldots, b_n) : a \in A, b_i \in B_k, \sum b_i + \varphi(a) = \overline{b}_k\}.$$

Note that A is the same for both models. The choice of a implies that $\varphi(a)$ must be subtracted from $\overline{b}_k$, society's endowment of the private good, which determines the total quantity of the private good available for individual consumption, $\sum b_i$. For the ordinal model, $B_{\text{ORD}} = \Re_+$ and $\overline{b}_{\text{ORD}} > \max\{\varphi(a) : a \in A\}$, i.e., it is feasible to produce any quantity of the public good. For the quasilinear model, $B_{\text{QL}} = \Re$ and $\overline{b}_{\text{QL}} = 0$. Thus, $A \times \Re_+$ is the domain for ordinal utility and $A \times \Re$ the domain for quasilinear utility.[3]

For either class of utility defined on its respective domain, the combinations of public and private goods yielding at least as much utility as (a, b) to an individual with tastes u is

$$R(a, b; u) = \{(a', b') : u(a', b') \geqslant u(a, b)\}.$$

In addition to the fact that (i) $B_{\text{QL}} = \Re$, the quasilinear model is distinguished by

(ii) $(a', b') \in R(a, b; u)$ if and only if $(a', b' + \alpha) \in R(a, b + \alpha; u), \forall \alpha \in \Re$.

It is well-known that a utility function representing such preferences can be written as $u(a, b) = v(a) + b$. Because the domain (i) is not bounded below, quasilinearity is not a special case of the ordinal model. Nevertheless, the ordinal model contains utilities satisfying (ii) on $A \times \Re_+$ and this fact will allow us to import certain conclusions derived from quasilinearity.

It will be important, below, to define the following partial ordering. The utility function u' is a *sharpening* of u at (a, b) if

$$(S) \qquad (a', b') \neq (a, b),\ u(a, b) = u(a', b') \text{ implies } (a', b') \notin R(a, b; u').$$

If u' is a sharpening of u at (a, b), then except for the common point (a, b), $R(a, b; u')$ lies entirely inside $R(a, b; u)$. With quasilinearity, $u' = v' + b$ is a sharpening of $u = v + b$ at (a, b) if $v(a') + b' = v(a) + b$ implies $v'(a') + b' < v'(a) + b$, or

$$(S_{ql}) \qquad v'(a') - v'(a) < v(a') - v(a).$$

Sharpenings include the possibility that u is not differentiable at (a, b), or v is not differentiable at a. (See Figure 1 in the Appendix.)

Preferences will be restricted to exhibit bounded rates of substitution between public goods and the private good. For $a, a' \in A$ and b, define $L(a, a', b)$ implicitly by

$$u(a, b) = u(a', b + L(a, a', b)) \tag{1}$$

as the quantity of the private good an individual with utility function u and quantity of the private good b would be willing to sacrifice (if $L < 0$) or require (if $L > 0$) to

[3] The absence of a lower bound on the private good in the quasilinear case is a technical simplification condition. It also implies that any $a \in A$ can be produced.

remain indifferent when society moves from a to a'. In the ordinal model, the quantity of the private good is bounded below by 0. If the individual is willing to forego all his private commodity to move to a' from a, the constraint $b \geqslant 0$ prevents us from keeping him at the same utility level. If $u(a', 0) > u(a, b)$, then define $L(a, a', b) = b$, i.e., L is defined to be the maximal amount that can be taken away from him. A Lipschitz condition is imposed on L:

$$|L(a, a', b)| < M\|a - a'\|. \tag{2}$$

For quasilinear utility, where $u(a, b) = v(a) + b$, there is no lower bound on the private good and the Lipschitz condition reduces to $|v(a) - v(a')| < M \| a - a' \|$. To bound the rate of change of the cost function, there is a similar restriction that

$$|\varphi(a) - \varphi(a')| < M \| a - a' \| .$$

Bounds on rates of substitution between public goods and the private good and on the rate of change of the cost function limit the extent to which one individual will be able to influence the outcome. In the ordinal model, below, the bound will also help to ensure the existence of prices. Note that the Lipschitz condition is consistent with the absence of differentiability in u and φ.

3 Quasilinear Utility

In this section utility functions are of the form $u(a, b) = v(a) + b$, where $b \in \Re$. Consequently, $u (= v + b)$ is identified by v and we shall denote the set $\mathcal{U}$ that are quasilinear and Lipschitz as $\mathcal{V}$. We shall not assume that v is concave; hence, $R(a, b; u)$ is not necessarily convex.

A key property of quasilinear utility is that efficiency implies that a must be chosen to maximize the sum of utilities,

$$f_0(\mathbf{u}) \in \arg\max_a \sum_{i=1}^{n} v_i(a) - \varphi(a) \tag{3}$$

where $b_i = f_i(\mathbf{u})$ and $\sum_{i=1}^{n} f_i(\mathbf{u}) = -\varphi(a)$. Throughout this Section f is assumed to be efficient and anonymous.

3.1 Domains

We describe restrictions on a domain that is sufficiently rich to characterize an anonymous efficient incentive compatible mechanism while also being non-vacuous. The latter condition will require that the domain be a subset of $\mathcal{V}^n$. Recalling condition (S_{ql}), denote by $\mathcal{V}^*(v,a)$ the set of functions $u'(a,b) = v'(a) + b$ where v' is a sharpening of v at a.

The following assumption says that if $\mathbf{u}$ belongs to the domain and the mechanism chooses $f_0(\mathbf{u})$, then the domain also includes all sharpenings of $\mathbf{u}$ at $f_0(\mathbf{u})$.

Definition 1 *$(f, \mathcal{D})$ is first order if $\mathbf{u} = (v_1 + b, \ldots, v_n + b) \in \mathcal{D}$ implies that*

$$\mathcal{V}^*(v_1, f_0(\mathbf{u})) \times \cdots \times \mathcal{V}^*(v_n, f_0(\mathbf{u})) \subset \mathcal{D}.$$

Note that if $\mathbf{u}' = (v_1' + b, \ldots, v_n' + b) \in \mathcal{V}^*(v_1, f_0(\mathbf{u})) \times \cdots \times \mathcal{V}^*(v_n, f_0(\mathbf{u}))$, then $f_0(\mathbf{u}) \in \arg\max_a \sum_{i=1}^n v_i'(a) - \varphi(a)$. The "first order" terminology is meant to convey the idea that individuals are always free to change their preferences from any $\mathbf{u} \in \mathcal{D}$ to any other preferences such that the "marginal conditions" for efficiency at $f_0(\mathbf{u})$ do not change.

The first order condition limits the allowable perturbations that individuals can make from $\mathbf{u}$, but the fact that they form a Cartesian product says that the first order perturbations for i are independent of those for $j \neq i$. The next condition includes the opposite kind of construction. It places no limits on the allowable perturbations any individual can make, but it permits only one-at-a-time departures.

Definition 2 *$(f, \mathcal{D})$ is comprehensive around $\overline{\mathbf{u}} = (\overline{v}_1 + b, \ldots, \overline{v}_n + b)$ if it is first order and*

$$\bigcup_i (\overline{\mathbf{u}}_{-i}, \mathcal{V}) \subset \mathcal{D}.$$

The fact that $(f, \mathcal{D})$ is comprehensive around $\overline{\mathbf{u}}$ is essential. It means that we can apply the full strength of the incentive compatibility condition at $\overline{\mathbf{u}}$.

The domain $\mathcal{D} = \mathcal{V}^n$ is comprehensive around *every* $\mathbf{u} \in \mathcal{D}$, but as demonstrated, below and elsewhere, such a domain precludes the existence of a desired mechanism. To obtain a valid characterization, we shall focus on the following:

Definition 3 *$(f, \mathcal{D})$ is minimally comprehensive around $\overline{\mathbf{u}} =$ if it is the minimal first order domain to contain*

$$\mathcal{D}(\overline{\mathbf{u}}, f_0(\overline{\mathbf{u}})) \quad = \quad \bigcup_i \Big\{ \ldots \times \mathcal{V}^*(\overline{v}_{i-1}, f_0(\overline{\mathbf{u}})) \times \mathcal{V} \times \mathcal{V}^*(\overline{v}_{i+1}, f_0(\overline{\mathbf{u}})) \times \ldots \Big\}$$

The minimally comprehensive set $(f, \mathcal{D})$ around $\overline{\mathbf{u}} = (\overline{v}_1 + b, \ldots, \overline{v}_n + b)$ is created in the following way. First, consider all economies of the form $\mathbf{u} = (v_1 + b, \ldots, v_n + b)$ where for some i, $v_i + b \in \mathcal{V}$ and for all $j \neq i$, v_j is a sharpening of $\overline{v}_j$ at $f_0(\overline{\mathbf{u}})$. For such an economy $\mathbf{u}$, if $f_0(\mathbf{u}) \neq f_0(\overline{\mathbf{u}})$, then all the economies $\mathbf{u}'$ where for each i, v_i' is a sharpening of v_i at $f_0(\mathbf{u})$ are also in $(f, \mathcal{D})$. By permitting sharpenings at any $\mathbf{u} \in \mathcal{D}$, a first order mechanism allows individuals to announce other utilities consistent with $f_0(\mathbf{u})$ being an efficient choice. In addition, the comprehensive condition around $\overline{\mathbf{u}}$ allows any one individual to deviate from $\overline{\mathbf{u}}$ by announcing any u_i' among those that are Lipschitz and monotonic in b. A minimally comprehensive $(f, \mathcal{D})$ is the smallest such domain.

3.2 Characterization

A key property of a mechanism satisfying our assumptions is that the cost of production must be shared equally. This strong conclusion is obtained even on a very restricted domain.

Theorem 1 *Let $(f, \mathcal{D})$ be first order. If f is incentive compatible on $\mathcal{D}$, then for every $\mathbf{u} \in \mathcal{D}$ and for every i,*

$$f_i(\mathbf{u}) = -\frac{\varphi(f_0(\mathbf{u}))}{n}.$$

We provide here an outline of the proof. (All proofs are in the Appendix). Efficiency requires $f_0(\mathbf{u}) \in \arg\max_a \sum_{i=1}^n v_i(a) - \varphi(a)$. To simplify, assume the outcome a^* achieving efficiency is unique. As observed above, sharpenings of $\mathbf{u}$ are consistent with the efficiency of a^*. Lemma 1 establishes that to preserve efficiency and incentive compatibility, sharpenings should not change a^* or the money commodity allocation of the person whose utility is sharpened. To show that $f_i(\mathbf{u}) = -\frac{\varphi(f_0(\mathbf{u}))}{n}$, we assume first that $\varphi \equiv 0$ and choose one individual, say 1, whose b commodity outcome is positive. We sharpen his utility at a^* to obtain a function that is sharpening of $v_i(a^*)$ for all i. By Lemma 1, his b outcome is still positive, so there is an individual, say 2, whose b outcome is negative. We replace his utility by the same sharpening. This will not change his outcome, which is still negative, and now the first two agents have the same utility, hence the same b outcome, which is now negative. After n such sharpenings we obtain the contradiction that all agents have the same utility with either all receiving a positive b (n is odd), or all receiving a negative b (n is even). This result is extended to $\varphi \not\equiv 0$ □

The following is the main result of this section:

Theorem 2 *Let $(f, \mathcal{D})$ be minimally comprehensive around $\overline{\mathbf{u}}$. Then f satisfies incentive compatibility if and only if it exhibits no change.*

To call Theorem 2 a characterization, the following must be demonstrated.

Theorem 3 *Let $(h, \mathcal{D})$ be minimally comprehensive around $\overline{\mathbf{u}}$. If h is anonymous efficient and incentive compatible on $\mathcal{D}$, then $h = f$.*

3.3 Possibility and Impossibility Results

To show that the characterization is not vacuous we exhibit some possibility results on a restricted domain $\mathcal{D}$.

Corollary 1 *Let $(f, \mathcal{D})$ be minimally comprehensive around $\overline{\mathbf{u}} = (\overline{v}_1 + b, \ldots, \overline{v}_n + b)$ and $f_0(\overline{\mathbf{u}}) = a^*$. If $\overline{\mathbf{u}}$ and a^* are such that for every a,*

$$\sum_{i=1}^{n} \frac{\left(v_i(a^*) - \frac{\varphi(a^*)}{n}\right) - \left(v_i(a) - \frac{\varphi(a)}{n}\right)}{\| a^* - a \|} > 4M \tag{4}$$

then there is an efficient anonymous incentive compatible mechanism f on the minimally comprehensive domain $\mathcal{D}(\mathbf{u}, f_0(\mathbf{u}))$, where $f_0(\mathbf{u}) = a^*$ and $f_i(\mathbf{u}) = -\frac{\varphi(a^*)}{n}$ for all $\mathbf{u} \in \mathcal{D}(\mathbf{u}, f_0(\mathbf{u}))$.

Inequality (4) establishes a sufficiently large "kink" at a^* among the individuals to nullify the ability of any one person to overturn its efficiency.

Remark 1 Note the following.

1. Corollary 1 requires $n \geqslant 3$. Otherwise, inequality (4) cannot be satisfied.
2. The condition in (4) does not imply that for every consumer, a^* is the optimal quantity.
3. Assuming differentiable production cost function φ, the condition of (4) implies that at least one consumer has nondifferentiable preferences at a^*, but it does not imply that all consumers have nondifferentiable preferences at this point.
4. The condition in (4) is sufficient, but not necessary. For example, if $\varphi(a) \equiv 0$, $n = 3$ and $v_1(a) = v_2(a) = v_3(a) = -4M \| a^* - a \| /7$, then the mechanism f such that $f_0(\mathbf{u}_{-i}, u) = a^*$ and $f_i(\mathbf{u}_{-i}, u) = 0$, $i = 1, 2, 3$, satisfies all the requirements in $\mathcal{D}(\mathbf{u}, a^*)$. ■

If an efficient mechanism created no externalities, the argument in the first paragraph of the Introduction says that it would be incentive compatible. Conversely, Theorem 2 implies that if a change in the public good allocation were required to accommodate a change in one individual's preferences, that would unavoidably create positive or negative externalities on the payoffs for others. And Theorem 2 also implies that the presence of externalities creates opportunities for strategic manipulation. To show how non-generic is the possibility of exact efficiency and incentive compatibility with a fixed finite population, the following suffices for impossibility:

Corollary 2 *If* $(f, \mathcal{D})$ *is comprehensive around each* $\mathbf{u} \in \mathcal{D}$ *and there exists no* $a \in A$ *such that* $f_0(\mathcal{D}) \equiv \{a\}$*, then* f *is not incentive compatible.*

To illustrate impossibility, first suppose $\varphi \equiv 0$. By Theorem 1, $b_i = 0$ for all i. Hence, no consumer is punished in his b-good compensation for announcing false preferences. It is easy to construct examples where $a^* = f_0(\mathbf{u})$ is not the best point for one consumer and this consumer has the ability to change this outcome by announcing false preferences. For example, suppose $k = 1$ (k is the number of public goods), $A = [0, 1]$, let $v_2(a) = \cdots = v_n(a) = -Ma/2n$, and let $v_1(a) = Ma/2n$. Let $u_1' = \frac{1}{2}Ma + b$ to obtain $f_0(\mathbf{u}) = 0$ but $f_0(\mathbf{u}_{-1}, u_1') = 1$. So consumer 1 is better off announcing u_1' instead of his true utility $u_1 = v_1 + b$.

When $\varphi \not\equiv 0$, consumer i would like a higher production of the public good iff his marginal utility from it exceeds his share of the marginal cost (that it, $\frac{1}{n}$ of it). Again, manipulation is easy. For example, suppose as before that $k = 1$ and $A = [0, 1]$, and suppose that $n > 2$ and φ is strictly increasing. Let $v_2(a) = \cdots = v_n(a) = \frac{\varphi(a)}{n+1}$. For $v_1(a) = \frac{\varphi(a)}{n-1}$, the mechanism produces $a = 0$, while for $v_1(a) = \frac{3\varphi(a)}{n+1}$, the mechanism produces $a = 1$, and person 1 is better off.

The fragility of the possibility result in Corollary 1 is demonstrated by Corollary 2. Nevertheless, Corollary 1 can also be used to point out why the elimination of externalities becomes more robust as the number of individuals increases. Consider the following extension that demonstrates possibility on a domain consisting of disjoint unions of minimally comprehensive domains.

Corollary 3 *Let* $n \geqslant 3$*. There is an efficient anonymous incentive compatible mechanism* f *on the domain*

$$\mathcal{D} = \bigcup_{a^*} \{\mathcal{D}(\mathbf{u}, a^*) : \mathbf{u}, \varphi, \text{ and } a^* \text{ satisfy inequality (4)} \\ \text{and } \mathcal{D}(\mathbf{u}, a^*) \text{ is minimally comprehensive}\}$$

The disjoint unions of minimally comprehensive domains highlighted in Corollary 3 cover only a small subset of $\mathcal{V}^n$. Nevertheless, they suggest how the collection $\bigcup_{a^*} \mathcal{D}(\mathbf{u}, a^*)$ can come closer to filling out $\mathcal{V}^n$ the larger is n, i.e., the set $\mathcal{V}^n \backslash \{\bigcup_{a^*} \mathcal{D}(\mathbf{u}, a^*)\}$ becomes smaller as n increases and, using the techniques employed in [9], it can be shown that it converges to a closed no-where dense set in the continuum limit.

4 Ordinal Utilities

We now turn to the case of ordinal preferences, so $u = u(a, b)$, where as before $a \in A \subset \Re^k$ for a compact convex A. We assume in this section that $b \in \Re_+$, and denote society's endowment of the private good by $\overline{b}$. The domain restrictions are similar to the quasilinear model with the important qualification that the set of utilities $\mathcal{U}$ are assumed to be quasi-concave (or even concave) — hence, $R(a, b; u)$ is always convex.

The quasiconcavity and Lipschitz restrictions on $u \in \mathcal{U}$ imply that for any (a, b) in its domain there is a hyperplane defined by $(p, 1) \in \Re^k \times \Re$ passing through (a, b) supporting $R(a, b; u)$, i.e.,

$$p \cdot a + b \leqslant (p, 1) \cdot R(a, b; u).$$

We shall also assume that φ is a convex function — hence, $C = \{(a, -b) : b \geqslant \varphi(a), a \in A\}$ is a convex set in $A \times \Re$. The combination $(a, -\varphi(a))$ is profit-maximizing at $\overline{p} \in \Re^k$ if

$$\overline{p} \cdot a - \varphi(a) \geqslant (\overline{p}, 1) \cdot C.$$

Note that when $\varphi \equiv 0$, any $(a, -\varphi(a)) = (a, 0)$ is profit-maximizing at $\overline{p} = 0$

The following result adapts well-known pricing and efficiency conditions for private to public goods.

Proposition 1 *Assume each u_i is quasiconcave, Lipschitz and increasing in b and φ is convex and Lipschitz. The feasible allocation $(a, b_1, \ldots, b_n)$ is efficient for $\mathbf{u}$ if and only if there exists $(p_1, \ldots, p_n)$, where $p_i \in \Re^k$, $i = 1, \ldots, n$, and $\bar{p} \in \Re^k$ such that*

1. $p_i \cdot a + b \leqslant (p_i, 1) \cdot R(a, b; u_i)$, $\forall i$,

2. $\overline{p} \cdot a - \varphi(a) \geqslant (\overline{p}, 1) \cdot C$,

3. $\bar{p} = \sum_i p_i$.

For the purely ordinal conclusions in Sections 4.1–4.3, we assume that there are no production costs, that is, $\varphi \equiv 0$. We explain below (section 4.4) how to adapt the analysis to the case of concave utilities with convex production costs.

With no costs of production, the set of feasible allocations is

$$\mathcal{F} = \{(a, b_1, \ldots, b_n) : a \in A, \ \sum_i b_i = \overline{b}\}.$$

4.1 Domains

Denote by $\mathcal{U}^*(u, a, b)$ the set of those $u' \in \mathcal{U}$ that are sharpenings of u at (a, b). Similarly to definitions 1–3, we define here $(f, \mathcal{D})$ to be first order if $\mathbf{u} \in \mathcal{D}$ implies

$$\mathcal{U}^*(u_1, f_0(\mathbf{u}), f_1(\mathbf{u})) \times \cdots \times \mathcal{U}^*(u_n, f_0(\mathbf{u}), f_n(\mathbf{u})) \subset \mathcal{D}$$

and we say that $(f, \mathcal{D})$ is comprehensive around $\overline{\mathbf{u}}$ if it is first order and

$$\bigcup_i (\overline{\mathbf{u}}_{-i}, \mathcal{U}) \subset \mathcal{D}$$

Finally, $(f, \mathcal{D})$ is minimally comprehensive around $\overline{\mathbf{u}}$ if it is the minimal first order domain to contain $\mathcal{D}(\overline{\mathbf{u}}, f(\overline{\mathbf{u}}))$, given by

$$\bigcup_i \Big\{ \ldots \times \mathcal{U}^*(\overline{u}_{i-1}, f_0(\overline{\mathbf{u}}), f_{i-1}(\overline{\mathbf{u}})) \times \mathcal{U} \ \times \ \mathcal{U}^*(\overline{u}_{i+1}, f_0(\overline{\mathbf{u}}), f_{i+1}(\overline{\mathbf{u}})) \times \ldots \Big\} \tag{5}$$

4.2 Characterization

The following result is similar to Theorem 1.

Theorem 4 *Let $(f, \mathcal{D})$ be first order. If f is incentive compatible on $\mathcal{D}$, then for every $\mathbf{u} \in \mathcal{D}$ and for every i,*

$$f_i(\mathbf{u}) = \frac{\overline{b}}{n}.$$

The analog of Theorem 2 is:

Theorem 5 *Let $(f, \mathcal{D})$ be minimally comprehensive around $\overline{\mathbf{u}} \in \mathcal{U}^n$. If f satisfies efficiency, anonymity, and incentive compatibility, then it exhibits no change.*

The proofs of Theorems 4 and 5 are in the Appendix. We briefly explain why they are different from those of Theorems 1 and 2. The source of difficulty is the characterization of efficiency. In the quasilinear model, efficiency is defined by eq. (3).

There may be many points at which $\sum v_i(a)$ is maximized. But once we sharpen, even slightly, one of the functions v_i at the outcome of the mechanism (denoted a^*), this point becomes the unique efficient outcome. Therefore, in the proofs of Theorems 1 and 2 we could generically assume a unique, well defined efficient outcome of the public good. The focus, therefore, was on the allocation of the private good. In the ordinal case analyzed in Theorems 4 and 5, this simplification is not available because many different (in fact, a continuum of) vectors of the public goods may represent efficient outcomes (Proposition 1), as the marginal rate of substitution between the public goods and the private good is not constant.

Similar remarks hold for the proof of Theorem 6, where one may wonder why the proof of this theorem is so difficult (see Appendix) while the equivalent result for the quasilinear case (Theorem 3) requires only a few lines. There too, the reason is that in the quasilinear case, efficiency generically determines a unique vector of public goods, while in the ordinal case, the conditions for efficiency are generically satisfied for a continuum of public goods vectors.

Next, we want to show that our analysis is really a characterization of efficient, incentive compatible mechanisms. We first observe that although we did not assume continuity, since the mechanism f on the domain $(f, \mathcal{D})$ exhibits no change, it is trivially continuous. To establish the uniqueness of the result, we confine our attention to mechanisms that are continuous. Define:

Continuity $\mathbf{u}^\ell \to \mathbf{u}$ implies $f(\mathbf{u}^\ell) \to f(\mathbf{u})$.

Suppose that the continuous mechanism g satisfies our assumptions on the domain $(f, \mathcal{D})$, which is comprehensive (with respect to f) around $\overline{\mathbf{u}}$. We show that on this domain, $h \equiv f$.

Theorem 6 *Let $(f, \mathcal{D})$ be minimally comprehensive around $\overline{\mathbf{u}}$. Suppose that a mechanism h on $\mathcal{D}$ satisfies continuity, efficiency, anonymity, and incentive compatibility. Then $h = f$.*

4.3 Possibility and Impossibility Results

Similarly to Section 3.3, we want to show that the analysis of the ordinal case is not vacuous by demonstrating the existence of comprehensive domains over which efficient anonymous incentive compatible mechanisms exist. Toward this goal, consider a minimally comprehensive domain $\mathcal{D}(\mathbf{u}, a^*)$. Recall the notation $L_i(a^*, a, b)$ which is the amount of private good compensation (positive or negative) a person with utility u_i will require to move from a^* to a, given that he holds b units of the private commodity. Similarly to Corollary 1, one can prove:

Corollary 4 *Let $(f, \mathcal{D})$ be minimally comprehensive around $\overline{\mathbf{u}}$ and $f_0(\overline{\mathbf{u}}) = a^*$. If $\mathbf{u}$ and a^* are such that for every a,*

$$\sum_{i=1}^{n} L_i(a^*, a, \tfrac{\bar{b}}{n}) < 2M \parallel a^* - a \parallel,$$

then there is an efficient anonymous incentive compatible mechanism f on the minimally comprehensive domain $\mathcal{D}(\mathbf{u}, a^*)$, where $f_0(\mathbf{u}) = a^*$ and $f_i(\mathbf{u}) = -\frac{\bar{b}}{n}$ for all $\mathbf{u} \in \mathcal{D}(\mathbf{u}, f_0(\mathbf{u}))$. [4]

Even though convexity is assumed in the ordinal model, the conclusion in Theorem 4 that no change characterizes efficiency and incentive compatibility means that Corollary 2 for the quasilinear model applies here as well.

Corollary 5 *If $(f, \mathcal{D})$ is comprehensive around each $\mathbf{u} \in \mathcal{D}$ and there exists no $a \in A$ such that $f_0(\mathcal{D}) \equiv \{a\}$, then f is not incentive compatible.*

The results for the quasilinear model are not a special case of the ordinal model because the constraints differ. Here $\sum b_i = \overline{b}$ and $b_i \geqslant 0$, while in Section 3, even when the production costs $\varphi \equiv 0$, the constraint is $\sum b_i = 0$ and b_i is not bounded. Nevertheless, the arguments for impossibility in the two environments are similar.

4.4 Costly Production and Concave Utilities

The conclusions of Sections 4.1–4.3 extend to costly production of public goods, provided concave utilities are assumed. In the proofs of Theorems 4–6, whenever the function u_i is concave (and not just quasiconcave), the manipulations used must also be concave. These Theorems therefore hold for the case where attention is restricted to concave functions with zero production costs.

Assume now nonzero production costs. Similarly to the proof of Theorem 1, we define for all i, $\mathfrak{u}_i(a, b) = u_i(a, b) - \frac{\varphi(a)}{n}$. As u_i is concave and φ convex, the function $\mathfrak{u}_i$ must also be concave. This function mimics the case of zero production costs, and similarly to the case of quasilinear utilities, extends the conclusions from costless to costly public goods.

[4] The quantity $2M$ replaces $4M$ in Corollary 1 because here $\varphi \equiv 0$.

5 Geometry of Efficient Incentive Compatibility

The purpose of this section is to restate the main conclusions of the previous sections in the language of price-taking equilibrium. Assume that $\mathcal{U}$ is quasiconcave. With quasilinearity, where $u(a,b) = v(a) + b$, u is quasiconcave if and only if v is concave.

Define the "inverse demand" for $(a, b; u)$ as

$$D(a,b;u) = \{(p,w) : p \cdot a + b = w, u(a,b) \geqslant u(a',b'), p \cdot a' + b' = w\},$$

the set of price and wealth combinations (p, w) such that an individual with utility u facing the budget constraint defined by $p \cdot a' + b' = w$ (the price of b is normalized to 1) would find (a, b) to be a utility-maximizing choice.

If $(p,w) \in D(a, b : u)$, then $w = p \cdot a + b = \min{(p,1)} \cdot R(a,b;u)$. Similarly, define the "inverse supply" of $a \in A$ as those prices for which a is a profit maximizing choice,

$$S(a) = \{\overline{p} : \overline{p} \cdot a - \varphi(a) \geqslant (\overline{p}, 1) \cdot C\}.$$

Because φ is convex and Lipschitz, $S(a)$ is non-empty for every $a \in A$.

It is readily seen that $D(a, b; u)$ is convex. When $D(a, b; u)$ is not a singleton, we shall say u has a *kink* at (a, b).

Define the size of the kink as

$$|D|(a,b;u) = \max\{\|p - p'\| : (p,w), (p',w') \in D(a,b;u)\}.$$

With quasiconcave utility, the definition of u' as a sharpening of u at (a, b) is compatible with $D(a, b; u) = D(a, b; u')$, for example, when u and u' are differentiable. But sharpening can also occur when $u(a', b') = u(a, b)$ and $(a', b') \neq (a, b)$ implies that there exists $(p', w') \in D(a, b; u')$ that is not in $D(a, b; u)$; hence, u' is kinked at (a, b) compared to u.

For comparison purposes, below, say that u exhibits a *flat* at (a, b) if there is an $(a', b') \neq (a, b)$ such that $D(a', b'; u) \cap D(a, b; u) \neq \emptyset$, i.e., the hyperplane containing (a, b) also includes (a', b'). Since $R(a, b; u)$ is convex, the indifference curve connecting convex combinations of (a, b) and (a', b') also lies on the hyperplane.

Because the mechanism chooses efficient allocations and, by Proposition 1, efficient allocations can be priced, an efficient mechanism can be given an as-if pricing interpretation: When individuals announce $\mathbf{u}$, we can "invert" the mechanism to say that instead of choosing $f_0(\mathbf{u})$ and $f_i(\mathbf{u})$ directly, the mechanism chooses prices $p_i(\mathbf{u})$, $\bar{p}(\mathbf{u})$ and wealths $w_i(\mathbf{u})$ to satisfy the following requirements,

- $(p_i(\mathbf{u}), w_i(\mathbf{u})) \in D(f_0(\mathbf{u}), f_i(\mathbf{u}); u_i)$,
- $\bar{p}(\mathbf{u}) \in S(f_0(\mathbf{u}))$,
- $\bar{p}(\mathbf{u}) = \sum_i p_i(\mathbf{u})$,

and then allows the individual consumers and the seller to maximize accordingly. Hence, the incentive compatibility restriction requires that these prices and wealths cannot be profitably manipulated by consumers. (The producer is not strategic.)

To show how this is accomplished, recall the no externalities property of the mechanism at $\mathbf{u}$ with respect to i:

$$u_j(f_0(\mathbf{u}), f_j(\mathbf{u})) = u_j(f_0(\mathbf{u}_{-i}, u_i'), f_j(\mathbf{u}_{-i}, u_i')), \ \forall j \neq i, \ \forall u_i' \in \mathcal{U}.$$

One way to eliminate externalities would be if variations in i's preferences do not change the effective utility maximizing budget constraint for any $j \neq i$, i.e.,

$$\{(p_j(\mathbf{u}), w_j(\mathbf{u}))\} = \bigcap_{u_i' \in \mathcal{U}} D(f_0(\mathbf{u}_{-i}, u_i'), f_j(\mathbf{u}_{-i}, u_i'); u_j). \tag{6}$$

We highlight this condition because its analog is exactly what is required in [9] for the characterization of no externalities with private goods, to exhibit the contrast with public goods. With private goods, a change in i's preferences do not change prices and the wealths of any $j \neq i$. Nevertheless, efficiency will typically require that a change in i's preferences leads to a change in i's allocation. With a finite number of consumers, the only way the change can be accommodated while preserving no externalities is for at least some of the indifference curves of $j \neq i$ to exhibit "flats." [If there are no flats, efficiency and incentive compatibility are impossible.] As the size of the population increases, the length of average flat necessary to eliminate externalities decreases and goes to zero in the continuum limit.

With private goods, efficiency prices are not personalized, but with public goods they are. To see why the efficiency conditions for public goods require that individual i's tastes will *unavoidably* influence the efficiency prices for j, consider the costless public goods case where efficiency requires

$$\sum_{j \neq i} p_j(\mathbf{u}) + p_i(\mathbf{u}) = \sum_{j \neq i} p_j(\mathbf{u}_{-i}, u_i') + p_i(\mathbf{u}_{-i}, u_i') = 0.$$

In addition, suppose the further restriction implied by Theorems 2 and 5 that i is unable to change the outcome. Therefore, the variation between $p_i(\mathbf{u})$ and $p_i(\mathbf{u}_{-i}, u_i')$

is entirely attributable to the change from u_i to u_i'. Because the change from u_i to u_i' is typically accompanied by changes in marginal rates of substitution, $p_i(\mathbf{u}) \neq p_i(\mathbf{u}_{-i}, u_i')$. To accommodate the change while preserving the equality above requires

$$\sum_{j \neq i} p_j(\mathbf{u}) \neq \sum_{j \neq i} p_j(\mathbf{u}_{-i}, u_i') \tag{7}$$

which is incompatible with condition (6).

With public goods, the pricing characterization of no externalities is the dual condition

$$\bigcup_{u_i' \in \mathcal{U}} \sum_{j \neq i} (p_j(\mathbf{u}_{-i}, u_i'), w_j(\mathbf{u}_{-i}, u_i')) \subset \sum_{j \neq i} D(f_0(\mathbf{u}), f_j(\mathbf{u}); u_j), \tag{8}$$

that there are some u_j sufficiently kinked at $(f_0(\mathbf{u}), f_j(\mathbf{u}))$ to accommodate possible variations in the tastes of i. I.e., for every $p_i(\mathbf{u}_{-i}, u_i')$, there are some j with $p_j(\mathbf{u}_{-i}, u_i')$ such that $(p_j(\mathbf{u}_{-i}, u_i'), w_j(\mathbf{u}_{-i}, u_i')) \in D(f_0(\mathbf{u}), f_j(\mathbf{u}); u_j)$ that helps to offset $p_i(\mathbf{u}_{-i}, u_i') - p_i(\mathbf{u})$. [If there are no kinks, efficiency and incentive compatibility are impossible.] Alternatively put, the implications of eliminating externalities with public goods is that to accommodate changes in the preferences of individual i, the (personalized) prices facing j must be allowed to vary. For the change in prices to be consistent with no externalities, quantities cannot vary.

The restrictions imposed by the inequality in (7) become less meaningful as the population increases. The larger the number of individuals, the smaller the adjustment required by each $j \neq i$ to offset a given $p_i(\mathbf{u}_{-i}, u_i') - p_i(\mathbf{u})$. Hence, as the number of individuals increases, the size of the average kink $|D|(f_0(\mathbf{u}), f_j(\mathbf{u}); u_j)$ necessary to satisfy (8) decreases, and in the limiting case of a continuum of individuals, the accommodation can be achieved when $|D| = 0$.

Appendix: Proofs

Proof of Theorem 1 We deal first with the case of no production costs. The condition (3) for efficiency becomes

$$f_0(\mathbf{u}) \in \arg\max_a \sum_{i=1}^{n} v_i(a) \tag{9}$$

and the statement of the theorem becomes

Suppose $\varphi \equiv 0$. Let $(f, \mathcal{D})$ be first order. If f is incentive compatible on $\mathcal{D}$, then f satisfies eq. (9)*, and for every $\mathbf{u} \in \mathcal{D}$ and for every i, $f_i(\mathbf{u}) = 0$.*

We prove the rest of this version of the Theorem in Lemmas 1 and 2, and extend it to the general case in Lemmas 3 and 4. We assume throughout that $u_i(a, b) = v_i(a)+b$.

The first step is to show that sharpenings by i do not change the outcome of the mechanism to i.

Lemma 1 *Let $\mathbf{u} \in \mathcal{D}$. Suppose v'_i is a sharpening of v_i at $\bar{a} = f_0(\mathbf{u})$, and let $u'_i(a, b) = v'_i(a) + b$. Then $f_0(\mathbf{u}_{-i}, u'_i) = f_0(\mathbf{u})$ and $f_i(\mathbf{u}_{-i}, u'_i) = f_i(\mathbf{u})$.*

Proof As $(f, \mathcal{D})$ is first order, $(\mathbf{u}_{-i}, u'_i) \in \mathcal{D}$. Replacing u_i with u'_i will not change f_0 because $\bar{a}$ is now the only point to maximize $\sum_{j \neq i} v_j + v'_i$ (recall that for all $a \neq \bar{a}$, $v'_i(\bar{a}) + \sum_{j \neq i} v_j(\bar{a}) = v_i(\bar{a}) + \sum_{j \neq i} v_j(\bar{a}) \geqslant v_i(a) + \sum_{j \neq i} v_j(a) > v'_i(a) + \sum_{j \neq i} v_j(a)$, where the first equality and the last inequality follow by the definition of sharpening, and the weak inequality follows by eq. (9)). It will not change f_i because of incentive compatibility: If $f_i(\mathbf{u}_{-i}, u'_i) > f_i(\mathbf{u})$, then person i with the true utility u_i will announce u'_i and if $f_i(\mathbf{u}_{-i}, u'_i) < f_i(\mathbf{u})$, then if his true utility is u'_i, person i will announce u_i instead. □

Lemma 2 *$\forall \mathbf{u} \in \mathcal{D}$ and $\forall i$, $f_i(\mathbf{u}) = 0$.*

Proof Fix $\mathbf{u} \in \mathcal{D}$. By eq. (2), there is $M' < M$ such that $|L(a, a', b)| < M' \|a - a'\|$. Let $a^* = f_0(\mathbf{u})$, and define

$$v^*(a) = -M \, \| \, a^* - a \, \|$$

Clearly $u^*(a, b) := v^*(a) + b \in \mathcal{V}$ and for every i, v^* is a sharpening of v_i at a^* (that is, $v^* \in \mathcal{V}^*(v_i, a^*)$, $i = 1, \dots, n$).

Observe that after u_i is replaced with u^*_i, a^* becomes the unique efficient quantity of the public good. Therefore, replacing u_i with u^* will not change the optimality of a^* and the payoffs b_i to person i (although it may change the payoffs to the other consumers).

Since $\sum_i f_i(\mathbf{u}) = 0$, it follows that if there exists an individual j such that $f_j(\mathbf{u}) \neq 0$, then for some i, $f_i(\mathbf{u}) > 0$. Say $i = 1$. Let $\mathbf{u}^1 = (\mathbf{u}_{-1}, u^*)$ and obtain by Lemma 1 that $f_0(\mathbf{u}^1) = f_0(\mathbf{u}) = a^*$ and $f_1(\mathbf{u}^1) = f_1(\mathbf{u}) > 0$. Since $f_1(\mathbf{u}^1) > 0$ and, by efficiency, $\sum_{i=1}^n f_i(\mathbf{u}^1) = 0$, there must be another person, say 2, such that $f_2(\mathbf{u}^1) < 0$. Let $\mathbf{u}^2 = (\mathbf{u}^1_{-2}, u^*)$. Again by Lemma 1, $f_2(\mathbf{u}^2) = f_2(\mathbf{u}^1)$ and $f_0(\mathbf{u}^2) = f_0(\mathbf{u}^1) = a^*$. But now consumers 1 and 2 have the same utility function, therefore, by anonymity, both receive the same (negative) amount of b. So there is another person, say 3, such that $f_3(\mathbf{u}^2) > 0$. Continuing to replace the utility functions of all individuals by u^* we get at the end that all consumers have the same utility function, but either for all i, $f_i > 0$ (n is odd), or for all i, $f_i < 0$ (n is even). In both cases we get a violation of $\sum b_i = 0$. □

We thus proved the theorem for the case $\varphi \equiv 0$. We now use this result to prove the general case, where production costs are not always zero. Let $(f, \mathcal{D})$ be first order and suppose that the mechanism f on $\mathcal{D}$ satisfies efficiency, anonymity, and incentive compatibility when the cost function is φ. We use this mechanism to create a mechanism g on a first order domain $\mathcal{D}'$ that satisfies efficiency, anonymity, and incentive compatibility when there are no production costs. Since Lemma 2 characterizes such mechanisms, we will be able to characterize the general mechanism.

For every $u_i(a,b) = v_i(a) + b \in \mathcal{D}$, define $\mathfrak{v}_i(a) = v_i(a) - \frac{\varphi(a)}{n}$, $\mathfrak{u}_i(a,b) = \mathfrak{v}_i(a) + b$, $i = 1, \ldots, n$, and let $\mathfrak{u} = (\mathfrak{u}_1, \ldots, \mathfrak{u}_n)$. Observe that since v_i and φ are Lipschitz, so is $\mathfrak{v}_i$ (even if with a different constant). Let $\mathcal{D}' = \{\mathfrak{u} = (\mathfrak{u}_1, \ldots, \mathfrak{u}_n) : \mathbf{u} = (u_1, \ldots, u_n) \in \mathcal{D}\}$, and define a mechanism g for the domain $\mathcal{D}'$ by

- $g_0(\mathfrak{u}) = f_0(\mathbf{u})$; and
- $g_i(\mathfrak{u}) = f_i(\mathbf{u}) + \frac{\varphi(f_0(\mathbf{u}))}{n}$

Lemma 3 $(g, \mathcal{D}')$ *is first order.*

Proof Let $\mathfrak{u} \in \mathcal{D}'$ and let $\mathfrak{v}_i'$ be a sharpening of $\mathfrak{v}_i$ at $g_0(\mathfrak{u})$. Since $g_0(\mathfrak{u}) = f_0(\mathbf{u})$, the function $v_i'(a) = \mathfrak{v}_i'(a) + \frac{\varphi(a)}{n}$ is a sharpening of v_i at $f_0(\mathbf{u})$. By the comprehensibility of $\mathcal{D}$, $\mathfrak{u}_i'(a,b) = \mathfrak{v}_i'(a) + b \in \mathcal{D}'$. □

Lemma 4 *The mechanism g on $\mathcal{D}'$ satisfies efficiency, anonymity, and incentive compatibility.*

Proof The proof follows by the fact that the mechanism f on $\mathcal{D}$ satisfies these three properties.

Efficiency: By the definition of g, the efficiency of f, and conditions of eqs. (3) and (9),

$$\begin{aligned} g_0(\mathfrak{u}) = f_0(\mathbf{u}) \quad \in \quad & \arg\max_a \sum_{i=1}^n v_i(a) - \varphi(a) = \\ & \arg\max_a \sum_{i=1}^n \left[v_i(a) - \tfrac{\varphi(a)}{n} \right] = \arg\max_a \sum_{i=1}^n \mathfrak{v}_i(a) \end{aligned}$$

Also,

$$\sum_{i=1}^{n} g_i(\mathbf{u}) = \sum_{i=1}^{n} \left[f_i(\mathbf{u}) + \tfrac{\varphi(f_0(\mathbf{u}))}{n} \right] = 0$$

Anonymity: $\mathfrak{u}_i = \mathfrak{u}_j$ implies $\mathfrak{v}_i = \mathfrak{v}_j$, hence $v_i = v_j$. Since f satisfies anonymity, $f_i(\mathbf{u}) = f_j(\mathbf{u})$ and $g_i(\mathbf{u}) = g_j(\mathbf{u})$.

Incentive Compatibility: Let $\mathfrak{u}'_i(a,b) = \mathfrak{v}'_i(a) + b$ and let $u'_i(a,b) = v'_i(a) + b$ where $v'_i(a) = \mathfrak{v}'_i(a) + \frac{\varphi(a)}{n}$. To simplify notation, denote $\mathbf{u}' = (\mathbf{u}_{-i}, u'_i)$ and $\mathbf{\mathfrak{u}}' = (\mathbf{u}_{-i}, \mathfrak{u}'_i)$. By definition and the incentive compatibility of f,

$$\begin{aligned}
&\mathfrak{u}_i(g(\mathbf{\mathfrak{u}}')) = \mathfrak{u}_i(g_0(\mathbf{\mathfrak{u}}'), g_i(\mathbf{\mathfrak{u}}')) = \\
&\mathfrak{u}_i(f_0(\mathbf{u}'), f_i(\mathbf{u}') + \tfrac{\varphi(f_0(\mathbf{u}'))}{n}) = \\
&\mathfrak{v}_i(f_0(\mathbf{u}')) + f_i(\mathbf{u}') + \tfrac{\varphi(f_0(\mathbf{u}'))}{n} = \\
&v_i(f_0(\mathbf{u}')) + f_i(\mathbf{u}') = \\
&u_i(f(\mathbf{u}')) \leqslant \\
&u_i(f(\mathbf{u})) = \\
&v_i(f_0(\mathbf{u})) + f_i(\mathbf{u}) = \\
&\mathfrak{v}_i(f_0(\mathbf{u})) + f_i(\mathbf{u}) + \tfrac{\varphi(f_0(\mathbf{u}))}{n} = \\
&\mathfrak{u}_i(f_0(\mathbf{u}), f_i(\mathbf{u}) + \tfrac{\varphi(f_0(\mathbf{u}))}{n}) = \\
&\mathfrak{u}_i(g_0(\mathbf{u}), g_i(\mathbf{u})) = \mathfrak{u}_i(g(\mathbf{u}))
\end{aligned}$$

□

We obtain that g satisfies all the requirements of Theorem 1 for the case $\varphi \equiv 0$, and therefore for all i, $g_i(\mathbf{u}) = 0$. Since for all i, $g_i(\mathbf{u}) = f_i(\mathbf{u}) + \frac{\varphi(f_0(\mathbf{u}))}{n}$, it follows that $f_i(\mathbf{u}) = -\frac{\varphi(f_0(\mathbf{u}))}{n}$. ■

Proof of Theorem 2 We assume throughout the proof that production costs are always zero. If production costs are not zero, then adjust the proof as in Theorem 1. By Lemma 2 we know that for the case of zero production costs, $f_i \equiv 0$, $i = 1, \ldots, n$. Denote $a^* = f_0(\overline{\mathbf{u}})$. Suppose that for some $\mathbf{u}, (\mathbf{u}_{-i}, u'_i) \in \mathcal{D}$, $f_0(\mathbf{u}) = a^*$ but $f_0(\mathbf{u}_{-i}, u'_i) = a' \neq a^*$, and assume, wlg, that $\| a' - a^* \| = 1$. Denote the corresponding v functions by $\overline{v}_i$, v_i, and v', and let $\mathbf{u}' = (\mathbf{u}_{-i}, u'_i)$. Assume, wlg, that a^* and a' are the unique efficient points for the corresponding economies. Otherwise, sharpen v_i and v'_i to make them unique.

As $f_0(\mathbf{u}) = f_0(\overline{\mathbf{u}}) = a^*$, it follows by the discussion after Def. 3 that all but one (at most) of the functions $v_1, \ldots, v_n$ are sharpenings of $\overline{v}_1, \ldots, \overline{v}_n$ at a^*. If v'_i

is a sharpening of $\overline{v}_i$ at a^*, then as $f_0(\mathbf{u}) = a^* = f_0(\overline{\mathbf{u}})$, it must be the case that $f_0(\mathbf{u}_{-i}, u_i') = a^*$ and not a', a contradiction. Therefore, for all $j \neq i$, v_j is a sharpening of $\overline{v}_j$ at a^*.

Our aim is to show a violation of IC, which is trivially obtained if $v_i(a') > v_i(a^*)$, as person i can shift the economy from a^* to a'. Otherwise, define for $\delta \in (0, M)$, $v_i^\delta(a) = -\delta \parallel a - a' \parallel$. If $f_0(v_1 + b, \ldots, v_i^\delta + b, \ldots, v_n + b) \neq a'$, then by switching from v_i^δ to v_i', person i will strictly improve his position by shifting the economy to a', a violation of IC. It thus follows that for every $\delta \in (0, M)$, $f_0(v_1 + b, \ldots, v_i^\delta + b, \ldots, v_n + b) = a'$. By efficiency, and as $\parallel a' - a^* \parallel = 1$,

$$\sum_{j \neq i} v_j(a') + v_i^\delta(a') \geqslant \sum_{j \neq i} v_j(a^*) + v_i^\delta(a^*) \Longrightarrow$$
$$\sum_{j \neq i} v_j(a^*) - \sum_{j \neq i} v_j(a') \leqslant v_i^\delta(a') - v_i^\delta(a^*) = \delta$$

This inequality holds for all $\delta > 0$, hence $\sum_{j \neq i} v_j(a^*) \leqslant \sum_{j \neq i} v_j(a')$.

Similarly, if his utility is $w_i^\delta(a) = -\delta \parallel a - a^* \parallel$, person i can force a^* by using v_i, hence $f_0(v_1 + b, \ldots, w_i^\delta + b, \ldots, v_n + b) = a^*$. As before we obtain $\sum_{j \neq i} v_j(a') \leqslant \sum_{j \neq i} v_j(a^*)$, therefore $\sum_{j \neq i} v_j(a^*) = \sum_{j \neq i} v_j(a')$.

Fix v_i^δ, and choose $\{v_j'\}_{j \neq i}$ such that they are sharpenings of $\{v_j\}_{j \neq i}$ at a^* and such that for $a \neq a^*$,

$$\sum_{j \neq i} v_j'(a) < \min \left\{ \sum_{j \neq i} v_j(a), \sum_{j \neq i} v_j(a^*) - 2\delta \parallel a - a^* \parallel \right\}$$

By the triangle inequality, for every $a \neq a^*$,

$$\sum_{j \neq i} v_j'(a) + v_i^\delta(a) <$$
$$\sum_{j \neq i} v_j(a^*) - 2\delta \parallel a - a^* \parallel - \delta \parallel a - a' \parallel <$$
$$\sum_{j \neq i} v_j(a^*) - \delta =$$
$$\sum_{j \neq i} v_j'(a^*) + v_i^\delta(a^*)$$

It thus follows that $f_0(v'_1+b,\ldots,v_i^\delta,\ldots,v'_n+b)=a^*$. Replace now v_i^δ with $v_i^{3\delta}$, recall that $\sum_{j\neq i} v'_j(a^*)=\sum_{j\neq i} v_j(a^*)=\sum_{j\neq i} v_j(a')$, and obtain

$$\begin{aligned}&\sum_{j\neq i} v'_j(a^*)+v_i^{3\delta}(a^*)=\\&\sum_{j\neq i} v_j(a^*)-3\delta<\\&\sum_{j\neq i} v_j(a^*)-2\delta=\\&\sum_{j\neq i} v'_j(a')+v_i^{3\delta}(a')\end{aligned}$$

Therefore $f_0(v'_1+b,\ldots,v_i^{3\delta},\ldots,v'_n+b)=a''\neq a^*$. If $\| a''-a' \|\geqslant 1$, then

$$\begin{aligned}&\sum_{j\neq i} v'_j(a'')+v_i^{3\delta}(a'')\leqslant\\&\sum_{j\neq i} v_j(a^*)-2\delta\,\| a''-a^* \| -3\delta<\\&\sum_{j\neq i} v_j(a^*)-2\delta=\\&\sum_{j\neq i} v'_j(a')+v_i^{3\delta}(a')\end{aligned}$$

Hence $\| a''-a' \|< 1$ and person i improved his position by replacing v_i^δ with $v_i^{3\delta}$, a violation of IC. ■

Proof of Theorem 3 The proof of this theorem is essentially a repetition of Lemmas 1–4, so we offer just an outline of it. Let $(f,\mathcal{D})$ be comprehensive around $\overline{\mathbf{u}}$, and suppose that for every $\mathbf{u}\in\mathcal{D}$, $\arg\max_a \sum[v_i(a)-\varphi(a)]$ is a singleton. Suppose that the mechanism h on $\mathcal{D}$ satisfies efficiency, anonymity, and incentive compatibility. By efficiency and eq. (3) it follows that for $\mathbf{u}\in\mathcal{D}$, $f_0(\mathbf{u})=h_0(\mathbf{u})$. Since $(f,\mathcal{D})$ is comprehensive, we can use Lemmas 1–4 to get that for all i, $h_i(\mathbf{u})=-\frac{\varphi(f_0(\mathbf{u}))}{n}$. In other words, $h\equiv f$ on $\mathcal{D}$. ■

Proof of Corollary 1 Incentive compatibility of the mechanism follows from the fact that the mechanism on this domain is constant. Anonymity follows by the fact that everyone receives the same quantity of b. Next we prove efficiency. Inequality (4) implies that for all $a\neq a^*$,

$$\sum v_i(a)-\varphi(a)<\sum v_i(a^*)-\varphi(a^*)$$

Let $\mathbf{u}' \in \mathcal{D}(\mathbf{u}, a^*)$, and suppose that v'_j is not a sharpening of v_j (as $f_0 \equiv a^*$, there is at most one such j). By the Lipschitz conditions, for all a and a',

$$\left|\left(v'_j(a) - \tfrac{\varphi(a)}{n}\right) - \left(v'_j(a') - \tfrac{\varphi(a')}{n}\right)\right| \leqslant 2M \, \| \, a - a' \, \|$$

Also, by the definition of sharpening, for $i \neq j$, $v'_i(a) - v'_i(a^*) \leqslant v_i(a) - v_i(a^*)$ for all a. Therefore

$$\begin{aligned}
&\sum_{i=1}^{n} \left(v'_i(a) - \tfrac{\varphi(a)}{n}\right) - \sum_{i=1}^{n} \left(v'_i(a^*) - \tfrac{\varphi(a^*)}{n}\right) = \\
&[v'_j(a) - v'_j(a^*)] + \sum_{i \neq j} [v'_i(a) - v'_i(a^*)] - \varphi(a) + \varphi(a^*) \leqslant \\
&[v'_j(a) - v'_j(a^*)] - [v_j(a) - v_j(a^*)] + \\
&\qquad \sum_{i=1}^{n} [v_i(a) - v_i(a^*)] - \varphi(a) + \varphi(a^*) = \\
&\left[\left(v'_j(a) - \tfrac{\varphi(a)}{n}\right) - \left(v'_j(a^*) - \tfrac{\varphi(a^*)}{n}\right)\right] - \\
&\qquad \left[\left(v_j(a) - \tfrac{\varphi(a)}{n}\right) - \left(v_j(a^*) - \tfrac{\varphi(a^*)}{n}\right)\right] + \\
&\qquad \sum_{i=1}^{n} \left[\left(v_i(a) - \tfrac{\varphi(a)}{n}\right) - \left(v_i(a^*)) - \tfrac{\varphi(a^*)}{n}\right)\right] < \\
&2M \, \| \, a^* - a \, \| + 2M \, \| \, a^* - a \, \| - 4M \, \| \, a^* - a \, \| = 0
\end{aligned}$$

Hence the mechanism is efficient. ■

Proof of Corollary 3 For $\mathbf{u}' \in \mathcal{D}(\mathbf{u}, a^*)$, the mechanism is given by $f_0(\mathbf{u}') = a^*$ and $f_1(\mathbf{u}') = \cdots = f_n(\mathbf{u}') = -\frac{\varphi(a^*)}{n}$. Note that for $\mathbf{u}' \in \mathcal{D}(\mathbf{u}, a^*)$, a^* is the only point to maximize $\sum v'_i$. Therefore, if $(\mathbf{u}, a^*)$ and $(\overline{\mathbf{u}}, a^{**})$ satisfy eq. (4), then $a^* \neq a^{**}$ implies $\mathcal{D}(\mathbf{u}, a^*) \cap \mathcal{D}(\overline{\mathbf{u}}, a^{**}) = \varnothing$. Note that for all $\mathbf{u}' \in \mathcal{D}(\mathbf{u}, a^*)$, $f(\mathbf{u}') = a^*$ and for all $\mathbf{u}' \in \mathcal{D}(\overline{\mathbf{u}}, a^{**})$, $f(\mathbf{u}') = a^{**}$. The domain $\mathcal{D}$ is therefore well-defined. ■

Proof of Theorem 4 We prove the theorem in a two lemmas. Observe first that by eq. (1), as u_i is continuous, so is the function $L(a, a', b)$. The domain of u_i is the compact set $A \times [0, \overline{b}]$; therefore there exists $M' < M$ such that for all a, a', and b,

$$L(a, a', b) < M' \, \| \, a - a' \, \|$$

Lemma 5 *Let $\mathbf{u} \in \mathcal{D}$. If u'_i is a sharpening of u_i at $(f_0(\mathbf{u}), f_i(\mathbf{u}))$, then $f_0(\mathbf{u}_{-i}, u'_i) = f_0(\mathbf{u})$ and $f_i(\mathbf{u}_{-i}, u'_i) = f_i(\mathbf{u})$.*

Proof Let $(a, b_i) = (f_0(\mathbf{u}), f_i(\mathbf{u}))$, and let $(a', b'_i) = (f_0(\mathbf{u}_{-i}, u'_i), f_i(\mathbf{u}_{-i}, u'_i))$. If $u_i(a', b'_i) > u_i(a, b_i)$, then person i with the true utility u_i will be better off announcing u'_i. Otherwise, $u_i(a', b'_i) \leqslant u_i(a, b_i)$. If $(a, b_i) \neq (a', b'_i)$, then, by the definition of a sharpening, $u'_i(a, b_i) > u'_i(a', b'_i)$, and person i with the true utility u'_i will be better off announcing u_i instead. This leaves (a, b_i) as the only possible outcome of $f_i(\mathbf{u}_{-i}, u'_i)$. □

Lemma 6 *Let* $\mathbf{u} \in \mathcal{D}$. *Then for all* i, $f_i(\mathbf{u}) = \frac{\overline{b}}{n}$.

Proof Let $\overline{a} = f_0(\mathbf{u})$ and suppose that for some i, say $i = 1$, $f_1(\mathbf{u}) \neq \frac{\overline{b}}{n}$. Define

$$u_{\overline{a}}(a, b) = b - M' \, \| a - \overline{a} \|$$

Clearly, $u_{\overline{a}}$ is a strict sharpening of all the functions u_i at $(\overline{a}, b)$ for all b, hence, by Lemma 5, $f_0(\mathbf{u}_{-1}, u_{\overline{a}}) = f_0(\mathbf{u}) = \overline{a}$, and $f_0(\mathbf{u}_{-1}, u_{\overline{a}}) = f_0(\mathbf{u}) > \frac{\overline{b}}{n}$.

Following the procedure used in the proof of Lemma 2, we obtain a contradiction, hence for all i, $f_i(\mathbf{u}) = \frac{\overline{b}}{n}$. ■

Proof of Theorem 5 Let $\mathbf{u}, \mathbf{u}' \in \mathcal{D}$, and suppose that $f_0(\mathbf{u}) = a^*$ but $f_0(\mathbf{u}') = a' \neq a^*$. As $\mathcal{D}$ is minimally comprehensive around $\overline{\mathbf{u}}$, and as we know that by Theorem 4, for all i, $f_i(\overline{\mathbf{u}}) = \frac{\overline{b}}{n}$, we can assume wlg that

1. $a^* = f_0(\overline{\mathbf{u}})$,
2. there exists i such that for all $j \neq i$, u_j is a sharpening of $\overline{u}_j$ at $(a^*, \frac{\overline{b}}{n})$,
3. $\mathbf{u}' = (\mathbf{u}_{-i}, u'_i)$.

By Lemma 5, for all j, $f_j(\mathbf{u}) = f_j(\mathbf{u}') = \frac{\overline{b}}{n}$. We assume first that none of the functions $u_1, \ldots, u_n, u'_i$ reaches the same value at $(a^*, \frac{\overline{b}}{n})$ and $(a', \frac{\overline{b}}{n})$.

By Proposition 1, efficiency implies that for all a,

$$\lim_{\delta \to 0} \sum_j u_j((1-\delta)a^* + \delta a, \tfrac{\overline{b}}{n}) \geqslant \textstyle\sum_j u_j(a^*, \frac{\overline{b}}{n})$$

If $(a, \frac{\overline{b}}{n}, \ldots, \frac{\overline{b}}{n})$ too is efficient, then

$$\lim_{\delta \to 0} \sum_j u_j((1-\delta)a + \delta a^*, \tfrac{\overline{b}}{n}) \geqslant \textstyle\sum_j u_j(a, \frac{\overline{b}}{n})$$

By quasiconcavity, $\sum_j u_j((1-\delta)a^* + \delta a), \frac{\overline{b}}{n})$ is then constant in δ along the $[0, 1]$ segment. Therefore, any sharpening of u_i at $(a^*, \frac{\overline{b}}{n})$, and likewise, any sharpening of u_i' at a', implies that a^* (for $\mathbf{u}$) and a' (for $\mathbf{u}'$) are the only efficient allocations for which each person receives $\frac{\bar{b}}{n}$ units of the b good. The arguments below relate essentially to a line in $\Re^k$, so we assume first one public good only. We show at the end of the proof how to extend it to many goods. Assume, wlg, $a' < a^*$.

First we want to replace, for all $j \neq i$, the function u_j with $\tilde{u}_j$ which will satisfy the following requirement.

1. It will be a sharpening of u_j at $(a^*, \frac{\bar{b}}{n})$;
2. It will be a sharpening of u_j at $(a', \frac{\bar{b}}{n})$;
3. The slopes of its indifference curves along the $[a', a^*]$ segment will be constant.

As $\sum b_j = \bar{b}$ and for all j, $b_j \geqslant 0$, the relevant domain over which all utilities are defined is compact. There exists therefore $M'' < M$ such that for all j, u_j satisfies the Lipschitz condition with respect to M''. Choose $M' \in (M'', M)$ and let the slope of all indifference curves of $\tilde{u}_j$ be $-M'$ to the left of a' and M' to the right of a^*. We assumed that for all $j \neq i$, $u_j(a^*, \frac{\bar{b}}{n}) \neq u_j(a', \frac{\bar{b}}{n})$. Suppose that $u_j(a', \frac{\bar{b}}{n}) < u_j(a^*, \frac{\bar{b}}{n})$. Let $b' < \frac{\bar{b}}{n}$ such that $u_j(a', \frac{\bar{b}}{n}) < u_j(a^*, b')$ and define the slope of the indifference curves of $\tilde{u}_j$ between a' and a^* to be $-(\frac{\bar{b}}{n} - b')/(a^* - a')$ (see Fig. 1). The construction of $\tilde{u}_j$ for the case $u_j(a', \frac{\bar{b}}{n}) > u_j(a^*, \frac{\bar{b}}{n})$ is similar.

Denote

- $\tilde{\mathbf{u}} = (\ldots, \tilde{u}_{i-1}, u_i, \tilde{u}_{i+1}, \ldots)$;
- $\tilde{\mathbf{u}}' = (\ldots, \tilde{u}_{i-1}, u_i', \tilde{u}_{i+1}, \ldots)$;

By Lemma 5, $f_0(\tilde{\mathbf{u}}) = a^*$ and $f_0(\tilde{\mathbf{u}}') = a'$. Recall that $a' < a^*$, and let

- $x = \sum_{j \neq i} \mathrm{MRS}^{b,a}(\tilde{u}_j(a, \frac{\bar{b}}{n}))$ for $a \in (a', a^*)$;
- $y = \lim_{a \uparrow a^*} \mathrm{MRS}^{b,a}(u_i(a, \frac{\bar{b}}{n}))$;
- $z = \lim_{a \downarrow a'} \mathrm{MRS}^{b,a}(u_i'(a, \frac{\bar{b}}{n}))$.

By the construction of $\tilde{u}_j$, x is constant along (a', a^*). As $f_0(\tilde{\mathbf{u}}) = a^*$ and $f_0(\tilde{\mathbf{u}}') = a'$, and moreover, as a^* and a' are the only points for which $(a, \frac{\bar{b}}{n}, \ldots, \frac{\bar{b}}{n})$ are efficient for $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{u}}'$, respectively, we obtain that $x + y < 0$ but $x + z > 0$. Our aim is to show that person i has the ability to manipulate f_0 to his benefit, a violation of IC.

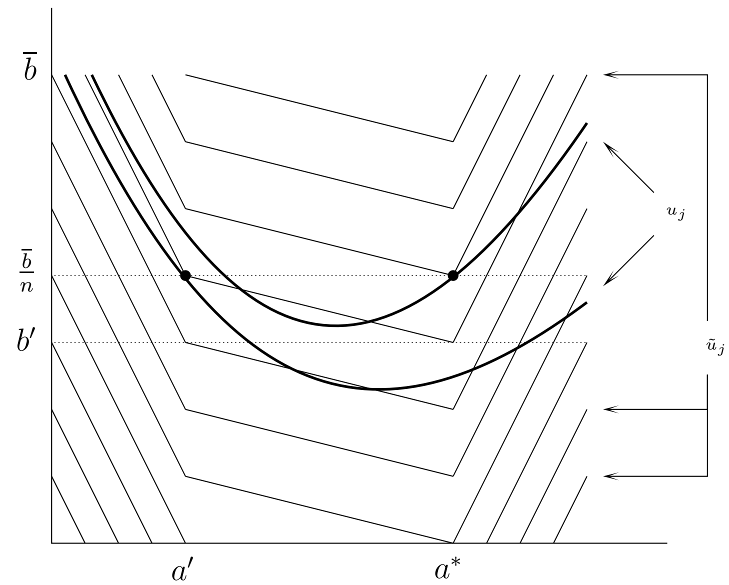

Figure 1: Before and after sharpenings at a' and a^*

As $x + y < 0$ but $x + z > 0$, and as $|y|, |z| < M'$, it follows that $x \in (-M', M')$. Let $\hat{u}_i^\eta$ be such that for all b,

$$\mathrm{MRS}^{b,a}(\hat{u}_i^\eta(a,b)) = \begin{cases} -M' & a < a' \\ -\eta & a \in (a', a^*) \\ M' & a > a^* \end{cases}$$

If $x > 0$, then person i with the utility $\hat{u}^{-x+\varepsilon}$, $0 < \varepsilon < \min\{M - M', |x|\}$, can switch the economy from a', where it will be if he declares $\hat{u}^{-x+\varepsilon}$, to a^*, which he favors, by stating $\hat{u}^{-x-\varepsilon}$. Similarly, if $x < 0$, person i with the true utility $\hat{u}^{-x-\varepsilon}$ can switch the economy from a^*, where it will be if he declares $\hat{u}^{-x-\varepsilon}$, to a', which he favors, by stating $\hat{u}^{-x+\varepsilon}$. If $x = 0$, sharpen slightly the utility of one person $j \neq i$ at a^* and follow the same analysis.

We assumed that for all $j \neq i$, $u_j(a^*, \frac{\bar{b}}{n}) \neq u_j(a', \frac{\bar{b}}{n})$. If for some $j \neq i$, $u_j(a^*, \frac{\bar{b}}{n}) = u_j(a', \frac{\bar{b}}{n})$, then let the slope of all the indifference curves of $\tilde{u}_j$ between a' and a^* be zero. We used Lemma 5 earlier to claim that $f_0(\tilde{\mathbf{u}}) = a^*$ and $f_0(\tilde{\mathbf{u}}') = a'$. This Lemma requires (strict) sharpenings which in the present case cannot be provided by $\tilde{u}_j$. But the choice of u_i and u_i' in the beginning of the proof of this Theorem guarantees that $f_0(\tilde{\mathbf{u}}) = a^*$ and $f_0(\tilde{\mathbf{u}}') = a'$.

Our analysis was done with respect to one public good only. To complete the proof we have to show that it extends to $\Re^k$. We only provide a sketch of such a proof, based on the above analysis. To extend the sharpening of Fig. 1 to $\Re^k$, recall that in Fig. 1 the vertical axis represents the private commodity, and the horizontal axis is the line through $(a', 0)$ and $(a^*, 0)$. The sharpening in the other directions will be to the slope of $-M'$ (or, when necessarily, $-M''$) towards the indifference curves of the picture. ■

Proof of Theorem 6 We first observe that by Theorem 5, the minimally comprehensive domain around $\overline{\mathbf{u}} \in \mathcal{Q}^n$ is $\mathcal{D}(\overline{\mathbf{u}}, f(\overline{\mathbf{u}}))$ (see eq. (5)). Given $\mathbf{u} \in \mathcal{D}(\overline{\mathbf{u}}, f(\overline{\mathbf{u}}))$ we may therefore assume, wlg, that

(∗) For all $i > 1$, u_i is a sharpening of $\overline{u}_i$ at $(f_0(\overline{\mathbf{u}}), \frac{\overline{b}}{n})$.

Similarly to Lemma 6, we can prove the following:

Fact 1 *If for some* $\mathbf{u} \in \mathcal{D}(\overline{\mathbf{u}}, f(\overline{\mathbf{u}}))$, $g_0(\mathbf{u}) = a^* := f_0(\overline{\mathbf{u}})$, *then for all* i, $g_i(\mathbf{u}) = \frac{\overline{b}}{n}$.

We say that the function $u(a, b)$ is locally quasilinear around (a^o, b^o) if there is $\varepsilon > 0$, a function $v : A \to \Re$, and an increasing function $h : \Re \to \Re$ such that

$$|u(a, b) - u(a^o, b^o)| < \varepsilon \Longrightarrow h(u(a, b)) = v(a) + b$$

Suppose that for some $\mathbf{u} \in \mathcal{D}(\overline{\mathbf{u}}, f(\overline{\mathbf{u}}))$, $g(\mathbf{u}) = a \neq a^* = f_0(\overline{\mathbf{u}})$. We first create an economy $\mathbf{u}'$ such that for all i

- $g_0(\ldots, u'_i, u_{i+1}, \ldots) \neq a^*$
- u'_i is quasiconcave and is a sharpening of u_i at $(a^*, \frac{\overline{b}}{n})$
- u'_i is locally quasilinear around $(a^*, \frac{\overline{b}}{n})$

Denote $\mathbf{u}^i = (\ldots, u'_i, u_{i+1}, \ldots)$. Suppose we created u'_{i-1} and construct u'_i.

Case 1. If

$$u_i(g_0(\mathbf{u}^{i-1}), g_i(\mathbf{u}^{i-1})) < u_i(a^*, \frac{\overline{b}}{n})$$

then replace u_i with $u'_i = u^*$, given by

$$u^*(a, b) = b - M' \parallel a - a^* \parallel \tag{10}$$

for some $M' \in (M'', M)$ where M'' is such that for all j, u_j satisfies the Lipschitz condition with respect to M''. Clearly, u^* is quasilinear and a sharpening of u_i at

$(a^*, \frac{\bar{b}}{n})$. By fact 1, if $g_0(\mathbf{u}^i) = a^*$, then $g_i(\mathbf{u}^i) = \frac{\bar{b}}{n}$ and person i benefits from replacing u_i with u_i', a violation of IC, hence $g_0(\mathbf{u}^i) \neq a^*$.

Case 2. If

$$u_i(g_0(\mathbf{u}^{i-1}), g_i(\mathbf{u}^{i-1})) = u_i(a^*, \tfrac{\bar{b}}{n}) \tag{11}$$

then, using the same technique as in the proof of Theorem 5 (see Fig. 1), replace u_i with u_i' which is a sharpening of u_i at $(a^*, \frac{\bar{b}}{n})$ and at $(g_0(\mathbf{u}^{i-1}),\ g_i(\mathbf{u}^{i-1}))$ such that eq. (11) is satisfied with u_i'. As eq. (11) restricts one indifference curve only, we can construct u_i' to be quasilinear, and by construction, u_i' is quasiconcave. The function u_i' is only a weak sharpening of u_i at $(g_0(\mathbf{u}^{i-1}), g_i(\mathbf{u}^{i-1}))$, but it is the limit of sharpenings $u_{i,m}'$ of u_i at $(g_0(\mathbf{u}^{i-1}), g_i(\mathbf{u}^{i-1}))$. The functions $u_{i,m}'$ are not sharpenings of u_i at $(a^*, \frac{\bar{b}}{n})$, but our domain permits one person's utility to be any function in $\mathcal{Q}$. If $i = 1$, then by condition $(*)$, for all other agent j, his utility is a sharpening of u_j at $(a^*, \frac{\bar{b}}{n})$. If $i > 1$, then person 1 utility was already replaced by a sharpening of u_1 at $(a^*, \frac{\bar{b}}{n})$ (see steps 1 and 3). As $u_{i,m}'$ is a sharpening of u_i at $(g_0(\mathbf{u}^{i-1}), g_i(\mathbf{u}^{i-1}))$, it follows by Lemma 6 that

$$g_0(\mathbf{u}_{-i}^{i-1}, u_{i,m}') = g_0(\mathbf{u}^{i-1}) \neq a^*$$

and by continuity, $g_0(\mathbf{u}^i) \neq a^*$.

Case 3. If

$$u_i(g_0(\mathbf{u}^{i-1}), g_i(\mathbf{u}^{i-1})) > u_i(a^*, \tfrac{\bar{b}}{n})$$

then replace u_i with u_i' which is depicted on Fig. 2. This function is locally quasilinear around $(a^*, \frac{\bar{b}}{n})$. It is quasiconcave, and a sharpening of u_i at $(a^*, \frac{\bar{b}}{n})$. It is also a sharpening of u_i at $(g_0(\mathbf{u}^{i-1}), g_i(\mathbf{u}^{i-1}))$. We thus obtain that $g_0(\mathbf{u}^i) = g_0(\mathbf{u}^{i-1}) \neq a^*$.

We now have the economy $\mathbf{u}'$ which is in the domain $\mathcal{D}(\overline{\mathbf{u}}, f(\overline{\mathbf{u}}))$. For each i, u_i' is a sharpening of u_i at $(a^*, \frac{\bar{b}}{n})$, and $g_0(\mathbf{u}') \neq a^*$. By efficiency it is not the case that for all i, $u_i' = u^*$ (see eq. (10)). Replace the utilities of those agents whose utility is not u^* with u^*, one at a time. There is an agent i such that replacing his utility u_i' with u^* will change the quantity of the public good from $a' \neq a^*$ to a^*. Denote the profile before i makes the change $\mathbf{u}''$. Consider the path $\{u_i^\alpha := \alpha u^* + (1-\alpha)u_i'\}$ where $\alpha \in [0, 1]$. Denote $\mathbf{u}^\alpha = (\mathbf{u}_{-i}'', u_i^\alpha)$. As both u^* and u_i' are quasiconcave, and as each indifference curve of u^* is a sharpening of an indifference curve of u_i' at a point along

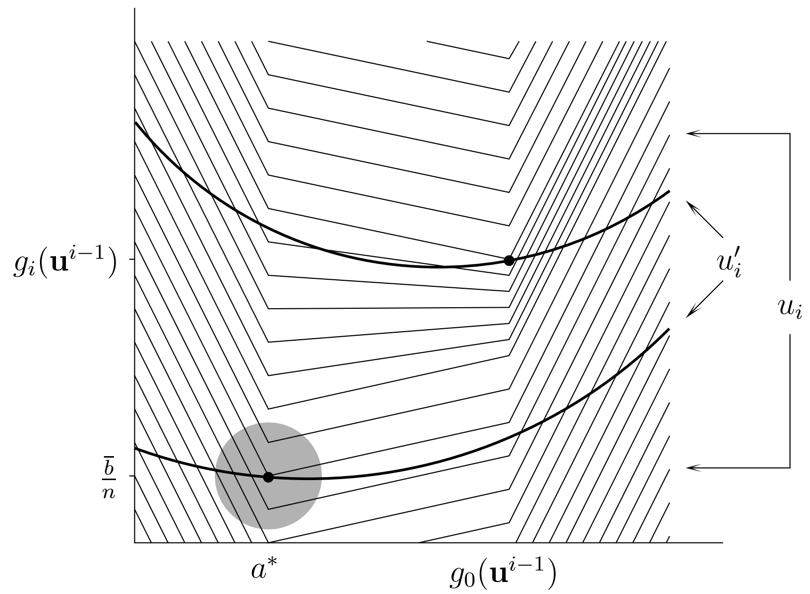

Figure 2: The function u_i' of Case 3. Area of local quasilinearity is shadowed.

the line $a = a^*$, it follows that each element of the path is a quasiconcave function. Moreover, as both u^* and u_i' are locally quasilinear around $(a^*, \frac{\bar{b}}{n})$, so is each element of the path.

The point a^* (together with equal allocation of the money commodity) is an efficient allocation for $\overline{u}$, as it is the outcome of the mechanism f for that profile. Its efficiency does not disappear when utility functions are replaced with sharpenings at $(a^*, \frac{\bar{b}}{n})$. Consider a profile $\tilde{\mathbf{u}}^\alpha$ where for every j, $\tilde{u}_j^\alpha$ is quasilinear, and shares with u_j^α the indifference curve through $(a^*, \frac{\bar{b}}{n})$. As in the analysis of the quasilinear case of section 3 and by Fact 1, $(a^*, \frac{\bar{b}}{n}, \ldots, \frac{\bar{b}}{n})$ is the only efficient IC outcome for $\tilde{\mathbf{u}}^\alpha$. But all the utility functions of the profile $\mathbf{u}^\alpha$ are locally quasilinear around $(a^*, \frac{\bar{b}}{n}, \ldots, \frac{\bar{b}}{n})$, and moreover, for each j, there is a constant segment along the line $a = a^*$ around $\frac{\bar{b}}{n}$ that belongs to the quasilinear region. Therefore, there is a neighborhood of $(a^*, \frac{\bar{b}}{n}, \ldots, \frac{\bar{b}}{n})$ where this point is the only efficient IC outcome for all α. As this point is not the outcome of the mechanism for $\alpha = 0$, but it is the outcome of the mechanism for $\alpha = 1$, we obtain a contradiction. Therefore on the domain $\mathcal{D}(\overline{\mathbf{u}}, f(\overline{\mathbf{u}}))$, $g_0 \equiv a^*$, and by Fact 1 the private good is allocated equally to all, just like the mechanism f. ■

References

[1] Barbera, S. and M. Jackson, 1994. "A characterization of strategy-proof social choice functions for economies with pure public goods," *Social Choice and Welfare*, 11:241–252.

[2] Clarke, E.H., 1971. "Multipart pricing of public goods," *Public Choice*, 11:17–33.

[3] Green, J and J.-J. Laffont, 1977. "Characterization of satisfactory mechanisms for the revelation of preferences for public goods," *Econometrica*, 45:427–438.

[4] Groves, T. and J. Ledyard, 1987. "Incentive compatibility since 1972," in T. Groves and R. Radner (eds.), *Essays in Honor of Leonid Hurwicz*. Minneapolis: University of Minnesota Press.

[5] Groves, T. and M. Loeb, 1975. "Incentives and public inputs," *Journal of Public Economics*, 4:211–226.

[6] Hammond, P., 1979. "Straightforward individual incentive compatibility in large economies," *Review of Economic Studies*, 46:263–282.

[7] Jehiel, P., M. Meyer-ter-Vehn, M., B. Moldovanu, and W. Zame, 2006. "The Limits of Ex-post Implementation," *Econometrica* 74:585-611.

[8] Makowski, L and J. M. Ostroy, 1992. "Groves mechanisms in continuum economies: Characterization and existence," *Journal of Mathematical Economics*, 21:1–35.

[9] Makowski, L., J.M. Ostroy, and U. Segal, 1999. "Efficient incentive compatible economies are perfectly competitive," *Journal of Economic Theory*, 85:169–225.

[10] Mitsui, T., 1983. "Asymptotic efficiency of the Pivot mechanism with a general project space,“ *Journal of Economic Theory*, 31:318–331.

[11] Moulin, H., 1980. "On strategy-proofness and single peakedness," *Public Choice*, 35: 437–455.

[12] Rob, R., 1982. "Asymptotic efficiency of the demand revealing mechanism," *Journal of Economic Theory*, 28:209–220.

[13] Samuelson, P., 1954. "The Pure Theory of Public Expenditure," *The Review of Economics and Statistics*, 387–389.

[14] Serizawa, S., 1999. "Strategy-proof and symmetric social choice functions for public good economies," *Econometrica*, 67:121–145.

[15] Tideman, T.N. and G. Tullock, 1976. "A new and superior process for making social choices," *Journal of Political Economy*, 84:1145–1159.

[16] W. Vickrey, W., 1961. "Counterspeculation, auctions, and competitive sealed tenders," *Journal of Finance*, 16:8–37.

[17] Walker, M., 1980. "On the nonexistence of a dominant-strategy mechanism for making optimal public decisions," *Econometrica*, 48:1521–1540.

[18] Zhou, L., 1994. "Impossibility of strategy-proof mechanisms in economies with pure public goods," *Review of Economic Studies*, 58:107–119.